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D-PROXIMITY SPACES

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1. INTRODUCTION:

It is well known that a pseudo metric space (X, d) provides a motivation for defining an EF-proximity viz.

(1.1) $A \,\delta\, B$ iff $d(A, B) = 0$.

The pseudo metric d induces a covering uniformity (on X) with a countable base $\{\mathcal{U}_n\colon n \in \mathbb{N}\}$. In terms of this base, the above proximity can also be defined by

(1.2) $A \,\delta\, B$ iff $\operatorname{St}(A, \mathcal{U}_n) \cap B \neq \emptyset$ for each $n \in \mathbb{N}$.

Now a developable space (X, τ) is a topological space with a development $\{\mathcal{U}_n\colon n \in \mathbb{N}\}$ which is a family of open covers such that for each $x \in X$ $\{\operatorname{St}(x, \mathcal{U}_n)\colon n \in \mathbb{N}\}$ is a nbhd. base at x. A developable space is one of the most important generalizations of a metric space and commands a vast literature. Hence it is natural to expect that a developable space would provide a motivation for another proximity – a generalization of an EF-proximity. Indeed if a proximity δ is defined by (1.2) for a development $\{\mathcal{U}_n\}$, then obviously δ is a compatible LO-proximity on (X, τ). Moreover, δ satisfies the additional condition:

(1.3) A non δ B implies the existence of subsets C, $\{C_n\colon n \in \mathbb{N}\}$ of X such that $B \subset C$, $X - C = \bigcup \{C_n\colon n \in \mathbb{N}\}$, A non δ C and C_n non δ C for each n.

To verify (1.3), we set $C = B^-$ and we know that, δ being a LO-proximity, A non δ C. We set $C_n = X - \operatorname{St}(C, \mathcal{U}_n)$. Clearly, C_n non δ C for each $n \in \mathbb{N}$. Also since $X - C$ is open, for each $x \in X - C$, there is an $n \in \mathbb{N}$ such that $\operatorname{St}(x, \mathcal{U}_n) \subset X - C$ i.e. $x \in C_n$. Thus $X - C = \bigcup \{C_n\colon n \in \mathbb{N}\}$. We will show later on that (1.3) is stronger than the LO-axiom and weaker than the EF-axiom.

Although pseudo metric spaces provide a motivation for EF-proximities via (1.1) or (1.2), a topological space (X, τ) has a compatible EF-proximity if and only if

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(X, τ) is completely regular (CR) i.e. homeomorphic to a subspace of the product of pseudometric spaces. Analogously, we will show that a topological space (X, τ) has a compatible D-proximity if and only if (X, τ) is D-completely regular (DCR) i.e. homeomorphic to a subspace of the product of developable spaces. These spaces were first discovered by Brandenburg [1], who along with Heldermann [5, 6] and others, also made a detailed study of these spaces. DCR spaces are generalizations of CR spaces and have analogues for most of the known results concerning CR spaces – see the above mentioned references (the readers will find a complete bibliography in Brandenburg [2]) as well as Di Maio-Naimpally-Pareek [3]. In this paper we continue this study with special reference to D-proximities.

In Section 2, we prove the compatibility of a DCR topology with a D-proximity and show that a D-proximity lies strictly between EF and LO proximities. In Section 3, we develop the theory of bases and subbases for D-proximities along the lines of the work of Sharma [10] who constructed such a theory for EF-proximities. In Section 4, we study a D-compactification of a D-proximity space which is a generalization of the well known Smirnov compactification of an EF-proximity space. This enables us to show how every D-proximity is generated by a family of USC pseudo-semimetrics. We also point out the important role played by closed G_δ sets in D-proximities.

In Section 5, we construct D-proximities in several ways: (i) by continuous functions into a developable space, (ii) by pseudo-semimetrics, (iii) by closed G_δ-bases etc. Following Mrówka [8], we show that D-proximities can be constructed from grills (or semi-ultrafilters) which satisfy certain conditions.

In Section 6, we study D-uniformities in relation to D-proximities which are analogous to Weil (W) or Alfsen-Njastad(AN) uniformities with reference to EF-proximities or Mozzochi (M) uniformities in relation to LO-proximities. Several results concerning continuity, p-continuity and u-continuity are similar to those in EF or LO proximities.

X, Y denote nonempty sets. $P(X)$ is the power set of X. If (X, τ), (Y, σ) are topological spaces, $C(X, Y)$ denotes the family of continuous functions on X to Y. If $Y = [0, 1]$, we just write $C(X)$ for $C(X, Y)$ and write $U(X)$ for the family of all USC functions $f: X \to [0, 1]$ such that $f^{-1}(0)$ is closed and $f^{-1}[r, 1]$ is a closed G_δ set for each $r \in (0, 1]$. $Z(X)$, $ZU(X)$ denote the zero sets of $C(X)$ and $U(X)$ respectively.

(1.4) **Definition:** A topological space (X, τ) is *pseudo-semimetrizable* iff there is a function $d: X \times X \to \mathbb{R}$ such that for all x, y in X

(a) $d(x, y) = d(y, x) \geqq 0$,

(b) $d(x, x) = 0$,

(c) $p \in A^-$ iff $d(p, A) = 0$

where $d(P, Q) = \text{Inf}\{d(p, q): p \in P,\ q \in Q\}$.

Furthermore, if

(d) $d(x, y) = 0$ impies $x = y$

then d is a *semimetric* and τ is *semimetrizable.*

(1.5) **Definition:** A T_1 topological space (R, τ) is called *developable* if and only if it satisfies either of the following conditions:

(a) There exists a development i.e. a family $\{\mathcal{U}_n: n \in \mathbb{N}\}$ of open covers of X with $\mathcal{U}_{n+1} < \mathcal{U}_n$ and $\{\mathrm{St}(x, \mathcal{U}_n): n \in \mathbb{N}\}$ a nbhd. base for each $x \in X$.

(b) (X, τ) has a compatible USC semimetric d.

[Gagrat-Naimpally [4]].

We will refer to d as the *natural* semimetric associated with the developable space (X, τ) and it is related to the development by

(1.6) $$d(x, y) = \mathrm{Inf}\{1/n + 1: y \in \mathrm{St}(x, \mathcal{U}_n)\}.$$

(1.7) **Definition:** A collection $\mathcal{B}$ of closed subsets of a topological space (X, τ) is called a *G_δ-collection* iff for each $B \in \mathcal{B}$, there exists a family $\{B_n: n \in \mathbb{N}\} \subset \mathcal{B}$ such that $X - B = \bigcup\{B_n: n \in \mathbb{N}\}$.

(1.8) **Definition:** A *G_δ-base* is a G_δ collection which is a base for closed subset of (X, τ).

Brandenburg [2] has constructed a T_1 developable space D_1 which is second countable and of cardinality c which serves as a model for all DCR spaces just as $[0, 1]$ serves for all CR spaces.

