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RELATION PRODUCTS OF CONGRUENCES AND FACTOR CONGRUENCES

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Let A_1, A_2 be algebras of the same type. Denote by π_i the canonical projection from the Cartesian product $A_1 \times A_2$ onto A_i, $i = 1, 2$. Then $\Pi_i = \operatorname{Ker} \pi_i$, $i = 1, 2$, are called *factor congruences* on $A_1 \times A_2$. A congruence Θ on $A_1 \times A_2$ is called a *subfactor congruence* whenever $\Theta \subseteq \Pi_1$ or $\Theta \subseteq \Pi_2$ hold.
Factor congruences and subfactor congruences were introduced by J. Hagemann [5]; see also H.-P. Gumm [4] for further information. The present paper studies four congruence properties on the product $A_1 \times A_2$ which are related to permutability and "3 1/2-permutability" of congruences on $A_1 \times A_2$ with factor congruences Π_1 and Π_2. For the sake of brevity, denote the sequence $p_1, \ldots, p_m$ by $\boldsymbol{p}$.

Definition 1. Let A_1, A_2 be algebras of the same type. We say that the projection π_1 *preserves congruences* on $A_1 \times A_2$ whenever $\pi_1 \times \pi_1(\Theta) = \{\langle a_1, a'_1 \rangle \in A_1 \times A_1;\ \langle a_1, a_2 \rangle\ \Theta\ \langle a'_1, a'_2 \rangle$ for some elements $a_2, a'_2, \in A_2\}$ is a congruence on A_1 for any congruence Θ on $A_1 \times A_2$.

The property π_2 *preserves congruences* on $A_1 \times A_2$ is introduced analogously.

Theorem 1. *For a variety V, the following conditions are equivalent*:

(1) *projections π_i, $i = 1, 2$, preserve congruences on $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$;*

(2) *$\Theta \circ \Pi_1 \circ \Theta \subseteq \Pi_1 \circ \Theta \circ \Pi_1$ holds for any congruence Θ on $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$;*

(3) *$\Pi_1 \circ \Theta \circ \Pi_1 = \Pi_1 \circ \Theta \circ \Pi_1 \circ \Theta = \Theta \circ \Pi_1 \circ \Theta \circ \Pi_1$ hold for any congruence Θ on $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$;*

(4) *there exist ternary terms $p_1, \ldots, p_m$, quaternary terms $q_1, \ldots, q_m$, and $(4 + m)$-ary terms $s_1, \ldots, s_n$ such that*

$$(\alpha) \begin{cases} x = s_1(x, y, y, z, \boldsymbol{p}(x, y, z)), \\ s_i(y, x, z, y, \boldsymbol{p}(x, y, z)) = s_{i+1}(x, y, y, z, \boldsymbol{p}(x, y, z)), \quad 1 \leqq i < n, \\ z = s_n(y, x, z, y, \boldsymbol{p}(x, y, z)), \end{cases}$$

$$(\beta) \quad s_i(y, x, u, z, \boldsymbol{q}(x, y, z, u)) = s_{i+1}(x, y, z, u, \boldsymbol{q}(x, y, z, u)), \quad 1 \leqq i \leqq n,$$

are identities in V.

Proof. (1) $\Rightarrow$ (2): Take an arbitrary congruence Θ on the product $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$. Let $\langle x_1, x_2 \rangle\ \Theta \circ \Pi_1 \circ \Theta\ \langle y_1, y_2 \rangle$. Then $\langle x_1, x_2 \rangle\ \Theta\ \langle z_1, z_2 \rangle\ \Pi_1$ $\langle z_1, z_2' \rangle\ \Theta\ \langle y_1, y_2 \rangle$ for some elements $z_1 \in A_1$, $z_2, z_2' \in A_2$. Since $\langle x_1, z_1 \rangle$, $\langle z_1, y_1 \rangle \in \pi_1 \times \pi_1(\Theta)$ we have $\langle x_1, y_1 \rangle \in \pi_1 \times \pi_1(\Theta)$, by hypothesis. This means that $\langle x_1, x_2 \rangle\ \Theta\ \langle y_1, y_2' \rangle$ holds for some $x_2', y_2' \in A_2$. Altogether we find that $\langle x_1, x_2 \rangle\ \Pi_1\ \langle x_1, x_2' \rangle\ \Theta\ \langle y_1, y_2' \rangle\ \Pi_1\ \langle y_1, y_2 \rangle$ which implies $\langle x_1, x_2 \rangle\ \Pi_1 \circ \Theta \circ \Pi_1\ \langle y_1, y_2 \rangle$. The inclusion $\Theta \circ \Pi_1 \circ \Theta \subseteq \Pi_1 \circ \Theta \circ \Pi_1$ is verified.

