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MEDIAL IDEMPOTENT GROUPOIDS I

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1. INTRODUCTION

In the class of the simplest algebras, groupoids, a special attention have received groupoids which are medial and idempotent, i.e. satisfying the conditions

(M) $\qquad (xy)(uv) = (xu)(yv)$

(I) $\qquad x^2 = x$

This is so, becouse these appear as a matter of fact in almost all branches of mathematics. For example, the arithemitical mean or the square root $\surd(xy)$ define such groupoids. Also, affine spaces over prime fields are equivalent to these groupoids. Accordingly, medial idempotent groupoids were studied by many authors and are known under various names like groupoid modes, binary modes instead of "medial" also the adjectives "entropic" or "abelian" are used (see [14] and cf. also [22]).

A remarkable result in this area is the description of all varieties of commutative medial idempotent groupoids (CIA-groupoids) given in Ježek and Kepka [14]. The noncommutative case turn out much more complicated, however.

In this paper we propose a new approach to the problem of description of medial idempotent groupoids. Namely, we propose to describe them (and clasify) by means of the number $p_2(G, \cdot)$ of essentially polynomial in $(G, \cdot)$. As a first step in this direction we have

Theorem. *Let $(G, \cdot)$ be a proper medial idempotent groupoid, i.e., xy is essentially binary. Then we have*

(i) $p_2(G, \cdot) = 1$ *if and only if $(G, \cdot)$ is either a semilattice or an affine space over $GF(3)$.*

(ii) $p_2(G, \cdot) = 2$ *if and only if $(G, \cdot)$ is either a diagonal semigroup or an n-polynomial groupoid or an affine space over $GF(4)$.*

(iii) $p_2(G, \cdot) = 3$ *if and only if $(G, \cdot)$ is either an affine space over $GF(5)$ or a nontrivial Płonka sum of affine spaces over $GF(3)$ which are not all singletons.*

All the groupoids appearing in the above theorem are well-known and the definitions and basic characterizations of them are recalled to the reader in § 2.

Here, as in [12], polynomially equivalent algebras (i.e., having the same sets of polynomials) are treated as identical. The proof of the part (i) is explicitely contained in [6] (It follows from Lemma 1 of [8], Theorem 1 and Theorem 8 of [6]). The proof of (ii) is analogous to the proof of the characterization theorem for distributive and idempotent groupoids (for details see [7]) and we also use the results of [10]. In this paper we prove the part (iii). First (§ 3) we characterize all medial commutative idempotent groupoids with $p_3(G, \cdot) = 3$. And then we prove (§ 4) that any noncommutative medial idempotent groupoid $(G, \cdot)$ with $p_2(G, \cdot) = 3$ is polynomially equivalent to some commutative medial idempotent groupoid.

Our terminology is standard (cf [12]).

Throughout, by xy^n we denote as usually the polynomial $(\ldots(xy)\ldots y)\,y$ with y appearing n-times. Instead of medial idempotent we write briefly MIG.

2. CHARACTERIZATIONS

We recall the definitions and basic characterizations of the groupoids appearing in Theorem.

1. Semilattices. A groupid $(S, \cdot)$ satisfying

1. $x^2 = x$
2. $xy = yx$
3. $(xy)\,z = x(yz)$

is called a semilattice. The variety of all semilattice will be denoted by $\mathscr{C}$.

It is well-known that with any semilattice $(S, \cdot)$ one can associate a partial ordered set $(S, \leqq)$ such that for every $a, b \in S$ there exists the least upper bound i.e., l.u.b. (a, b) and conversely (see e.g., [12]).

It is also known that an algebra $\mathfrak{A}$ is a nontrivial semilattice if and only if $p_0(\mathfrak{A}) = 0$ and $p_n(\mathfrak{A}) = 1$ for all $n \geqq 1$. For the definitions of p_n-sequences and other used in this paper we refer to [12].