(1.9) **Theorem:** *A T_1 topological space (X, τ) is DCR if and only if any one of the following equivalent conditions is satisfied:*

(a) *For each $z \notin D^-$ and distinct points p, q in $\mathbb{D}_1$, there is an $f \in C(X, \mathbb{D}_1)$ such that $f(z) = p$ and $f(D^-) = q$. $\mathbb{D}_1$ can be replaced by any T_1 developable space.*

(b) *There exists a G_δ-base $\mathcal{B}$ for closed subsets of (X, τ).*

We note that $\mathcal{B} \subset ZU(X)$ (see 1.11).

(1.10) **Lemma:** *If $X - C = \bigcup\{C_n: n \in \mathbb{N}\}$ where C, C_n are closed subsets of (X, τ) then there is an USC function $f: X \to [0, 1]$ such that $C = f^{-1}(0)$, $f(C_1) = 1$, $f(C_n) \subset [1/n, 1]$.*

Proof. Define $f(C) = 0$, $f(C_1) = 1$ and inductively, $f[C_n - \bigcup\{C_m: m < n\}] = 1/n$ for $n \geqq 2$.

(1.11) **Corollary:** *If C, C_n are closed G_δ sets, then, $f \in U(X)$.*

In analogy with CR spaces versus normal spaces, Brandenburg [1] defined D-normal spaces. The following statement gives the information that we need.

(1.12) **Theorem:** *The following conditions are equivalent for a T_1-space* (called a D-*normal space*)

(a) *A, B are disjoint closed sets implies there is an $f \in U(X)$ such that $f(A) = 0$, $f(B) = 1$.*

[*This is a bit stronger than what is known, but it follows easily from* (1.11)]

(b) *Disjoint closed sets in (X, τ) are contained in disjoint closed G_δ-sets.*

(c) *A, B are disjoint closed sets in X, implies there is an $f \in C(X, \mathbb{D}_1)$ such that $f(A) = p$, $f(B) = q$, $p \neq q$.*

2. D-PROXIMITY:

In Section 1, we provided a motivation for the D-proximity axiom (1.3) via the developable spaces. Here we provide another motivation which, in additon, supplies us with a D-proximity compatible with a DCR space. An analogy is provided by a CR space (X, τ) which has a G_δ base $Z(X)$; if we define two sets as *far* iff they are contained in disjoint members of $Z(X)$ then the resulting proximity δ_F is EF and compatible with τ. In the present case, suppose (X, τ) is a DCR space; then it has a G_δ base $\mathscr{B}$ for closed sets [1.9(b)]. We may assume that $\mathscr{B}$ is a ring i.e. closed under finite unions and finite intersections. Then $\mathscr{B}$ is a *separating* ring i.e. $z \notin D^-$ iff z and D^- are contained in disjoint members of $\mathscr{B}$. If we consider disjoint members of $\mathscr{B}$ as *far*, then we arrive at the following:

(2.1) A non δ B iff A, B are contained in disjoint members of $\mathscr{B}$.

It is easy to show that δ is a compatible *basic proximity* viz.

(2.2) (a) $A \,\delta\, B$ implies $B \,\delta\, A$

(b) $A \,\delta\, B$ implies $A \neq \emptyset$, $B \neq \emptyset$

(c) $A \cap B \neq \emptyset$ implies $A \,\delta\, B$

(d) $A \,\delta\, (B \cup C)$ iff $A \,\delta\, B$ or $A \,\delta\, C$

δ is also *separated* i.e.

(e) $\{x\} \,\delta\{y\}$ implies $x = y$.

Since $\mathscr{B}$ is a G_δ base, for each $D \in \mathscr{B}$, there exists $\{D_n: n \in \mathbb{N}\} \subset \mathscr{B}$ such that $X - D = \bigcup\{D_n: n \in \mathbb{N}\}$, we find that δ as defined by (2.1) also satisfies the *D-axiom* below

(2.3) (D) A non δ B implies there is a $D \subset X$ such that A non δ D, $B \subset D$, $X - D = \bigcup\{D_n: n \in \mathbb{N}\}$ and D_n non δ D for each $n \in \mathbb{N}$.

(2.4) **Definition:** A binary relation δ on $P(X)$ is called a *D-proximity* iff δ is a basic proximity satisfying the D-axiom (2.3).

Thus we have proved:

(2.5) **Theorem:** *Every DCR space (X, τ) has a compatible separated D-proximity.*

We now recall the EF and LO axioms.

(2.6) **Definition:** A basic proximity δ on X is called

(EF) if A non δ B implies there is a $D \subset X$ such that A non δ D and $(X - D)$ non δ B.

(LO) if $A \,\delta\, B$ and $b \,\delta\, C$ for each $b \in B$, then $A \,\delta\, C$.

It is known (see for example Sharma [10] that every EF proximity δ is induced by a family of pseudometrics $\{d_i: i \in I\}$. i.e. A non δ B iff there is an $i \in I$ such that $d_i(A, B) > 0$. Set $S_i(A, \varepsilon) = \{x \in X: d_i(x, a) < \varepsilon$ for some $a \in A\}$.

(2.7) **Theorem:** EF $\Rightarrow$ D $\Rightarrow$ LO.

Proof. (EF) $\Rightarrow$ (D). Suppose δ is EF and A non δ B. Then there is an $i \in I$ such that $d_i(A, C) > 0$ where $C = \{x \in X: d_i(x, B) = 0\}$. Then A non δ C and $B \subset C$. Set $C_n = X - S_i(B, 1/n)$. Then $X - C = \bigcup \{C_n: n \in \mathbb{N}\}$ and C_n non δ C. Hence δ is D. To prove (D) $\Rightarrow$ (LO), suppose (D) holds, $A \,\delta\, B$, $b \,\delta\, C$ for each $b \in B$ but A non δ C. Then there is a $D \subset X$ such that $X - D = \bigcup \{D_n: n \in \mathbb{N}\}$, A non δ D and D_n non δ C for each n. Since $A \,\delta\, B$ and A non δ D, $B \not\subset D$. Hence there is a $b \in B \cap D_n$ for some n. Since $b \,\delta\, C$, D_n non δ C, a contradiction.

(2.8) **Lemma:** *If A non δ B in a D-proximity space (X, δ), then there exists sets D, $\{D_n: n \in \mathbb{N}\}$ which are closed G_δ sets in $\tau(\delta)$, such that A non δ D, $B \subset D$, $X - D = \bigcup \{D_n: n \in \mathbb{N}\}$ and D_n non δ D, for each $n \in \mathbb{N}$.*

Proof. In (2.3) we have A^- non δ D^-, D_n^- non δ D and so $X - D^- \subset X - D = \bigcup \{D_n: n \in \mathbb{N}\} \subset \bigcup \{D_n^-: n \in \mathbb{N}\} \subset X - D^-$. Hence $X - D^- = \bigcup \{D_n^-: n \in \mathbb{N}\}$. Thus in (2.3) D can be chosen to be a closed G_δ set. By symmetry A non δ B implies the existence of closed G_δ sets C, D such that $A \subset C$, $B \subset D$ and C non δ D. Continuing further, since D non δ D_n we may replace D_n by a closed G_δ-set.