(2) $\Rightarrow$ (3): The assumed inclusion $\Theta \circ \Pi_1 \circ \Theta \subseteq \Pi_1 \circ \Theta \circ \Pi_1$ yields $\Pi_1 \circ \Theta \circ \Pi_1 \circ \Theta \subseteq \Pi_1 \circ \Pi_1 \circ \Theta \circ \Pi_1 = \Pi_1 \circ \Theta \circ \Pi_1$. Further, $\Theta \circ \Pi_1 \circ \Theta \circ \Pi_1 = (\Pi_1 \circ \Theta \circ \Pi_1 \circ \Theta)^{-1} \subseteq (\Pi_1 \circ \Theta \circ \Pi_1)^{-1} = \Pi_1 \circ \Theta \circ \Pi_1$.

(3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (4): Take $A_1 = F_V(x, y, z)$, $A_2 = F_V(x, y, z, u)$, the V-free algebras over three and four free generators, respectively. Let $\Theta = \Theta(\langle\langle x, x \rangle, \langle y, y \rangle\rangle, \langle\langle y, z \rangle, \langle z, u \rangle\rangle)$. Then $\langle x, x \rangle\ \Theta \circ \Pi_1 \circ \Theta\ \langle z, u \rangle$ since $\langle x, x \rangle\ \Theta\ \langle y, y \rangle\ \Pi_1\ \langle y, z \rangle\ \Theta\ \langle z, u \rangle$. In virtue of hypothesis (2) we get that $\langle x, x \rangle\ \Pi_1 \circ \Theta \circ \Pi_1\ \langle z, u \rangle$, which means that $\langle x, x \rangle\ \Pi_1\ \langle x, c \rangle\ \Theta\ \langle z, d \rangle\ \Pi_1\ \langle z, u \rangle$ for some elements $c, d \in A_2$. Applying the standard argument, see e.g. [2], to the relation $\langle\langle x, c \rangle, \langle z, d \rangle\rangle \in \Theta(\langle\langle x, x \rangle, \langle y, y \rangle\rangle, \langle\langle y, z \rangle, \langle z, u \rangle\rangle)$ we obtain the identities (4).

(4) $\Rightarrow$ (1): Let Ψ be a congruence on the product $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$, and let $\langle a, b \rangle\ \Psi\ \langle c, d \rangle$, $\langle c, g \rangle\ \Psi\ \langle h, k \rangle$. Substitute $x = a$, $y = c$, $z = h$ in the identities (3) (α) and $x = b$, $y = d$, $z = g$, $u = k$ in the identities (3) (β). Then

(α) $a = s_1(a, c, c, h, \boldsymbol{p}(a, c, h))$,

(β) $s_1 = s_1(b, d, g, k, \boldsymbol{q}(b, d, g, k))$,

(α) $s_i(c, a, h, c, \boldsymbol{p}(a, c, h)) = s_{i+1}(a, c, c, h, \boldsymbol{p}(a, c, h))$,

(β) $s_i(d, b, k, g, \boldsymbol{q}(b, d, g, k)) = s_{i+1}(b, d, g, k, \boldsymbol{q}(b, d, g, k))$, $1 \leqq i < n$,

(α) $h = s_n(c, a, h, c, \boldsymbol{p}(a, c, h))$,

(β) $s_n = s_n(d, b, k, g, \boldsymbol{q}(b, d, g, k))$,

from which the required conclusion $\langle a, s_1 \rangle\ \Psi\ \langle h, s_n \rangle$ readily follows. The proof is complete.

Remark 1. It is evident that conditions (2), (3) from Theorem 1 can be replaced by

(2') $\Theta \circ \Pi_2 \circ \Theta \subseteq \Pi_2 \circ \Theta \circ \Pi_2$ *holds for any congruence* Θ *on* $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$;

(3') $\Pi_2 \circ \Theta \circ \Pi_2 = \Pi_2 \circ \Theta \circ \Pi_2 \circ \Theta = \Theta \circ \Pi_2 \circ \Theta \circ \Pi_2$ *hold for any congruence* Θ *on* $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$.

