Ladislav Nebeský

Route systems and bipartite graphs

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 2, 260–264

Persistent URL: http://dml.cz/dmlcz/102457

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

ROUTE SYSTEMS AND BIPARTITE GRAPHS

LADISLAV NEBESKÝ, Praha

(Received January 3, 1989)

Route systems, which are defined in the present paper, generalize the systems of all shortest paths of connected graphs. The route systems which are the systems of all shortest paths of connected bipartite graphs will be characterized here.

We first make some conventions concerning sequences. Let V be a finite nonempty set. We denote by $\mathscr{S}_N(V)$ the set of all sequences $(u_1, \ldots, u_j)$ such that $j \geqq 1$ and $u_1, \ldots, u_j \in V$. Let $k \geqq 1$, and let $v_1, \ldots, v_k \in V$; if we denote $\alpha = (v_1, \ldots, v_k)$, then we shall write $|\alpha| = k$ and $\bar{\alpha} = (v_k, \ldots, v_1)$. Let $m \geqq 2$, let $n_1, \ldots, n_m \geqq 1$, and let $w_{11}, \ldots, w_{1n_1}, \ldots, w_{m1}, \ldots, w_{mn_m} \in V$; if we denote

$$\beta_1 = (w_{11}, \ldots, w_{1n_1}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\beta_m = (w_{m1}, \ldots, w_{mn_m}),$$

then we shall denote by $(\beta_1, \ldots, \beta_m)$ the sequence

$$(w_{11}, \ldots, w_{1n_1}, \ldots, w_{m1}, \ldots, w_{mn_m}).$$

Moreover, we denote by $\mathscr{S}(V)$ the set $\mathscr{S}_N(V) \cup \{\omega\}$, where ω is the empty sequence with the properties that $|\omega| = 0$, $\bar{\omega} = \omega$, and $(\omega, \gamma) = (\gamma, \omega) = \gamma$ for any $\gamma \in \mathscr{S}(V)$. We denote by $\mathscr{A}_N(V)$ the set of all sequences $(u_1, \ldots, u_j)$ such that $j \geqq 1$, and $u_1, \ldots, u_j$ are mutually distinct elements of V. Finally, we denote $\mathscr{A}(V) = \mathscr{A}_N(V) \cup \{\omega\}$.

We shall say that an ordered pair $(V, \mathscr{R})$ is a *route system* if V is a finite nonempty set, $\mathscr{R} \subseteq \mathscr{A}_N(V)$, and the following axioms are fulfilled:

I. if $\alpha \in \mathscr{R}$, then $\bar{\alpha} \in \mathscr{R}$;
II. if $\alpha, \gamma \in \mathscr{A}(V)$, $\beta \in \mathscr{A}_N(V)$, and $(\alpha, \beta, \gamma) \in \mathscr{R}$, then $\beta \in \mathscr{R}$;
III. if $u, v \in V$, $\alpha, \beta, \gamma, \delta \in \mathscr{A}(V)$, and $(\alpha, u, \beta, v, \gamma)$, $(u, \delta, v) \in \mathscr{R}$, then $(\alpha, u, \delta, v, \gamma) \in \mathscr{R}$;
IV. for any distinct $u, v \in V$ there exists $\alpha \in \mathscr{A}(V)$ such that $(u, \alpha, v) \in \mathscr{R}$.

Let $(V, \mathscr{R})$ be a route system. Then elements of V will be referred to as *vertices*, and elements of $\mathscr{R}$ will be referred to as *routes*. Let u and v be vertices, and α be a route; if either $u = v$ and $\alpha = (u)$ or $u \neq v$ and there exists $\beta \in \mathscr{A}(V)$ such that $\alpha = (u, \beta, v)$, then we say that α is an $u - v$ route.

By a graph we mean a finite undirected graph with no loop or multiple edge, i.e.

a graph in the sense of [1], for example. Let $(V, \mathscr{R})$ be a route system. By the graph of $(V, \mathscr{R})$ we mean the graph G such that the vertex set $V(G)$ of G is identical with V and the edge set $E(G)$ of G is defined as follows:

$$uv \in E(G) \quad \text{if and only if} \quad (u, v) \in \mathscr{R}\,.$$

for any distinct $u, v \in V$.