(2.9) **Theorem:** *If (X, δ) is a separated D-proximity space, then $\tau(\delta)$ is DCR.*

Proof. From Lemma (2.8) it follows that

$$\mathscr{B} = \{A \subset X: X - A = \bigcup \{A_n: n \in \mathbb{N}\},\ A, A_n \text{ closed } G_\delta\text{-sets in } \tau(\delta) \text{ and } A_n \text{ non } \delta\ A\}$$

is a G_δ-collection as well as a base for closed sets in $(X, \tau(\delta))$. We also note that $\delta(\mathscr{B})$ as defined by (2.1) is precisely δ.

Combining (2.5) and (2.9) we have the main result of this section.

(2.10) **Theorem:** *A topological space (X, τ) is DCR if and only if X has a compatible separated D-proximity.*

Since there are T_1 spaces which are not DCR and DCR spaces which are not CR, we have:

(2.11) **Theorem:** *D-proximity is distinct from both LO-proximity and EF-proximity.*

3. BASES AND SUBBASES:

The study of D-proximity bases and subbases provide us with a powerful tool to construct compatible D-proximities on DCR spaces. In several constructions, the union axiom (2.2) (d) is either not satisfied or is rather tricky to prove. With the help of a base or a subbase, these situations are handled easily. Our study of bases and subbases is along the lines similar to the study of EF-proximity bases and subbases by Sharma [10], wherein the reader will find further details. For the most part, we sketch the proofs only when they are different from Sharma's. I_m denotes the set of first m natural numbers.

(3.1) **Definition:** A *D-proximity base* on X is a binary relation $\mathscr{B}$ on $P(X)$ satisfying:

(B.1) $(\emptyset, X) \notin \mathscr{B}$

(B.2) $A \cap B \neq \emptyset$ implies $(A, B) \in \mathscr{B}$

(B.3) $(A, B) \in \mathscr{B}$ implies $(B, A) \in \mathscr{B}$

(B.4) If $(A, B) \in \mathscr{B}$ and $A \subset A'$, $B \subset B'$, then $(A', B') \in \mathscr{B}$

(B.5) If $(A, B) \notin \mathscr{B}$, then there exist subsets E, $\{E_n: n \in \mathbb{N}\}$ of X such that, $B \subset E$, $X - E = \bigcup \{E_n: n \in \mathbb{N}\}$ and $(A, E) \notin \mathscr{B}$ and $(E, E_n) \notin \mathscr{B}$ for each $n \in \mathbb{N}$.

Furthermore, a D-proximity base $\mathscr{B}$ is *separated* iff

(B.6) $(x, y) \in \mathscr{B}$ implies $x = y$.

Now we show how a D-proximity base generates a D-proximity.

(3.2) **Theorem:** *Suppose $\mathscr{B}$ is a D-proximity base on X and suppose a binary relation $\delta = \delta(\mathscr{B})$ on $P(X)$ is defined by:*

(3.3) *$A \,\delta\, B$ iff given any finite covers $\{A_i: i \in I_m\}$, $\{B_j: j \in I_n\}$ of A, B respectively, there exists an $(i, j) \in I_m \times I_n$ such that $(A_i, B_j) \in \mathscr{B}$.*

Then δ is the coarsest D-proximity on X which is finer than $\mathscr{B}$. Furthermore, δ is separated if and only if $\mathscr{B}$ is separated.

Proof. Obviously $\delta > \mathscr{B}$ and from Sharma's paper it follows that δ is a basic proximity and that it is separated iff $\mathscr{B}$ is separated. So we need prove only the axiom (D). Suppose A non $\delta\, B$, then there exist finite covers $\{A_i: i \in I_m\}$, $\{B_j: j \in I_n\}$ of A, B respectively such that $(A_i, B_j) \notin \mathscr{B}$ for each $(i, j) \in I_m \times I_n$. By (B.5) there exist countably many sets E_{ij}, $\{E^n_{ij}: n \in \mathbb{N}\}$ such that $B_j \subset E_{ij}$, $X - E_{ij} = \bigcup \{E^n_{ij}: n \in \mathbb{N}\}$, $(A_i, E_{ij}) \notin \mathscr{B}$ and $(E^n_{ij}, E_{ij}) \notin \mathscr{B}$ for $i \in I_m$, $j \in I_n$, $n \in \mathbb{N}$. Set $E_j = \bigcap \{E_{ij}: i \in I_m\}$ and $E = \bigcup \{E_j: j \in I_n\}$. Then $(A_i, E_j) \notin \mathscr{B}$ for each i, j and hence A non $\delta\, E$. Now denote by $E^{n(i)}_{ij}$ any element of the family $\{E^n_{ij}\}$. Clearly $B \subset E$ and $[\bigcup \{E^{n(i)}_{ij}: i \in I_m\}]$ non $\delta\, E_{ij}$ for each j. Hence $[\bigcap_{j \in I_n} (\bigcup \{E^{n(i)}_{ij}: i \in I_m\})]$ non $\delta\, E$ since δ is a basic proximity. And the family of subsets on the left side is countable and covers $X - E$. Thus the axiom (D) is satisfied. That δ is the coarsest D-proximity finer than $\mathscr{B}$ can be proved as in Sharma [10].

If $\mathscr{B}$ is a D-proximity base we say that $\delta(\mathscr{B})$ is *generated* by $\mathscr{B}$.

(3.4) **Definition:** A *D-proximity subbase* on X is a binary relation $\mathscr{S}$ on $P(X)$ satisfying (S.1) and (S.2):

(S.1) $A \cap B \neq \emptyset$ implies $(A, B) \in \mathscr{S}$.

(S.2) If $(A, B) \notin \mathscr{S}$, then there exists a countable family of subsets of X, namely E, $\{E_n: n \in \mathbb{N}\}$ such that $B \subset E$, $X - E = \bigcup \{E_n: n \in \mathbb{N}\}$, $(A, E) \notin \mathscr{S}$ and $(E_n, E) \notin \mathscr{S}$ for each $n \in \mathbb{N}$.

Furthermore, $\mathscr{S}$ is *separated* iff

(S.3) $x \neq y$ and $(x, y) \in \mathscr{S}$, then there are sets P, Q of X such that $x \in P$, $y \in Q$, $(P, Q) \notin \mathscr{S}$ or $(Q, P) \notin \mathscr{S}$.

Several of the results in the rest of this section follow from appropriate modifications in Sharma's proofs [10]. Hence we state them without proofs.

(3.5) **Theorem:** *Let $\mathscr{S}$ be a D-proximity subbase on X. Then the binary relation $\mathscr{B} = \mathscr{B}(\mathscr{S})$ on $P(X)$ defined by*

(3.6) $(A, B) \in \mathscr{B}$ *iff* $A \neq \emptyset$, $B \neq \emptyset$ *and for any* $A' \supset A$, $B' \supset B$ *both* (A', B') *and* (B', A') *are in* $\mathscr{S}$

is a D-proximity base on X. Furthermore, the D-proximity $\delta = \delta(\mathscr{S})$ generated by $\mathscr{B}$ is the coarsest D-proximity finer than $\mathscr{S}$ and is separated iff $\mathscr{S}$ is separated.