Theorem 2. *For a variety* V, *the following conditions are equivalent*:

(1) *projections* π_i, $i = 1, 2$, *preserve subfactor congruences on* $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$;

(2) $\Theta_2 \circ \Pi_1 \circ \Theta_2 \subseteq \Pi_1 \circ \Theta_2 \circ \Pi_1$ *holds for any subfactor congruence* $\Theta_2 \subseteq \Pi_2$ *on* $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$;

(3) $\Pi_1 \circ \Theta_2 \circ \Pi_1 = \Pi_1 \circ \Theta_2 \circ \Pi_1 \circ \Theta_2 = \Theta_2 \circ \Pi_1 \circ \Theta_2 \circ \Pi_1$ *hold for any subfactor congruence* $\Theta_2 \subseteq \Pi_2$ *on* $A_1 \times A_2$, $A_i \in \boldsymbol{V}$, $i = 1, 2$;

(4) *there exist ternary terms* $p_1, \ldots, p_m$, *binary terms* $q_1, \ldots, q_m$, *and* $(4 + m)$*-ary terms* $t_1, \ldots, t_n$ *such that*

$$(\alpha) \begin{cases} x = t_1(x, y, y, z, \boldsymbol{p}(x, y, z)), \\ t_i(y, x, z, y, \boldsymbol{p}(x, y, z)) = t_{i+1}(x, y, y, z, \boldsymbol{p}(x, y, z)), \quad 1 \leqq i < n, \\ z = t_n(y, x, z, y, \boldsymbol{p}(x, y, z)), \end{cases}$$

$$(\beta) \quad t_i(x, x, y, y, \boldsymbol{q}(x, y)) = t_{i+1}(x, x, y, y, \boldsymbol{q}(x, y)), \quad 1 \leqq i \leqq n,$$

are identities in $\boldsymbol{V}$.

Proof. We omit the proof of part (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) as it runs in the same way as that of the foregoing Theorem 1.

(2) $\Rightarrow$ (4): Choose $A_1 = F_V(x, y, z)$, $A_2 = F_V(x, y)$ and $\Theta = \Theta(\langle\langle x, x\rangle, \langle y, x\rangle\rangle, \langle\langle y, y\rangle, \langle z, y\rangle\rangle)$. Apparently Θ is a subfactor congruence on $A_1 \times A_2$, in particular $\Theta \subseteq \Pi_2$. Since $\langle x, x\rangle\, \Theta\, \langle y, x\rangle\, \Pi_1\, \langle y, y\rangle\, \Theta\, \langle z, y\rangle$ the hypothesis (2) yields that $\langle x, x\rangle\, \Pi_1\, \langle x, c\rangle\, \Theta\, \langle z, c\rangle\, \Pi_1\, \langle z, y\rangle$ for an element $c \in A_2$. Applying [2] to the relation $\langle\langle x, c\rangle, \langle z, c\rangle\rangle \in \Theta\, (\langle\langle x, x\rangle, \langle y, x\rangle\rangle, \langle\langle y, y\rangle, \langle z, y\rangle\rangle)$ we immediately find the required identities (4).

(4) $\Rightarrow$ (1): Let $\Psi \subseteq \Pi_2$ be a subfactor congruence pn $A_1 \times A_2$, $A_i \in \boldsymbol{V}$, $i = 1, 2$. Assume that $\langle a, b\rangle\, \Psi\, \langle c, b\rangle$, $\langle c, g\rangle\, \Psi\, \langle h, g\rangle$. Setting $x = a$, $y = c$, $z = h$ in the identities (4) (α) and $x = b$, $y = g$ in the remaining identities (4) (β) we obtain

$$\begin{aligned}
&(\alpha) \quad a = t_1(a, c, c, h, \boldsymbol{p}(a, c, h)), \\
&(\beta) \quad t_1 = t_1(b, b, g, g, \boldsymbol{q}(b, g)), \\
&(\alpha) \quad t_i(c, a, h, c, \boldsymbol{p}(a, c, h)) = t_{i+1}(a, c, c, h, \boldsymbol{p}(a, c, h)), \\
&(\beta) \quad t_i(b, b, g, g, \boldsymbol{q}(b, g)) = t_{i+1}(b, b, g, g, \boldsymbol{q}(b, g)), \quad 1 \leqq i < n, \\
&(\alpha) \quad h = t_n(c, a, h, c, \boldsymbol{p}(a, c, h)), \\
&(\beta) \quad t_1 = t_n(b, b, g, g, \boldsymbol{q}(b, g)).
\end{aligned}$$

In this way we obtain $\langle a, t_1\rangle\, \Psi\, \langle h, t_1\rangle$ which establishes the transitivity of $\pi_1 \times \times \pi_1(\Psi)$. The proof is complete.