2. Diagonal semigroups. Following [16] a diagonal semigroup is a groupoid $(G, \cdot)$ satisfying

1. $x^2 = x$
2. $(xy)\,z = x(yz)$
3. $(xy)\,z = xz$

By Δ we denote the variety of all diagonal semigroups.

Using the characterization theorem for diagonal algebras from [16] we see that a groupoid $(G, \cdot)$ is a diagonal semigroup if and only if there exist two sets G_1 and G_2 such that $G = G_1 \times G_2$ and the operation $\cdot$ is defined as follows

$$\langle x_1, y_1\rangle \,.\, \langle x_2, y_2\rangle = \langle x_1, y_2\rangle$$

where $x_1, x_2 \in G_1$ and $y_1, y_2 \in G_2$.

It is clear that for a proper diagonal semigroup $(G, \cdot)$ we have $p_0(G, \cdot) = p_1(G, \cdot) = 0$, $p_2(G, \cdot) = 2$ and $p_n(G, \cdot) = 0$ for all $n \geqq 3$. We also have an algebra $\mathfrak{A}$ is a proper diagonal semigroup if and only if $\mathfrak{A}$ represents the sequence $\langle 0, 0, 2, 0, \ldots \rangle$ (for details see [16]).

3. n-polynomial groupoids. By Σ_i $(i = 1, 2, 3)$ we denote the varieties of groupoids $(G, \cdot)$ defined by the following identities:

$$\begin{aligned}
\Sigma_1:\quad & x^2 = x\,, \\
& (xy)\,z = x(yz)\,, \\
& x(yz) = x(zy)\,; \\
\Sigma_2:\quad & x^2 = x\,, \\
& (xy)\,z = (xz)\,y\,, \\
& x(yz) = xy\,, \\
& (xy)\,y = xy\,; \\
\Sigma_3:\quad & x^2 = x\,, \\
& (xy)\,z = (xz)\,y\,, \\
& x(yz) = xy\,, \\
& (xy)\,y = x\,.
\end{aligned}$$

(For details see [12], pp 394–395 and [19]).

In [19] a characterization theorem for groupoids from the above varieties is given (see also [3], [20] and [21]).

Recall that in [19] it is shown that if a groupoid $(G, \cdot)$ satisfies $p_n(G, \cdot) = n$ for all n, then $(G, \cdot)$ belongs to the variety Σ_i or its dual for some $i = 1, 2, 3$.

It is also easy to see that a proper groupoid $(G, \cdot)$ being not a semilattice belonging to Σ_i $(i = 1, 2, 3$ or its dual) satisfies $p_n(G, \cdot) = n$ for all n. Thus we shall call a groupoid $(G, \cdot)$ an n-polynomial groupoid (cf [7]) if

$$p_n(G, \cdot) = n \quad \text{for all} \quad n\,.$$

4. Affine spaces (modules). By an affine module G over the ring Z_d of integers modulo d, for odd d, we mean the groupoid $\mathfrak{G} = (G, \cdot)$ associated with an abelian group $(G, +)$ of exponent d where $\cdot$ is defined by the formula

$$xy = \frac{d+1}{2}\,(x + y)$$

for all $x, y \in G$. The polynomials of $\mathfrak{G}$ are in fact the idempotent polynomials of the group $(G, +)$ and therefore $\mathfrak{G}$ is also called the (full) idempotent reduct of $(G, +)$ (cf [18]). On the other hand, $\mathfrak{G}$ is equivalent to the algebra generated by the polynomials

$$a_1x_1 + \ldots + a_nx_n$$

of the module $(G, +)$ over Z_d satisfying $\sum_{i=1}^{n} a_i = 1$. Algebras of this type introduced in [15] and called affine modules, in accordance with the notion of an affine space (the case when Z_a is a field) (see [1] and [2]). In general, the ring Z_a can be replaced by any ring in the above definition.