(3.7) **Theorem:** *Suppose $\{\delta_i: i \in I\}$ is a nonempty family of D-proximities on a set X. Then the proximity δ generated by the D-proximity base $\mathscr{B} = \bigcap \{\delta_i: i \in I\}$ is the coarsest D-proximity finer than each δ_i. We denote $\delta = \operatorname{Sup} \{\delta_i: i \in I\}$.*

(3.8) **Corollary:** *Let $\{\delta_i: i \in I\}$ be a nonempty family of D-proximities on a set X. Then*

$$\tau[\operatorname{Sup} \{\delta_i: i \in I\}] = \bigvee \{\tau(\delta_i): i \in I\}$$

(3.9) **Theorem:** *Let $\{\delta_i: i \in I\}$ be a nonempty collection of D-proximities on a set X. Then there exists a finest D-proximity δ on X such that δ is coarser than each δ_i.*

From Theorems (3.7) and (3.9) we have

(3.10) **Theorem:** *The collection of all D-proximities on a set X forms a complete lattice under the natural ordering $\geqq$.*

We recall that a function f from one proximity space (X, δ_1) to another (Y, δ_2) is p-continuous iff $A\, \delta_1\, B$ implies $f(A)\, \delta_2\, f(B)$.

A proof of the following result is similar to Sharma's (3.7) [10]:

(3.11) **Theorem:** *Let (X, δ_1), (Y, δ_2) be two D-proximity spaces and let $\mathscr{S}$ be a subbase for δ_2. A function $f\colon X \to Y$ is p-continuous if and only if $(A, B) \notin \mathscr{S}$ implies $f^{-1}(A)$ non $\delta_1 f^{-1}(B)$.*

Next we consider the problem of defining an initial D-proximity δ on a set X when we are given D-proximity spaces $\{(Y_f, \delta_f) : f : X \to Y_f \text{ a function}\}$.

(3.12) **Theorem:** *Let F be a nonempty family of functions each $f \in F$ being a function on X to a D-proximity space (Y_f, δ_f). Then the proximity δ generated by the D-proximity base $\mathscr{B}$ defined by:*

(3.13) $\quad (A, B) \in \mathscr{B}$ *iff* $f(A)\, \delta_f\, f(B)$ *for each* $f \in F$ *is the coarsest D-proximity on X such that each member of F is p-continuous.*

In Theorem (3.12) we may replace δ_f by a base $\mathscr{B}_f$ or even a subbase $\mathscr{S}_f$ and the result remains true. A special case of Theorem (3.12) is the construction of a D-proximity for the product of D-proximity spaces. This, of course, is the coarsest D-proximity such that each projection is p-continuous. If (Y, δ_2) denotes the product of D-proximity spaces $\{(Y_i\delta_i) : i \in I\}$, then a function $g : (X, \delta_1) \to (Y, \delta_2)$ is p-continuous if and only if $p_i \circ g$ is p-continuous for each projection p_i.

4. D-COMPACTIFICATION:

Brandenburg [2] has shown that every DCR space (X, τ) has a D-compactificalon. In this section, we improve this result by showing that every separated D-proximity space (X, δ) is proximally isomorphic to a subspace of a D-compact space and that δ is obtained from the D-proximity δ_0 on its compactification where

$$A\, \delta_0\, B \quad \text{iff} \quad A^- \cap B^- \neq \emptyset.$$

This is analogous to the well known results: every separated EF (LO) proximity space (X, δ) is a proximal subspace of a compact Hausdorff (respectively compact T_1) space (X^*, δ_0) (Naimpally-Warrack (9), Mozzochi-Gagrat-Naimpally [7]). We then show that in a D-proximity space far away sets are separated by a p-continuous function to $\mathbb{D}_1$ and that each D-proximity δ is generated by (i) a family of USC pseudo-semimetrics as well as by (ii) a G_δ base for closed sets.

(4.1) **Definition:** A topological space is called *D-compact* iff every open cover has a finite refinement consisting of open F_σ sets (Brandenburg [2]).

Every compact Hausdorff space is D-compact a T_1 topological space is D-compact if and only if it is compact and DCR. Brandenburg [2] has shown that the Wallman-Frink compactification $\alpha(\mathbb{D}_1)$ of $\mathbb{D}_1$ is a D-compactification. We may suppose that $\mathbb{D}_1 \subset \alpha\mathbb{D}_1$ and p, q are two distinct points of $\mathbb{D}_1$. If (X, δ) is a separated D-proximity space, then A non δ B implies there exist C, D in $\mathscr{B}$ (Theorem 2.9) such that $A \subset C$, $B \subset D$ and C non δ D, C, D are D-closed sets and $f(C) = p$, $f(D) = q$, $p \neq q$, $p, q \in \mathbb{D}_1$ is continuous on $C \cup D$. By a result of Brandenburg [2], f has a continuous extension from $X \to \mathbb{D}_1 \subset \alpha\mathbb{D}_1$. Hence

(4.2) $\quad A$ non δ B implies there is a function $f = f_{A,B} \in C(X, \alpha\mathbb{D}_1)$ such that $f(A) = p$, $f(B) = q$.

We claim that the evaluation map $e: X \to Y = \Pi\{Y_f : f = f_{A,B} \in C(X, \alpha\mathbb{D}_1)$ and $Y_f = \alpha\mathbb{D}_1$ for each $f\}$ is a proximal isomorphism on X to $e(X)$. Clearly, e is an injective homeomorphism on $(X, \tau(\delta)) \to e(X)$ with the subspace topology induced by Y. Also if A non δ B, then $f = f_{A,B}$ exists such that $f(A) = p, f(B) = q$; hence $e(A)^- \cap \cap e(B)^- = \emptyset$. On the other hand, if $A\ \delta\ B$, then for each $f \in C(X, \mathbb{D}_1) f(A)^- \cap \cap f(B)^- \neq \emptyset$ i.e. $e(A)^- \cap e(B)^- \neq \emptyset$ in $e(X)^-$ which we denote by αX. Hence our claim that e is a proximal isomorphism is proven. Thus we have shown:

(4.3) **Theorem:** *For every separated D-proximity space (X, δ), there exists a D-compact space αX and a proximal isomorphism $e: (X, \delta) \to (\alpha X, \delta_0)$.*

Every compact Hausdorff space has a unique compatible EF-proximity δ_0 but may have more than one compatible D-proximities.

(4.4) **Example:** The compact Hausdorff space $[0, 1]$ has two compatible D-proximities δ_0 and δ where the latter is defined by

$$A\ \delta\ B \quad \text{iff} \quad A\ \delta_0\ B \quad \text{or} \quad A, B \quad \text{are both infinite}.$$

In an EF-proximity space (X, δ) if A non δ B then there is a *p-continuous* function $f: X \to [0, 1]$ such that $f(A) = 0$, $f(B) = 1$. We now prove an analogous result for D-proximity spaces.