We say that a route system $(V, \mathscr{R})$ is on a graph G, if G is the graph of $(V, \mathscr{R})$. Obviously, every route system is on exactly one graph.

Let $(V, \mathscr{R})$ be a route system, and let G be its graph. As follows from Axiom IV, G is connected. If we denote by $\mathscr{P}$ the set of all paths of G, then $\mathscr{R} \subseteq \mathscr{P} \subseteq \mathscr{A}_N(V)$. (Sometimes, paths of a graph are considered to be alternating sequence of vertices and edges. But in the present paper, paths of a graph are considered to be sequences of vertices).

Let G be a connected graph. Put $V = V(G)$. If $u, v \in V$, then we denote by $d(u, v)$ the distance between u and v in G. Let $w_1, w_2 \in V$, and let α be a $w_1 - w_2$ path of G; obviously, $|\alpha| \geqq d(w_1, w_2) + 1$; we say that α is a shortest $w_1 - w_2$ path of G if $|\alpha| = d(w_1, w_2) + 1$. Let β be a path of G; we shall say that β is a shortest path of G if there exist $u, v \in V$ such that β is a shortest $u - v$ path of G. Let $\mathscr{D}$ denote the set of all shortest paths of G. It is easy to show that $(V, \mathscr{D})$ is a route system. We shall say that $(V, \mathscr{D})$ is the *basic system* on G. If G is not a tree, then there exists a route system on G different from the basic system on G. It is not difficult to prove that for every spanning tree T of G there exists $\mathscr{R}_T \subseteq \mathscr{A}_N(V)$ such that each path of T belongs to $\mathscr{R}_T$ and $(V, \mathscr{R}_T)$ is a route system on G.

Let $(V, \mathscr{R})$ be a route system. For any $u, v \in V$ we denote by $\#_0(u, v)$ the set of all $w \in V$ such that there exists $t \in V - \{w\}$ with the properties that $(w, t) \in \mathscr{R}$, t belongs to a $w - u$ route, and t belongs to no $w - v$ route. Moreover, we define

$$\#(w_1, w_2) = \#_0(w_1, w_2) \cup \{w_1\}$$

for any $w_1, w_2 \in V$. Under the condition that $(V, \mathscr{R})$ is the basic system of a connected graph, the mapping $\#$ has been studied in [2].

Proposition 3 in [2] can be reformulated as follows: if $(V, \mathscr{R})$ is the basic system on a connected graph G, then G is bipartite if and only if $\#(u, v) \cap \#(v, u) = \{u, v\}$ for any distinct $u, v \in V$ such that $uv \in E(G)$. In the present paper we shall prove a much more general result:

Theorem. *Let $(V, \mathscr{R})$ be a route system, and let G be the graph of $(V, \mathscr{R})$. Then the following statements are equivalent:*

(1) $\#(u, v) \cap \#(v, u) = \{u, v\}$ *for any distinct $u, v \in V$ such that $uv \in E(G)$;*

(2) *for any mutually distinct $x, y, z \in V$ such that $yz \in E(G)$ there exists $\alpha \in \mathscr{A}(V)$ with the property that either $(x, \alpha, y, z) \in \mathscr{R}$ or $(x, \alpha, z, y) \in \mathscr{R}$;*