4.1. Affine spaces over $GF(3)$. Using the main result of [13] we have

a groupoid $(G, \cdot)$ is an affine space over $GF(3)$ if and only if $(G, \cdot)$ belongs to the following variety defined by the identities

$$x^2 = x\,, \quad xy = yx\,, \quad (xy)\,y = x \quad \text{and} \quad (xy)(uv) = (xu)(yv)\,.$$

We also see (cf. [13]) that for such nontrivial affine spaces we have

$$p_n = \frac{2^n - (-1)^n}{3} \quad \text{for all} \quad n\,.$$

In general, for any affine space over $GF(p)$ we have

$$p_n = \frac{(p-1)^n - (-1)^n}{p} \quad \text{for all} \quad n$$

(cf. [1]).

4.2. Affine spaces over $GF(5)$. Analogously as in [13] one can prove the following:

Fact. An algebra (G, F) is an affine space over $GF(5)$ if and only if there exists a binary polynomial $\cdot$ over (G, F) such that $(G, \cdot)$ satisfies the following identities

$$x^2 = x\,, \quad xy = yx\,, \quad ((xy)\,y)\,x = y \quad \text{and} \quad (xy)(uv) = (xu)(yv)\,.$$

Let us add that for any nontrivial affine space over $GF(5)$ we have

$$p_n = \frac{4^n - (-1)^n}{5}$$

for all n.

4.3. Affine spaces over $GF(4)$. Let G be an affine space over a four-element field $K = \{a, b, 0, 1\}$. Take into account the following groupoid $(G, \circ)$ where $x \circ y = ax + by$. Then it is easy to check that the groupoid $(G, \circ)$ satisfies

$$x \circ x = x\,, \quad (x \circ y) \circ x = y \quad \text{and} \quad (x \circ y) \circ z = (z \circ y) \circ x\,.$$

The groupoid $(G, \circ)$ is called an affine space over $GF(4)$. It is clear that such groupoids are medial and also quasigroups.

Following [10] we have

A groupoid $(G, \cdot)$ is an affine space over $GF(4)$ if and only if $(G, \cdot)$ satisfies

$$xx = x\,, \quad (xy)\,x = y \quad \text{and} \quad (xy)\,z = (zy)\,x\,.$$

3. THE COMMUTATIVE CASE

In this section we prove (iii) of the theorem for medial commutative idempotent groupoids. We start with (all groupoids in this section are commutative)

Lemma 3.1. *If $(G, \cdot)$ is MIG such that $p_2(G, \cdot) > 1$, then $p_2(G, \cdot) \geqq 3$. Moreover, if $p_2(G, \cdot) = 3$, then $(G, \cdot)$ satisfies either*

$$xy = xy^3 \quad \text{or} \quad xy^3 = yx^2 .$$

Proof. Consider the standard polynomial $x \circ y = xy^2$. It is easy to see that $p_2(G, \cdot) > 1$ proves that $x \circ y$ is essentially binary. If $x \circ y$ is commutative, then applying Theorem 8 of [6] we deduce that $(G, \cdot)$ is a semilattice which contradicts $p_2(G, \cdot) > 1$. Since $p_2(G, \cdot) = 3$ we infer that xy, $x \circ y$ and $y \circ x$ are the only essentially binary polynomials over $(G, \cdot)$. Take now the polynomial $x * y = xy^3$. Applying Theorem 1 of [5] we conclude that $x * y \neq y$. If $x * y = x$ (it is clear that xy^2 is essentially binary), then applying Theorem 3 of [5] we get $p_2(G, \cdot) \geqq 5$, a contradiction. Thus we have proved that $(G, \cdot)$ satisfies either

$$xy = xy^3 \quad \text{or} \quad xy^3 = yx^2 \quad \text{or} \quad xy^2 = xy^3 .$$

The last identity according to Theorem 8 of [6] cannot happen which completes the proof.