(4.5) **Theorem:** *In a separated D-proximity space (X, δ) if A non δ B, then there is a p-continuous function $f: X \to \mathbb{D}_1$ such that $f(A) = p, f(B) = q$.*

Proof: If A non δ B, then $e(A)^- \cap e(B)^- = \emptyset$ in the D-compactification αX. Since αX is D-normal, there is a continuous function $g: \alpha X \to \mathbb{D}_1$ such that $g\ e(A) = p$, $g\ e(B) = q$. If αX is assigned the D-proximity δ_0, then g is p-continuous and hence $f = ge$ is also p-continuous.

If in the proof of Theorem (4.5) d_f is defined by

(4.6) $$d_f(x, y) = d(f(x), f(y))$$

where d is the natural semimetric on $\mathbb{D}_1$, then d_f is an USC pseudo-semi-metric on X and $\delta(d_f) \leqq \delta$. Clearly $\delta = \text{Sup}\{\delta(d_f) : f: X \to \mathbb{D}_1\ p\text{-continuous}\}$. Hence we have the following result which may be compared to Sharma's result (3.12) [10]:

(4.7) **Theorem:** *If δ is any D-proximity on a set X, then there exists a nonempty collection $\{\delta_i : i \in I\}$ of USC pseudo-semimetric proximities on X such that $\delta = \text{Sup}\{\delta_i : i \in I\}$.*

(4.8) From Lemma (2.8) we have: If A non δ B in a D-proximity space then A, B are contained in closed G_δ sets G_A, G_B respectively, such that G_A non δ G_B.

In any EF-proximity space (X, δ) if $A^- \cap B^- = \emptyset$ and one of them is compact, then A non δ B. We now prove an analogous result for D-proximity spaces.

(4.9) **Definition:** A closed subset E of a DCR space (X, τ) is called *G-compact* iff every cover of E by closed G_δ sets has a finite subcover.

The following result follows easily from (4.8):

(4.10) **Theorem:** *In any D-proximity space* (X, δ), $A^- \cap B^- = \emptyset$ *and one of them is G-compact implies* A non δ B.

5. CONSTRUCTION OF D-PROXIMITIES:

It is well known that every CR space (X, τ) has a compatible finest EF proximity δ_F defined by

(5.1) $\quad A$ non δ_F B iff there is an $f \in C(X)$ such that $f(A) = 0$, $f(B) = 1$.

In addition, a CR space (X, τ) need not have a coarsest compatible EF-proximity; however, it has a coarsest compatible EF proximity δ_{CE} iff X is locally compact where

(5.2) $\quad A \, \delta_{CE} \, B$ iff $A^- \cap B^- \neq \emptyset$ or both A^-, B^- are non compact.

In the case of a T_1 space (X, τ), the finest and the coarsest compatible LO-proximities δ_0, δ_{CL} respectively always exist and are given by:

(5.3) $\quad A \, \delta_0 \, B$ iff $A^- \cap B^- \neq \emptyset$.

(5.4) $\quad A \, \delta_{CL} \, B$ iff $A \, \delta_0 \, B$ or A, B are both infinite.

In this section we make a study of the construction of compatible D-proximities on a DCR space (X, τ). We show the existence of the finest compatible D-proximity δ_U and investigate the existence of the coarsest compatible D-proximity δ_{CD}. We compare these with their analogous in EF and LO proximities.

It follows easily from Theorem (2.9) that if δ is *any* D-proximity on X, then

(5.5) $\quad A$ non δ B implies there is an $f \in U(X)$ such that $f(A) = 0$, $f(B) = 1$.

This can be compated to the well known result in EF-proximities wherein $f \in C(X)$. The existence of the finest proximity δ_U on a DCR space (X, τ) follows from Theorem (3.7).

We now give several equivalent ways of describing δ_U:

(5.6) **Theorem:** *On a DCR space* (X, τ) *the following are equivalent, and describe the finest compatible D-proximity* δ_U.

(a) $X \times X - \mathscr{B} = \{(A, B) \in P(X) \times P(X)$: there is an $f \in C(X, \mathbb{D}_1)$ such that $f(A) = p$, $f(B) = q$, $p \neq q\}$.

(b) $X \times X - \mathscr{B} = \{(A, B) \in P(X) \times P(X)$: *there is an* $f \in C(X, \mathbb{D}_1)$ *such that* $d(f(A), f(B)) > 0$ *where* d *is the natural semimetric on* $\mathbb{D}_1\}$.

(c) $X \times X - \mathscr{B} = \{(A, B) \in P(X) \times P(X)$: *there is an* $f \in C(X, \mathbb{D}_1)$ *and an* $n \in \mathbb{N}$ *such that* $\mathrm{St}\,(f(A), \mathscr{U}_n) \cap B = \emptyset$ *where* $\{\mathscr{U}_n: n \in \mathbb{N}\}$ *is the development on* $\mathbb{D}_1\}$.

(d) $X \times X - \mathscr{B} = \{(A, B) \in P(X) \times P(X)$: *there is an* $f \in C(X, \mathbb{D}_1)$ *such that* $f(A)$ non $\delta\, f(B)$ *where* δ *is defined by* $(1.2)\}$.

Proof: (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) is trivial. To show that (b) $\Rightarrow$ (a) we note that $\mathbb{D}_1$ is D-normal and so there is a $g \in C(\mathbb{D}_1, \mathbb{D}_1)$ such that $g(f(A)) = p$, $g(f(B)) = q$. The last statement is obvious.

(5.7) **Theorem:** *A DCR space* (X, τ) *is D-normal if and only if* $\delta_U = \delta_0$.

The above result may be compared to the proposition: A CR space is normal iff $\delta_F = \delta_0$.

(5.8) **Theorem:** *If* (X, τ) *is a CR space, then*

$$\delta_0 \geqq \delta_U \geqq \delta_F .$$

We now give some examples to clarify the relationships among δ_0, δ_U and δ_F.

(5.9) **Example:** If (X, τ) is DCR but not D-normal (Brandenburg (5.8) [2]), then $\delta_0 \neq \delta_U$.

(5.10) **Example:** If (X, τ) is D-normal but not CR, then $\delta_0 = \delta_U \neq \delta_F$.

Having considered the finest compatible D-proximity on a DCR space (X, τ), we now take up the study of the coarsest one. The discussion preceeding (4.1) suggests that the coarsest compatible D-proximity may not always exist and that it exists only when (X, τ) is *locally G-compact* i.e. each point is contained in a G-compact set. An example of a locally G-compact space is a T_1 topological space (X, τ) in which each singleton set is a G_δ.

(5.11) **Theorem:** *Let* (X, τ) *be a locally G-compact DCR space. Then*

(5.12) $\quad A$ non $\delta_{CD}\, B$ *iff* $A^- \cap B^- = \emptyset$ *and at least one of* A^-, B^- *is G-compact*

defines the coarsest compatible D-proximity on X.