(3) *$(V, \mathscr{R})$ is the basic system on G and G is bipartite.*

Proof. (1) $\Rightarrow$ (2). Let (1) hold. Consider any mutually distinct $x, y, z \in V$ such that $yz \in E(G)$. We wish to prove that there exists $\alpha \in \mathscr{A}(V)$ such that either (x, α, y, z) or (x, α, z, y) is a route. To the contrary, we assume that neither (x, α_1, y, z) nor (x, α_1, z, y) is a route for any $\alpha_1 \in \mathscr{A}(V)$. According to (1), $x \notin \#(y, z) \cap \#(z, y)$. Without loss of generality we assume that $x \notin \#(y, z)$. Denote $x_1 = x$. There exists $\beta_1 \in \mathscr{A}(V)$ such that (x_1, β_1, y) is a route. Since (x_1, β_1, y, z) is not a route and $x_1 \notin \#(y, z)$, there exist $\gamma_1, \delta_1, \beta_2 \in \mathscr{A}(V)$ and $x_2 \in V$ such that $\beta_1 = (\gamma_1, x_2, \delta_1)$, $(x_1, \gamma_1, x_2, \beta_2, z)$ is a route, y does not belong to β_2, and no vertex of δ_1 belongs to any $x_1 - z$ route. Since neither (x_1, α_1, y, z) nor (x_1, α_1, z, y) is a route for any $\alpha_1 \in \mathscr{A}(V)$, Axiom III implies that neither (x_2, α_2, y, z) nor (x_2, α_2, z, y) is a route for any $\alpha_2 \in \mathscr{A}(V)$. Since no vertex of δ_1 belongs to any $x_1 - z$ route, Axiom III implies that no vertex of δ_1 belongs to any $x_2 - z$ route, and therefore, $x_2 \in \#(y, z)$. As follows from (1), $x_2 \notin \#(z, y)$. Obviously, (x_2, β_2, z) is a route. Since (x_2, β_2, z, y) is not a route, there exist $\gamma_2, \delta_2, \beta_3 \in \mathscr{A}(V)$ and $x_3 \in V$ such that $\beta_2 = (\gamma_2, x_3, \delta_2)$, $(x_2, \gamma_2, x_3, \beta_3, y)$ is a route, z does not belong to β_3, and no vertex of δ_2 belongs to any $x_2 - y$ route. Since neither (x_2, α_2, z, y) nor (x_2, α_2, y, z) is a route for any $\alpha_2 \in \mathscr{A}(V)$, Axiom III implies that neither (x_3, α_3, z, y) nor (x_3, α_3, y, z) is a route for any $\alpha_3 \in \mathscr{A}(V)$. Since no vertex of δ_2 belongs to any $x_2 - y$ route, Axiom III implies that no vertex of δ_2 belongs to any $x_3 - y$ route, and therefore $x_3 \in \#(z, y)$. As follows from (1), $x_3 \notin \#(y, z)$. Since $(x_1, \gamma_1, x_2, \delta_1, y)$ and $(x_2, \gamma_2, x_3, \beta_3, y)$ are routes, Axiom III implies that x_1, x_2, x_3 are mutually distinct.

If we continue our construction, we get a sequence $(x_1, x_2, x_3, \ldots, x_{|V|+1})$ of mutually distinct vertices of V, which is a contradiction. Hence there exists $\alpha \in \mathscr{A}(V)$ such that (x, α, y, z) or (x, α, z, y) is a route, and (2) holds.

(2) $\Rightarrow$ (3). Let (2) hold. Obviously, G is connected. If $(V, \mathscr{R})$ is the basic system on G, then it easily follows from (2) that G is bipartite. Therefore, we need to prove that $(V, \mathscr{R})$ is the basic system on G.

Consider arbitrary $u, v \in V$. We wish to prove that for any shortest $u - v$ path ξ of G and any $u - v$ route ζ it holds that ξ is a route and ζ is a shortest path of G. We proceed by induction on $d(u, v)$. The case when $d(u, v) = 0$ is obvious. Let $d(u, v) \geqq 1$.

Consider an arbitrary shortest $u - v$ path ξ and arbitrary $u - v$ route ζ. Put $m = d(u, v)$ and $n = |\zeta| - 1$. Obviously, $m = |\xi| - 1$ and $m \leqq n$. It ξ and ζ have a common vertex different from u and v, then – by using the induction assumption and Axiom III – it is not difficult to prove that ξ is a route and ζ is a shortest path. We shall now assume that ξ and ζ have no common vertex different from u and v. Consequently, there exist mutually distinct vertices $x_1, \ldots, x_m, y_1, \ldots, y_n$ such that

$$\xi = (x_1, \ldots, x_m, y_1) \quad \text{and}$$
$$\zeta = (x_1, y_n, \ldots, y_1) .$$

Obviously, $x_1 = u$ and $y_1 = v$. Denote

$$\xi_k = (x_k, \ldots, x_m, y_1, \ldots, y_k) \quad \text{and} \quad \zeta_k = (x_k, \ldots, x_1, y_n, \ldots, y_k)$$

for each $k \in \{1, \ldots, m\}$. Clearly, $\xi_1 = \xi$ and $\zeta_1 = \zeta$. Since ξ is a shortest path and ζ is a route, we can see that ξ_k and ζ_k are paths of G, for each $k \in \{1, \ldots, m\}$.