Using a characterization theorem of [6] we get

Lemma 3.2. *Let $(G, \cdot)$ be MIG. Then $(G, \cdot)$ is a Płonka sum of affine spaces over $GF(3)$ if and only if $(G, \cdot)$ satisfies the following identity $xy = xy^3$.*

Lemma 3.3. *Let $(G, \cdot)$ be MIG with $xy^3 = yx^2$. Then the following are equivalent:*

(i) *$(G, \cdot)$ is a semilattice,*
(ii) *$(G, \cdot)$ satisfies $xy^2x = xy^2$,*
(iii) *$(G, \cdot)$ satisfies $xy^2x = yx^2$,*
(iv) *$(G, \cdot)$ satisfies $xy^2x = yx^2y$.*

Proof. It is clear that the condition (i) implies each of the remaining ones. Assume that (ii) holds. Then we have

$$\begin{aligned} xy^2 &= (xy^2)((xy^2)x) = ((xy^2)x)((xy)y) = \\ &= ((xy^2)(xy))(xy) = y(xy)^3 = (xy)y^2 = xy^3 = yx^2 . \end{aligned}$$

Thus $(G, \cdot)$ satisfies $xy^2 = yx^2$. Applying Theorem 8 of [6] we see that $(G, \cdot)$ is a semilattice.

Let now $(G, \cdot)$ satisfy $xy^2x = yx^2$ (iii). Then we have

$$y(yx)^2 = ((xy)y^2)(xy) = (xy^3)(xy) = (yx^2)(xy) = x(xy)^2 .$$

Thus we have

$$\begin{aligned} xy^2 &= yx^3 = (xy)x^2 = x(xy)^3 = (x(xy)^2)(xy) = \\ &= (y(yx)^2)(xy) = y(yx)^3 = yx^2 , \end{aligned}$$

which again proves that $(G, \cdot)$ is a semilattice.

If $xy^2x = yx^2y$ holds in $(G, \cdot)$, then using the medial law we get $xy^2x = xy$ and hence

$$xy^2 = ((xy)\,y^2)\,(xy) = xy^3xy = (yx^2)\,(xy) = ((yx)\,y)\,x = xy^2x\,.$$

Thus $xy^2 = xy^2x$ and therefore $xy^2 = yx^2$ which proves (as above) that $(G, \cdot)$ is a semilattice. The proof of the lemma is completed.

Lemma 3.4. *Let $(G, \cdot)$ be MIG. Then*

(i) $(G, \cdot)$ *is an affine space over $GF(3)$ if and only if $(G, \cdot)$ satisfies $xy^2x = x$.*

(ii) $(G, \cdot)$ *is an affine space over $GF(5)$ if and only if $(G, \cdot)$ satisfies $xy^2x = y$.*

Proof. If $(G, \cdot)$ is an affine space over $GF(3)$, then clearly $xy^2x = x$ holds in $(G, \cdot)$. One can also check that the identity $xy^2x = y$ holds in any affine space over $GF(5)$. We give here only the proof of the condition (ii) (The proof of (i) is similar). Assume that $xy^2x = y$ holds in $(G, \cdot)$. Using Theorem 10 of [6] we infer that $(G, \cdot)$ satisfies $xy^4 = x$. Applying now Theorem 1 of [6] we infer that $(G, \cdot)$ is an affine module over Z_d where $d|15$. If card $G = 1$, then clearly $(G, \cdot)$ is an affine space over $GF(5)$. If card $G > 1$, then using the identity $xy^2x = y$ and the fact that $d|15$ one gets that $d = 5$ and therefore $(G, \cdot)$ is an affine space over $GF(5)$. The proof of the lemma is completed.

Proposition 3.5. *Let $(G, \cdot)$ be MIG. Then $p_2(G, \cdot) = 3$ if and only if $(G, \cdot)$ is either a nontrivial sum of affine spaces being not all singletons or a nontrivial affine space over the field $GF(5)$.*

Proof. Assume that $p_2(G, \cdot) = 3$ (The converse is obvious). Applying Lemma 3.1 we infer that $(G, \cdot)$ satisfies either

$$xy = xy^3 \quad \text{or} \quad xy^3 = yx^2\,.$$

If $(G, \cdot)$ satisfies the first identity, then applying Lemma 3.2 we prove that $(G, \cdot)$ is a nontrivial Płonka sum of some affine spaces over $GF(3)$ which are not all one-element (we use here also the fact that a partition function $x \circ y = xy^2$ is essentially binary and noncommutative, for details see [6] and [17]).