Proof. That δ_{CD} is a separated basic proximity is straightforward. To verify the D-axiom suppose A non $\delta_{CD}\, B$ i.e. $A^- \cap B^- = \emptyset$ and A^- is G-compact. For each $x \in A^-$, there is a G-compact set G_x such that $x \in G_x$ and G_x non $\delta_{CD}\, B$. $A^- \subset \subset \{G_x: x \in A^-\}$ and hence $A^- \subset \{G_x: x \in J\}$ where J is a finite subset of A^-. If $G = \bigcup \{G_x: x \in J\}$, then G non $\delta_{CD}\, B$ and $X - G = \bigcup \{G_n: n \in \mathbb{N}\}$, since G is a closed G_δ set. Also G non $\delta_{CD}\, G_n$ for each n.

To show that δ_{CD} is compatible with τ we note that since (X, τ) is locally G-compact, $z \notin D^-$ implies $z \in G_z$, $D^- \subset G_D$ where G_z is G-compact, G_D is closed and $G_z \cap G_D = \emptyset$. That δ_{CD} is the coarsest compatible D-proximity follows from Theorem (4.11).

(5.13) **Theorem:** *In a CR space* (X, τ)

$$\delta_0 \geqq \delta_U \geqq \delta_F \geqq \delta_{CE} \geqq \delta_{CD} \geqq \delta_{CL} .$$

(5.14) **Corollary:** *In a DCR space in which singletons are* G_δ, $\delta_{CD} = \delta_{CL}$.

(5.15) **Example:** In $[0, 1]$ with the usual topology

$$\delta_{CL} = \delta_{CD} < \delta_{CE} = \delta_0 .$$

(5.16) **Example.** We now give an example of a completely normal compact space X which has no coarsest compatible D-proximity. X is the Alexandroff suqare (Steen-Seeback [11]) Page 120. No point on the diagonal, different from the endpoints, has a closed G-compact set containing it which separates it from a disjoint closed set. So X does not have the coarsest compatible D-proximity, although it has the coarsest compatible LO-proximity as well as the coarsest compatible EF-proximity (in fact, it has a unique compatible EF-proximity).

We now conclude this section by giving a method of construction of D-proximities on the lines of Mrówka's [8] construction of EF-proximities.

(5.17) **Definition:** A grill $\mathscr{G}$ on a set X is a union of ultrafilters. A semi-ultrafilter $\mathscr{S}$ on X is a grill such that it contains at most one singleton.

The following result is similar to Mrówka's – see Theorem (5.20) Naimpally-Warrack [9]:

(5.18) **Theorem:** *Let* $\mathscr{C}$ *be a family of semi-ultrafilters on a set X satisfying:*

(a) *Suppose* $A, B \in P(X)$. *If for each* $C \in P(X)$, $B \subset C$, $X - C = \bigcup \{C_n : n \in \mathbb{N}\}$, *there is a* $\sigma \in \mathscr{C}$ *such that either* $\{A, C\} \subset \sigma$ *or* $\{C_n, C\} \subset \sigma$ *for some* $n \in \mathbb{N}$, *then there is a* $\sigma' \in \mathscr{C}$ *such that* $\{A, B\} \subset \sigma'$.

(b) *Each ultrafilter on X is contained in some* $\sigma \in \mathscr{C}$.

Then there exists a DCR topology τ *on X and a compatible D-proximity* δ *on X such that each* $\sigma \in \mathscr{C}$ *is a bunch in* (X, τ).

Proof: In the proof of Mrówka's theorem concerning EF-proximities, an EF-proximity can be directly defined using semi-ultrafilters, since it involves clusters. In our case here, we must go to a D-proximity via a topology. So we define a topology on X via a closure operator defined by

(5.19) $\quad p \in A^-$ iff there is a $\sigma \in \mathscr{C}$ such that $\{p, A\} \subset \sigma$.

It is easy to show that (i) $\phi^- = \phi$ (ii) $A \subset A^-$ (iii) $(A \cup B)^- = A^- \cup B^-$. We now prove (iv) $(A^-)^- = A^-$. Suppose this is not true and $p \in (A^-)^-$ but $p \notin A^-$. Then for each $\sigma \in \mathscr{C}$ $\{p, A\} \not\subset \sigma$. There exist C, C_n such that $A \subset C$, $X - C = \bigcup \{C_n: n \in \mathbb{N}\}$ such that for each $\sigma \in \mathscr{C}$, $\{p, C\} \not\subset \sigma$ and $\{C_n, C\} \not\subset \sigma$ for each $n \in \mathbb{N}$. $A^- \not\subset C$ for otherwise, $\{p, A^-\} \subset \sigma$ which implies $\{p, C\} \subset \sigma$ a contradiction. So for some $n \in \mathbb{N}$, $A^- \cap C_n \neq \emptyset$. Suppose $z \in A^- \cap C_n$. Then $\{z, A\} \subset \sigma$ for some $\sigma \in \mathscr{C}$ and this, in turn, implies $\{C_n, A\} \subset \sigma$, a contradiction.

Next we define a proximity δ by

(5.20) $A\,\delta\,B$ iff there is a $\sigma \in \mathscr{C}$ containing $\{A^-, B^-\}$.

It is easy to show that δ is a compatible LO-proximity on (X, τ). Condition (a) makes δ a D-proximity. Clearly each $\sigma \in \mathscr{C}$ is a bunch.

6. D-UNIFORMITY:

It is well known that every EF(LO) proximity δ on X is induced by a Weil (respectively Mozzochi) uniformity. There are also generalizations of Weil uniformities, namely AN uniformities and correct uniformities which also induce EF proximities (Naimpally-Warrack [9]). In this section we investigate generalizations of uniformities which induce D-proximities. We follow the entourage forms of uniformities for the most part and indicate briefly how the covering forms are handled.

(6.1) **Definition:** A *basic uniformity* $\mathscr{U}$ is a family of symmetric subsets of $X \times X$ such that

(a) $\Delta \subset \bigcap \{U : U \in \mathscr{U}\}$.

(b) For each $A \in P(X)$, $U, V \in \mathscr{U}$, there is a $W \in \mathscr{U}$ such that $W[A] \subset U[A] \cap \cap V[A]$.

(c) If $U \in \mathscr{U}$ and $U \subset V = V^{-1}$, then $V \in \mathscr{U}$.

A basic uniformity is *separated* iff it satisfies

(d) $\Delta = \bigcap \{U : U \in \mathscr{U}\}$.

Without the condition (c) we get a basic uniformity *base*. We call members of $\mathscr{U}$ *entourages*.

(6.2) **Theorem:** *A family $\mathscr{U}$ of symmetric subsets of $X \times X$ is a basic uniformity if and only if $\delta = \delta(\mathscr{U})$ defined by*

(6.3) A non $\delta\, B$ *iff* $U[A] \cap B = \emptyset$ *for some* $U \in \mathscr{U}$
is a basic proximity. $\mathscr{U}$ is separated iff δ is separated.

Proof. This is a part of Theorem (2.2) of Mozzochi-Gagrat-Naimpally [7].