Let R be a binary relation on A, $a \in A$. Then $[a]\, R$ denotes the subset $\{x \in A;\ \langle a, x\rangle \in R\}$ of A.

Definition 2. Let A_1, A_2 be algebras of the same type. We say that the projection π_1 *preserves blocks of congruences* on $A_1 \times A_2$ whenever $\pi_1([\langle a_1, a_2\rangle]\, \Theta) = [\pi_1(\langle a_1, a_2\rangle)]\, \pi_1 \times \pi_1(\Theta)$ holds for any elements $a_i \in A_i$, $i = 1, 2$, and any congruence Θ on $A_1 \times A_2$.

The property π_2 *preserves blocks of congruences* on $A_1 \times A_2$ is introduced analogously.

Theorem 3. *Let* A_1, A_2 *be algebras of the same type. The following conditions are equivalent*:

(1) *projection* π_1 *preserves blocks of congruences on* $A_1 \times A_2$;

(2) $\Pi_1 \circ \Theta = \Theta \circ \Pi_1$ *holds for any congruence* Θ *on* $A_1 \times A_2$.

Proof. (1) $\Rightarrow$ (2): Let $\langle x_1, x_2 \rangle \Pi_1 \circ \Theta \langle y_1, y_2 \rangle$. Then $\langle x_1, x_2 \rangle \Pi_1 \langle x_1, x_2' \rangle \Theta \langle y_1, y_2 \rangle$ for an element $x_2' \in A_2$. In particular we have $\langle x_1, y_1 \rangle \in \pi_1 \times \pi_1(\Theta)$, which can be expressed as $\langle \pi_1(\langle x_1, x_2 \rangle), \pi_1(\langle y_1, y_2 \rangle) \rangle \in \pi_1 \times \pi_1(\Theta)$ or, equivalently, $\pi_1(\langle y_1, y_2 \rangle) \in [\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta)$. By hypothesis $[\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta) = \pi_1([\langle x_1, x_2 \rangle] \Theta)$ and so $\pi_1(\langle y_1, y_2 \rangle) \in \pi_1([\langle x_1, x_2 \rangle] \Theta)$. Hence $\langle y_1, y_2' \rangle \Theta \langle x_1, x_2 \rangle$ for some $y_2' \in A_2$. In this way we get $\langle x_1, x_2 \rangle \Theta \langle y_1, y_2' \rangle \Pi_1 \langle y_1, y_2 \rangle$, which proves the inclusion $\Pi_1 \circ \Theta \subseteq \Theta \circ \Pi_1$. Consequently also $\Theta \circ \Pi_1 = (\Pi_1 \circ \Theta)^{-1} \subseteq (\Theta \circ \Pi_1)^{-1} = \Pi_1 \circ \Theta$. Altogether $\Pi_1 \circ \Theta = \Theta \circ \Pi_1$, as required.

(2) $\Rightarrow$ (1): We want to verify the equality $\pi_1([\langle x_1, x_2 \rangle] \Theta) = [\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta)$ for any elements $x_i \in A_i$, $i = 1, 2$, and any congruence Θ on $A_1 \times A_2$.

(i) The inclusion $\pi_1([\langle x_1, x_2 \rangle] \Theta) \subseteq [\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta)$ is trivial.

(ii) Conversely, let $y_1 \in [\pi_1(\langle x_1, x_2 \rangle)] \pi_1 \times \pi_1(\Theta) = [x_1] \pi_1 \times \pi_1(\Theta)$. Equivalently $\langle x_1, y_1 \rangle \in \pi_1 \times \pi_1(\Theta)$, which means that $\langle y_1, y_2 \rangle \Theta \langle x_1, x_2' \rangle$ for some elements $y_2, x_2' \in A_2$. Since $\langle y_1, y_2 \rangle \Theta \langle x_1, x_2' \rangle \Pi_1 \langle x_1, x_2 \rangle$ we get $\langle y_1, y_2 \rangle \Pi_1 \langle y_1, y_2' \rangle \Theta \langle x_1, x_2 \rangle$ for an element $y_2' \in A_2$, by hypothesis. In other words $\langle y_1, y_2' \rangle \in [\langle x_1, x_2 \rangle] \Theta$ and so $y_1 = \pi_1(\langle y_1, y_2' \rangle) \in \pi_1([\langle x_1, x_2 \rangle] \Theta)$. The proof is complete.