Assume now that $xy^3 = yx^2$ holds in $(G, \cdot)$. Take into account the polynomial $x \square y = xy^2x$. By Lemmas 3.1 and 3.4 and the assumption $p_2(G, \cdot) = 3$ we obtain $xy^2x = y$. Applying again Lemma 1.4 we deduce that $(G, \cdot)$ is a nontrivial affine space over $GF(5)$ which completes the proof of the proposition.

4. NONCOMMUTATIVE CASE

Recall that a groupoid $(G, \cdot)$ is distributive if $(G, \cdot)$ satisfies both right and left-sided distributive laws

$$(xy)\,z = (xz)\,(yz) \quad \text{and} \quad z(xy) = (zx)\,(zy)\,.$$

It is clear that if $(G, \cdot)$ is medial and idempotent, then $(G, \cdot)$ is distributive. In this section as was previous ammounced we prove (iii) of the Theorem from § 2 for medial noncommutative idempotent groupoids.

First we prove

Lemma 4.1. *Let $(G, \cdot)$ be an idempotent groupoid. Then the following are equivalent*

(i) *$(G, \cdot)$ is a diagonal semigroup,*
(ii) *$(G, \cdot)$ is medial and $(G, \cdot)$ satisfies $(xy)\,x = x$,*
(iii) *$(G, \cdot)$ is distributive and $(G, \cdot)$ satisfies $(xy)\,x = x$.*

Proof. The implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious. Assume (iii). Then we have

$$(xy)\,z = (xz)\,(yz) = (x(yz))\,(z(yz)) = (x(yz))\,z\,.$$

Thus we get $(xy)\,z = (x(yz))\,z$. Putting in this identity yz for x we get $yz = (yz)\,z$. Using this identity we obtain

$$(xy)\,z = (xz)\,(yz) = ((xz)\,y)\,((xz)\,z) = ((xz)\,y)\,(xz) = xz\,.$$

Hence we get $(xy)\,z = xz$. Analogously one proves that $(G, \cdot)$ satisfies $x(yz) = xz$ and hence $(G, \cdot)$ is a diagonal semigroup.

Lemma 4.2. *If $(G, \cdot)$ is MIG; satisfying $(xy)\,x = y$, then $p_2(G, \cdot) \neq 3$.*

Proof. If $(G, \cdot)$ is commutative, then $(G, \cdot)$ is a Steiner quasigroup and therefore $p_2(G, \cdot) \leqq 1$. Suppose that $(G, \cdot)$ is noncommutative and $p_2(G, \cdot) = 3$. Take into account the standard polynomial $(xy)\,y$. First observe that $(xy)\,y \neq y$. Indeed, if $(xy)\,y = y$, then $x = y(xy) = ((xy)\,y)\,(xy) = y$, a contradiction.

If $(xy)\,y = x$, then $yx = ((yx)\,y)\,y = xy$ which gives $xy = yx$, a contradiction.

The identity $(xy)\,y = xy$ yields $xy = x$. (In fact, the groupoid $(G, \cdot)$ is a quasigroup.)

If $(xy)\,y = yx$, then $y(yx) = y((xy)\,y) = xy$ and hence $(yx)\,(xy) = y$. For such groupoids one can check that $p_2 \leqq 2$.

It remains to consider the case

$$(xy)\,y = (yx)\,x\,.$$

In this case we consider the polynomial $x(xy)$ and keep in mind the assumption $p_2(G, \cdot) = 3$ and $(xy)\,x = y$.

We prove that $x(xy)$ is none of the polynomials:

$$x, y, xy, yx \quad \text{and} \quad (xy)\,y\,.$$

We have

$x(xy) = x$ implies $x = xx = (x(xy))\,x = xy$,
$x(xy) = y$ implies $yx = (x(xy))\,x = xy$,
$x(xy) = xy$ implies $y = (xy)\,x = (x(xy))\,x = xy$,
$x(xy) = yx$ implies $(yx)\,x = (x(xy))\,x = xy$ which proves that $(G, \cdot)$ is commutative, a contradiction.