(6.4) **Definition.** A basic uniformity $\mathscr{U}$ is

(M) if for every $A, B \in P(X)$ and $U \in \mathscr{U}$ if $V[A] \cap B \neq \emptyset$ for each $V \in \mathscr{U}$ then there exists an $x \in B$ and a $W \in \mathscr{U}$ such that, $W[x] \subset U[A]$.

(D) if for every $U \in \mathscr{U}$, $A \subset X$, there exist V, $\{V_n : n \in \mathbb{N}\}$ in $\mathscr{U}$ and subsets C, $\{C_n : n \in \mathbb{N}\}$ of X such that $X - U[A] \subset C$, $X - C = \bigcup \{C_n : n \in \mathbb{N}\}$, $V_n[C_n] \subset X - C$ and $V[C] \subset X - A$.

(C) if for each $U \in \mathscr{U}$, $A \subset X$, there are $V, W \in \mathscr{U}$ such that $W[V[A]] \subset U[A]$.

(AN) if for each finite family $\{A_i : i \in I_m\} \subset P(X)$ and $\{U_i : i \in I_m\} \subset \mathscr{U}$, there is a $U \in \mathscr{U}$ such that $U[A_i] \subset U_i[A_i]$, $i \in I_m$ and the triangle inequality viz. for $W \in \mathscr{U}$, there is a $V \in \mathscr{U}$ such that $V^2 \subset W$.

(W) if for each $U, V \in \mathscr{U}$, $U \cap V \in \mathscr{U}$ and the triangle inequality.

We now prove the relation between D-uniformities and D-proximities.

(6.5) **Theorem:** *A basic uniformity $\mathscr{U}$ is a D-uniformity if and only if $\delta = \delta(\mathscr{U})$, as defined by (6.3), is a D-proximity. Furthermore, δ is separated if and only if $\mathscr{U}$ is separated.*

Proof. Suppose $\mathscr{U}$ is a D-uniformity and $\delta = \delta(\mathscr{U})$ is defined by (6.3). If A non δ B, then there exists a $U \in \mathscr{U}$ such that $U[A] \cap B = \emptyset$. By (6.4) (D) there exist V, $\{V_n\colon n \in \mathbb{N}\}$ in $\mathscr{U}$ and subsets C, $\{C_n\colon n \in \mathbb{N}\}$ of X such that $B \subset X - U[A] \subset C$, $X - C = \bigcup \{C_n\colon n \in \mathbb{N}\}$, $V_n[C_n] \subset X - C$ and $V[C] \subset X - A$. So C non δ A and C_n non δ C for each n, showing thereby that δ is a D-proximity.

Conversely, suppose $\mathscr{U}$ is a basic proximity such that δ is a D-proximity. Suppose $U \in \mathscr{U}$ and $A \in P(X)$. Set $B = X - U[A]$. A non δ B and so there are C, $C_n \in P(X)$ such that, $B \subset C$, $X - C = \bigcup \{C_n\colon n \in \mathbb{N}\}$, A non δ C and C_n non δ C for each n. So there are V, $\{V_n\colon n \in \mathbb{N}\}$ in $\mathscr{U}$ such that $V_n[C_n] \cap C = \emptyset$ and $V[C] \cap A = \emptyset$. Hence $\mathscr{U}$ is a D-uniformity.

(6.6) **Corollary:** (W) $\Rightarrow$ (AN) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (M) *for any basic uniformity.*

Proof. (W) $\Rightarrow$ (AN) $\Rightarrow$ (C) see Naimpally-Warrack Theorem (13.14) [9]. Since every EF-proximity is a D-proximity, (C) $\Rightarrow$ (D) for a basic uniformity. Similary since every D-proximity is a LO-proximity, (D) $\Rightarrow$ (M) for a basic uniformity.

(6.7) **Remarks:** If any basic uniformity base satisfies (P), the uniformity generated by the base also satisfies (P), where $P \in \{M, D, C, AN, W\}$.

Every (LO) EF proximity space (X, δ) has a compatible totally bounded (M) W-uniformity. The following is an analogue for D-proximity spaces.

(6.8) **Theorem:** *Every (separated) D-proximity space (X, δ) has a compatible (separated) totally bounded D-uniformity.*

Proof. By Theorem (2.22) of Mozzochi-Gagrat-Naimpally [7], (X, δ), being also a LO-proximity space, has a compatible totally bounded M-uniformity $\mathscr{U}_1(\delta)$ having a base: $\{U_{A,B}\colon A \text{ non } \delta\ B\}$ where

$$(6.9) \qquad U_{A,B} = X \times X - [A \times B \cup B \times A].$$

By (6.5) and (6.7), $\mathscr{U}_1(\delta)$ is a totally bounded D-uniformity.

(6.10) **Corollary:** *A topology τ on X is induced by some separated D-uniformity if and only if (X, τ) is DCR.*

(6.11) **Definition:** *If (X, δ) is a D-proximity space then*

$$\Pi(\delta) = \{\mathscr{U} : \mathscr{U} \text{ a D-uniformity such that } \delta(\mathscr{U}) = \delta\}$$

is the *D-proximity class of uniformities.*

(6.12) **Theorem:** *If (X, δ) is a D-proximity space then $\mathscr{U}_1(\delta)$, as constructed in Theorem (6.8), is the coarsest member of $\Pi(\delta)$.*

Proof. Suppose $\mathscr{U} \in \Pi(\delta)$ and $U_{A,B} \in \mathscr{U}_1(\delta)$, where A non δ B. Then there esists a $V \in \mathscr{U}$ such that $(A \times B) \cap V = \emptyset$ and so $V \subset U_{A,B}$ i.e. $U_{A,B} \in \mathscr{U}$.

Having shown the existence of the coarsest member of $\Pi(\delta)$, we now show the existence of the finest member of $\Pi(\delta)$.

(6.13) **Theorem:** *If (X, δ) is a separated D-proximity space, then the union $\mathscr{B}$ of members of $\Pi(\delta)$ is a base for separated D-uniformity, which turns out to be finest element of $\Pi(\delta)$.*

Proof. By Theorem (2.29) and Corollary (2.30) of Mozzochi-Gagrat-Naimpally [7] $\mathscr{B}$ is a base for an M-uniformity $\mathscr{U}_0$ which is the finest M-uniformity compatible with δ. By (6.5), $\mathscr{U}_0 \in \Pi(\delta)$ and is the finest member.

(6.14) **Definition:** $\mathscr{S}$ is a subbase for a (separated) D-uniformity $\mathscr{U}$ on X iff the set $\mathscr{B}$ of all finite intersections of elements of $\mathscr{S}$ is a bse for $\mathscr{U}$.

(6.15) **Example:** (2.36) of Mozzochi-Gagrat-Naimpally [7] shows that a base for a D-uniformity need not be a subbase.