We now prove the following auxiliarly statement:

(4) if ζ_j is a route, then ξ_j is a route, for any $j \in \{1, \ldots, m\}$.

Consider an arbitrary $j \in \{1, \ldots, m\}$. There exist mutually distinct vertices $u_1, \ldots, u_m, v_1, \ldots, v_n$ such that

(5) $\xi_j = (u_1, \ldots, u_m, v_1)$ and

(6) $\zeta_j = (u_1, v_n, \ldots, v_1)$.

Clearly, $\{u_1, \ldots, u_m, v_1, \ldots, v_n\} = \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$. Now, let ζ_j be a route. Obviously, if $m = 1$, then ξ_j is a route. Assume that $m \geqq 2$. Let $d(u_1, v_1) \leqq m - 1$; according to the induction assumption ζ_j is a shortest $u_1 - v_1$ path of G, and thus $|\zeta_j| \leqq m$; but $|\zeta_j| = n + 1 > m$, which is a contradiction. Thus, we may assume that $d(u_1, v_1) = m$. Therefore, ξ_j is a shortest path. It follows from the induction assumption that $(u_1, \ldots, u_m)$ is a route. According to (2) there exists $\gamma \in \mathscr{A}(V)$ such that either (u_1, γ, v_1, u_m) or (u_1, γ, u_m, v_1) is a route. We first assume that (u_1, γ, v_1, u_m) is a route; since ζ_j is a route, Axiom III implies that $(u_1, v_n, \ldots, v_1, u_m)$ is also a route; since $d(u_1, u_m) = m - 1$, it follows from the induction assumption that $(u_1, v_n, \ldots, v_1, u_m)$ is a shortest $u_1 - u_m$ path; thus $n + 1 = m - 1$, which is a contradiction. We now assume that (u_1, γ, u_m, v_1) is a route; since $(u_1, \ldots, u_m)$ is a route, Axiom III implies that $\xi_j = (u_1, \ldots, u_m, v_1)$ is also a route, which completes the proof of (4).

Recall that $\xi = \xi_1$ and $\zeta = \zeta_1$. Since ζ is a route, it follows from (4) that ξ is also a route. It remains to prove that ζ is a shortest path of G; in other words, to prove that $m = n$. To the contrary, we suppose that $n > m$.

Denote

$$\zeta_{m+1} = (y_1, x_m, \ldots, x_1, y_n, \ldots, y_{m+1}).$$

We distinguish the following cases and subcases:

1. Assume that ζ_{m+1} is a route. Since ζ_1 is a route, Axiom III implies that $(y_1, \ldots, y_n, x_1, y_n, \ldots, y_{m+1})$ is a route. But $(y_1, \ldots, y_n, x_1, y_n, \ldots, y_{m+1}) \notin \mathscr{A}_N(V)$, which is a contradiction with the definition of a route system.

2. Assume that ζ_{m+1} is not a route. Since ζ_1 is a route, there exists $j \in \{1, \ldots, m\}$ such that

(7) ζ_j is a route but ζ_{j+1} is not a route.

We express ξ_j and ζ_j as in (5) and (6), respectively. Since ζ_j is a route, it follows from (4) that ξ_j is a route.