Remark 2. The equivalent conditions from the foregoing Theorem 3 defined the so called *factor permutable* varieties (briefly: FP-varieties). Mal'cev characterizations of FP-varieties were given by J. Hagemann [5] and by H.-P. Gumm [4].

Definition 3. Let A_1, A_2 be algebras of the same type. A congruence Θ on $A_1 \times A_2$ is called *factorable* whenever $\Theta = \Theta_1 \times \Theta_2 = \{\langle \langle a_1, a_2 \rangle, \langle a_1', a_2' \rangle \rangle;\ a_i \Theta_i a_i', i = 1, 2\}$ for some congruences Θ_i on A_i, $i = 1, 2$.

A variety V has *factorable subfactor congruences* whenever any subfactor congruence on $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$, has this property.

Theorem 4. *For a variety* V, *the following conditions are equivalent*:

(1) *projections* π_i, $i = 1, 2$, *preserve blocks of subfactor congruences on* $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$;

(2) $\Theta_1 \circ \Pi_2 = \Pi_2 \circ \Theta_1$ *holds for any subfactor congruence* $\Theta_1 \subseteq \Pi_1$ *on the product* $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$;

(3) $\Theta_1 \circ \Theta_2 = \Theta_2 \circ \Theta_1$ *holds for any subfactor congruences* $\Theta_i \subseteq \Pi_i$, $i = 1, 2$, *on the product* $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$;

(4) V *has factorable subfactor congruences.*

Proof: (1) $\Leftrightarrow$ (2): See the proof of Theorem 3.

(3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (4): Consider the principal subfactor congruence $\Theta_1 = \Theta(\langle \langle x, x \rangle, \langle x, y \rangle \rangle) \subseteq \Pi_1$ on the product $A_1 \times A_2 = F_V(x, y) \times F_V(x, y)$. Since $\langle x, y \rangle \Pi_2$

$\langle y, y\rangle$ holds evidently we get $\langle x, x\rangle\, \Theta_1 \circ \Pi_2\, \langle y, y\rangle$. By hypothesis (2) also $\langle x, x\rangle\, \Pi_2 \circ \Theta_1\, \langle y, y\rangle$, which means that $\langle x, x\rangle\, \Pi_2\, \langle s_1, s_2\rangle\, \Theta_1\, \langle y, y\rangle$ for some elements $s_i \in A_i$, $i = 1, 2$. Then $s_1 = y$, $s_2 = x$ and so $\langle\langle y, x\rangle, \langle y, y\rangle\rangle \in \Theta_1 = \Theta(\langle\langle x, x\rangle, \langle x, y\rangle\rangle)$. The last condition implies condition (4), as was already shown by J. Hagemann [5] and by I. Chajda [1].

(4) $\Rightarrow$ (3): Let $\Theta_i \subseteq \Pi_i$, $i = 1, 2$, be arbitrary subfactor congruences on the product $A_1 \times A_2$, $A_i \in V$, $i = 1, 2$. Take $\langle x_1, x_2\rangle\, \Theta_1 \circ \Theta_2\, \langle y_1, y_2\rangle$. Then $\langle x_1, x_2\rangle\, \Theta_1\, \langle s_1, s_2\rangle\, \Theta_2\, \langle y_1, y_2\rangle$ for some elements $s_i \in A_i$, $i = 1, 2$. Since Θ_1, Θ_2 are subfactor congruences we have $s_1 = x_1$ and $s_2 = y_2$. By hypothesis (4) $\langle x_1, x_2\rangle\, \Theta_1\, \langle x_1, y_2\rangle$ implies $\langle y_1, x_2\rangle\, \Theta_1\, \langle y_1, y_2\rangle$ and $\langle x_1, y_2\rangle\, \Theta_2\, \langle y_1, y_2\rangle$ implies $\langle x_1, x_2\rangle\, \Theta_2\, \langle y_1, x_2\rangle$. Altogether we have $\langle x_1, x_2\rangle\, \Theta_2\, \langle y_1, x_2\rangle\, \Theta_1\, \langle y_1, y_2\rangle$, i.e. $\langle x_1, x_2\rangle\, \Theta_2 \circ \Theta_1 \langle y_1, y_2\rangle$, which establishes the permutability of subfactor congruences Θ_1, Θ_2. The proof is complete.

Remark 3. Mal'cev conditions for varieties with factorable subfactor congruences can be found in J. Hagemann [5] and also in I. Chajda [1].

Examples. Any variety with permutable congruences as well as any variety with factorable congruences evidently have all properties listed in the above Theorem 1, 2, 3, 4.

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