(6.16) **Definition:** A (separated) D-uniformity $\mathscr{U}$ is *p-correct* iff there exists a (separated) D-proximity δ such that $\mathscr{S} = \{U_{A,B}\colon A \text{ non } \delta\ B\}$ is a subbase for $\mathscr{U}$. D-proximity δ is the *generator proximity* for $\mathscr{U}$.

(6.17) **Theorem:** *Suppose $(X, \mathscr{U})$ is a p-correct (separated) D-uniformity. Then $(X, \mathscr{U})$ is totally bounded and has an open base. Furthermore, $\mathscr{U}_1(\delta)$ has an open base.*

Proof. By Theorem (3.9) of Mozzochi-Gagrat-Naimpally [7], $\mathscr{U}$ is an M-uniformity which is totally bounded. Since $\mathscr{U}$ is a D-uniformity, the first assertion is obvious. The second assertion follows from Theorem (3.10) (l.c.)

(6.18) **Corollary:** *Let (X, δ) be a (separated) D-proximity space. Then there exists in $\Pi(\delta)$ a unique p-correct D-uniformity $\mathscr{U}_2(\delta)$ which is generated by the subbase $\{U_{A,B}\colon A \text{ non } \delta\ B\}$.*

Proof. This follows from Theorems (3.16), (5.4) of Mozzochi-Gagrat-Naimpally [7].

(6.19) **Remarks:** Unlike the EF-proximity case, in which there is a unique totally bounded compatible W-uniformity, a D-proximity class of D-uniformities $\Pi(\delta)$ may contain two distinct totally bounded D-uniformities viz. $\mathscr{U}_1(\delta)$ and $\mathscr{U}_2(\delta)$. As an example we may consider reals with the usual EF-proximity which is also a D-proximity.

Having studied the D-uniformities from the standpoint of *entourages*, we now briefly describe them from the point of view of *covers*, and relate them to para-uniformities of Brandenburg [2].

If $\mathscr{U}$ is a D-uniformity on X, then for each $U \in \mathscr{U}$ we set

(6.20) $\quad \alpha(U) = \{U(x) : x \in X\}$, a cover of X.

Then $\mu = \{\alpha(U) : U \in \mathscr{U}\}$ is a family of covers of X.

(6.21) **Definition:** Suppose μ is a family of covers of a set X. Then (X, μ) is called *parauniform space* iff

(N.1) If $\mathscr{U} \in \mu$ and $\mathscr{U}$ refines $\mathscr{V}$, then $\mathscr{V} \in \mu$.
(N.2) If $\mathscr{U}, \mathscr{V} \in \mu$, then $\mathscr{U} \wedge \mathscr{V} \in \mu$.
(N.3) If $\mathscr{U} \in \mu$, then $\operatorname{int}_\mu \mathscr{U}\{\operatorname{int}_\mu U : U \in \mathscr{U}\} \in \mu$

where $\operatorname{int}_\mu U = \{x \in U : \operatorname{St}(x, \mathscr{V}) \subset U$ for some $\mathscr{V} \in \mu\}$.

(P.U) For each $\mathscr{U} \in \mu$, there exists a countable *kernel-normal* subcollection $\beta = \beta(\mathscr{U})$ of μ i.e. $\{\mathscr{U}_n : n \in \mathbb{N}\} \subset \beta$ such that $\mathscr{U}_1 = \mathscr{U}$ and for each $\mathscr{U}_n \in \beta$ there is an $\mathscr{U}_m \in \beta$ such that $\mathscr{U}_m$ refines $\operatorname{int}_\beta \mathscr{U}$.

The following result is easy to prove:

(6.22) **Theorem:** *If $\mathscr{U}$ is a D-uniformity on X, then (X, μ) where $\mu = \{\alpha(U) : U \in \mathscr{U}\}$ as defined by* (6.20) *satisfies* (N.1), (N.2), (N.3) *and*

(CD) *for each $\mathscr{V} \in \mu$, there exists a countable family $\{\mathscr{V}_n : n \in \mathbb{N}\} \subset \mu$ such that $\mathscr{V}_1 = \mathscr{V}$, $\mathscr{V}_{n+1}$ refines $\mathscr{V}_n$ for each $n \in \mathbb{N}$.*

[If μ satisfies (N.1)–(N.3) and (CD), we call (X, μ) *a covering D-uniformity.*]

(6.23) **Corollary:** (X, δ) *is a D-proximity space if and only if it has a compatible covering D-uniformity μ, where $A \,\delta\, B$ iff $\operatorname{St}(A, \mathscr{U}) \cap B \neq \emptyset$ for every $\mathscr{U} \in \mu$.*

Obviously (P.U) $\Rightarrow$ (CD). Hence we have

(6.24) **Corollary:** *Every parauniform space (X, μ) induces a compatible D-proximity on X.*

Since a finer uniformity induces a finer proximity, we have from Brandenburg [2] Theorem (2.13):

(6.25) **Corollary:** *The family μ_f of all kernel-normal open covers of a DCR space (X, τ) is compatible with the finest D-proximity δ_U on X.*

We conclude this section with a discussion of continuity, p-continuity and (uniform or) u-continuity. Suppse $(X, \mathscr{U}_1)$, $(Y, \mathscr{U}_2)$ are two separated D-uniform spaces, $\delta_i = \delta(\mathscr{U}_i)$ $i = 1, 2$ the induced D-proximities and $\tau_i = \tau(\delta_i)$, the induced DCR topologies. It is easy to show that for a function $f: X \to Y$

u-continuity $\Rightarrow$ p-continuity $\Rightarrow$ continuity

but that the converses are not true. Now we investigate when the converses hold.

The following is an analogue of a result concerning EF proximities viz. $f: (X, \delta_F) \to (Y, \delta_2)$ is p-continuous iff it is continuous.

(6.26) **Theorem:** *Suppose* $f: (X, \tau_1) \to (Y, \tau_2)$ *is continuous and* $\delta_1 = \delta_U$ *the finest compatible D-proximity. Then* $f: (X, \delta_1) \to (Y, \delta_2)$ *is p-continuous.*

Proof. If C non δ D in Y, then there is a $g \in C(Y, \mathbb{D}_1)$ such that $g[C] = p, g[D] = q, p \neq q$. By Theorem (5.6) $(f^{-1}(C), f^{-1}(D)) \in X \times X - \mathcal{B}$. Hence $f^{-1}(C)$ non $\delta_U f^{-1}(D)$.

In EF-proximities it is well known that $f: (X, \delta_1) \to (Y, \delta_2)$ is p-continuous and $\mathcal{U}_2$ is totally bounded, then $f: (X, \mathcal{U}_1) \to (Y, \mathcal{U}_2)$ is u-continuous. An analogous result is true for D-proximities which follows from Theorem (5.8) of Mozzochi-Gagrat-Naimpally [7].

(6.27) **Theorem:** *Suppose* $f: (X, \delta_1) \to (Y, \delta_2)$ *is p-continuous and* $\mathcal{U}_2$ *is the coarsest element of* $\Pi(\delta_2)$, *then* $f(X, \mathcal{U}_1) \to (Y, \mathcal{U}_2)$ *is u-continuous.*

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