If again $x(xy)$ is commutative, then clearly $x(xy) = (xy)\,y$ and hence

$$xy = (x(xy))\,x = ((xy)\,y)\,x = ((xy)\,x)\,(yx) = y(yx)\,.$$

Thus $xy = yx$, a contradiction which completes the proof of the lemma

Lemma 4.3. *If* $(G, \cdot)$ *is MIG satisfying*

$$(xy)\,x = xy\,,$$

then $p_2(G, \cdot) \neq 3$.

Proof. If $(G, \cdot)$ is commutative, then $(G, \cdot)$ is a near-semilattice, i.e.,commutative idempotent and satisfying $xy^2 = xy$ and hence $p_2(G, \cdot) \leqq 1$ (According to Theorem 8 of [6] the groupoid $(G, \cdot)$ is even a semilattice).

If $(G, \cdot)$ is improper, then clearly $p_2(G, \cdot) \neq 3$ since $p_2(G, \cdot) = 0$. Further, assume that $(G, \cdot)$ is a (proper) medial idempotent noncommutative groupoid with $p_2(G, \cdot) = 3$.

First observe that

$$x(yz) = (xy)\,(xz) = ((xy)\,x)\,((xy)\,z) = (xy)\,((xy)\,z)\,.$$

Thus we get

$$x(yz) = (xy)\,((xy)\,z)\,.$$

Take now into account the polynomial $x(xy)$.

If $x(xy) = y$, then $x(yz) = z$, a contradiction.

If $x(xy) = x$, then $x(yz) = xy$ and hence

$$(xy)\,z = (xz)\,(yz) = (xz)\,y\,.$$

Such groupoids there exist (see e.g. [21]) and it is not difficult to prove that p_2 for these groupoids is either even or infinite.

If $x(xy) = xy$ holds in $(G, \cdot)$, then $(G, \cdot)$ satisfies $(xy)\,z = x(yz)$. Thus $(G, \cdot)$ is a noncommutative idempotent semigroup. One can easily check that for such semigroups we have $p_2 \neq 3$.

If $x(xy) = yx$, then we get

$$x(yz) = (xy)\,((xy)\,z) = z(xy)$$

and hence

$$xy = y(xy) = (yx)\,y = yx$$

which proves that $(G, \cdot)$ is commutative, a contradiction.

It remains to consider the case

$$x(xy) = y(yx)\,.$$

Then we have

$$x(yz) = (xy)\,((xy)\,z) = z(z(xy))\,.$$

This gives $xy = y(y(xy)) = y((yx)\,y) = y(yx)$ and therefore $(G, \cdot)$ is commutative which is impossible. The proof of the lemma is completed.

Lemma 4.4. *If* $(G, \cdot)$ *is MIG satisfying*

$$(xy)\, x = yx\,,$$

then $p_2(G, \cdot) \neq 3$.

Proof. As above we have

$$(xy)\, z = (x(yz))\,(yz)\,.$$

Consider the polynomial $(xy)\, y$. Further the proof of this lemma runs similarly as the proof of the preceding lemma. For instance, if $(G, \cdot)$ satisfies

$$(xy)\, y = (yx)\, x$$

then we have

$$(xy)\, z = (x(yz))\,(yz) = ((yz)\, x)\, x\,.$$

This gives

$$yz = ((yz)\, y)\, y = (zy)\, y$$

which proves that $(G, \cdot)$ is commutative, a contradiction.

Lemma 4.5. *If* $(G, \cdot)$ *is MIG, then the groupoid* $(G, \circ)$ *where* $x \circ y = (xy)\, x$ *is also MIG. Moreover there exist medial noncommutative idempotent groupoids* $(G, \cdot)$ *such that* $p_2(G, \cdot) = 3$ *and* $x \circ y = y \circ x$.