2.1. Assume that $m = 1$. It follows from (2) that there exists $\delta \in \mathscr{A}(V)$ such that either (v_n, δ, u_1, v_1) or (v_n, δ, v_1, u_1) is a route. Recall that $(u_1, v_n, \ldots, v_1)$ is a route. If (v_n, δ, u_1, v_1) is a route, then according to Axiom III $(v_n, \delta, u_1, v_n, \ldots, v_1)$ is a route, but $(v_n, \delta, u_1, v_n, \ldots, v_1) \notin \mathscr{A}_N(V)$, which is a contradiction. Similarly, if (v_n, δ, v_1, u_1) is a route, then $(v_n, \delta, v_1, \ldots, v_n, u_1)$ is a route, which is also a contradiction.

2.2. Assume that $m \geqq 2$. As follows from (2), there exists $\varphi \in \mathscr{A}(V)$ such that either (u_2, φ, v_2, v_1) or (u_2, φ, v_1, v_2) is a route.

2.2.1. Assume that (u_2, φ, v_2, v_1) is a route. Since $d(u_2, v_1) \leqq m - 1$, it follows from the induction assumption that (u_2, φ, v_2, v_1) is a shortest path, and therefore, $|\varphi| \leqq m - 3$. Since $(u_1, u_2, \ldots, u_m, v_1)$ and (u_2, φ, v_2, v_1) are routes, we have that $(u_1, u_2, \varphi, v_2, v_1)$ is a route. Since $|\varphi| \leqq m - 3$ and (u_1, u_2, φ, v_2) is a route, we have that $d(u_1, v_2) \leqq m - 1$. Since $(u_1, v_n, \ldots, v_2)$ is a route and $d(u_1, v_2) \leqq m - 1$, it follows from the induction assumption that $(u_1, v_n, \ldots, v_2)$ is a shortest path. Thus, $n \leqq m$, which is a contradiction.

2.2.2. Assume that (u_2, φ, v_2, v_1) is not a route. Then (u_2, φ, v_1, v_2) is a route. According to (2), there exists $\psi \in \mathscr{A}(V)$ such that either (v_2, ψ, u_2, u_1) or (v_2, ψ, u_1, u_2) is a route.

2.2.2.1. Assume that (v_2, ψ, u_2, u_1) is a route. Since (u_2, φ, v_1, v_2) is a route, we have that $(v_2, v_1, \varphi, u_2, u_1)$ is a route. Since $(u_1, v_n, \ldots, v_1)$ is a route, we have that $(v_2, v_1, v_2, \ldots, v_n, u_1)$ is a route, which is a contradiction.

2.2.2.2. Assume that (v_2, ψ, u_2, u_1) is not a route. Then $(v_2, \psi, u_1, u_2)$is a route. Since $(u_1, v_n, \ldots, v_2)$ is a route, we have that $(u_2, u_1, v_n, \ldots, v_2)$ is a route. This means that ζ_{j+1} is a route; which is a contradiction with (7).

Thus, we have received that $m = n$. Consequently, we have proved that (3) holds.

(3) $\Rightarrow$ (1) Let (3) hold. Obviously, G is connected. If $|V| \leqq 2$, then (1) holds trivially. Let $|V| \geqq 3$. Consider arbitrary adjacent vertices u and v of G. Let $w \in V - \{u, v\}$. Since G is connected and bipartite, we can see that either $d(w, u) = d(w, v) - 1$ or $d(w, v) = d(w, u) - 1$. We first assume that $d(w, u) = d(w, v) - 1$. It is obvious that if α is a shortest $w - u$ path of G, then (α, v) is a shortest $w - v$ path of G. Hence, $w \notin \#(u, v)$. Similarly, we can show that if $d(w, v) = d(w, u) - 1$, then $w \notin \#(v, u)$. Thus, we have that $w \notin \#(u, v) \cap \#(v, u)$. This means that (1) holds.

The proof of the theorem is complete.

References

[1] *M. Behzad, G. Chartrand,* and *L. Lesniak-Foster:* Graphs & Digraphs. Prindle, Weber & Schmidt, Boston 1979.

2] *L. Nebeský:* On certain extensions of intervals in graphs. Časopis pěst. mat. *115*, 1990, 171–177.

Author's address: 116 38 Praha 1, nám. J. Palacha 2, Czechoslovakia (Filozofická fakulta Univerzity Karlovy).