Proof. The first statement is a direct consequence of the medial law for the operation. To prove the second assertion it suffices to consider an affine space over $GF(5)$ treated as a groupoid $(G, \cdot)$ with $xy = 2x + 4y$.

Lemma 4.6. *If* $(G, \cdot)$ *is a medial noncommutative idempotent groupoid satisfying* $(xy)\, x = (yx)\, y$ *and* $p_2(G, \cdot) = 3$, *then the groupoids* $(G, \cdot)$ *and* $(G, \circ)$ *where* $x \circ y = (xy)\, x$ *are polynomially equivalent.*

Proof. We prove this lemma in several steps considering the polynomial $(xy)\, y$. Some of the proofs of some steps will be omitted. First we have

(i) Let $(G, \cdot)$ be an idempotent groupoid satisfying $(xy)\, y = y$. Then $(G, \cdot)$ is medial if and only if $(G, \cdot)$ satisfies

$$(xy)\, z = yz \quad \text{and} \quad x(yz) = y(xz)\,.$$

Thus one can easily see that if $(G, \cdot)$ is a medial idempotent groupoid satisfying $(xy)\, y = y$, then $p_2(G, \cdot) \neq 3$.

(ii) If $(G, \cdot)$ satisfies $(xy)\, y = x$, then $xy = (x \circ y) \circ y$ where $x \circ y = (xy)\, x$ and hence the groupoids $(G, \cdot)$ and $(G, \circ)$ are polynomially equivalent.

Indeed, putting xy for x in the identity $(xy)\, x = (yx)\, y$ we get

$$x(xy) = ((xy)\, y)\,(xy) = (y(xy))\, y = y((xy)\, y) = yx$$

(we use also the ditributive laws) and hence $(yx)\,(xy) = x$.

Further we have

$$(x \circ y) \circ y = (((xy)\, x)\, y)\,((xy)\, x) = (((xy)\, x)\,(xy))\,(yx) =$$
$$= ((x(xy))\, x)\,(yx) = ((yx)\, x)\,(yx) = y(yx) = xy\,.$$

Thus $(x \circ y) \circ y = xy$ and therefore $(G, \cdot)$ and $(G, \circ)$ are polynomially equivalent.

(iii) $(xy)\, y \neq xy$.

If $(xy)\, y = xy$ holds, then as above putting xy for x in $(xy)\, x = (yx)\, y$ one proves that $xy = yx$.

(iv) If $(xy)\, y = yx$, then $yx = (x \circ y) \circ y$.

(v) If $(xy)\, y = (yx)\, x$, then $(G, \cdot)$ is a semilattice (Thus in our groupoid $(xy)\, y \neq (yx)\, x$).

Indeed, first we have

$$(xy)\, x = (yx)\, y = (xy)\, y = (yx)\, x\,.$$

This gives

$$xy = (xy)(xy) = ((xy)\, x)((xy)\, y) = (xy)\, y = xy^2\,.$$

Thus $(G, \cdot)$ is a commutative idempotent groupoid satisfying $xy = xy^2$. According to Theorem 8 of [6] the groupoid $(G, \cdot)$ is a semilattice. This completes the proof of the lemma.

Lemma 4.7. *If $(G, \cdot)$ is a medial noncommutative idempotent groupoid satisfying $p_2(G, \cdot) = 3$, then the binary polynomial $(xy)\, x$ is commutative.*

Proof. The proof follows from Lemmas 4.1 – 4.4 and the assumption $p_2(G, \cdot) = 3$.

Proposition 4.8. *If $(G, \cdot)$ is a medial noncommutative idempotent groupoid satisfying $p_2(G, \cdot) = 3$, then $(G, \cdot)$ is polynomially equivalent to a medial commutative idempotent groupoid.*

Proof. It follows from Lemmas 2.5, 2.6 and 2.7.

Remark. Now the proof of (iii) of the Theorem is an immidiate consequence of Propositions 3.5 and 4.8.

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