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PSEUDO-COMPLEMENTED DISTRIBUTIVE GROUPOIDS

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1. INTRODUCTION

A considerable amount of work on pseudo-complemented semilattices has appeared since Frink's paper in 1962 [2]. More recently, the study of distributive groupoids has begun to flourish with a number of papers appearing in the literature by various investigators such as, for example, J. Ježek, J. Kepka, and P. Němec [4, 5]. The purpose of this article is to examine pseudo-complementation from the viewpoint of distributive groupoids. The class of distributive groupoids that will be exploited in this paper are those which are left separative, as will be defined later. These groupoids are idempotent and have an associated partial ordering that is a corresponding analogue of the standard partial ordering for semilattices. The particular subclass of these groupoids on which a pseudo-complementation can be defined have a structure that permits one to extend a number of results from the theory of pseudo-complemented semilattices. It will be shown that a theorem of the Glivenko-Frink type can be obtained, that is, the set of closed elements in the groupoid form a boolean lattice and that the set of dense elements is a dual ideal. Also, assuming an additional restriction, the closed and dense elements will be used to give a complete description of the structure of a certain class of pseudo-complemented distributive groupoids. Finally, it should be pointed out that one effort has been made to extend the work on pseudo-complemented semilattices to other types of groupoids. The paper by K. Nirmala Kumai Amma [1] considered the problem for intraregular groupoids $(S, \cdot)$, that is, having the property that $J(x) \cap J(y) = J(xy)$ for each $x, y \in S$, where $J(x)$ denotes the principle ideal generated by the element x. A remark concerning systems of this type as compared with those of the type studied in this paper will be made in Section 3.

2. ABIAN'S ORDER RELATION

Throughout this article, S will denote a distributive groupoid, that is, $a(bc) = (ab)(ac)$ and $(bc)a = (ba)(ca)$ for each $a, b, c \in S$. Also, $\leqq$ will denote Abian's relation given by $x \leqq y$ if and only if $x^2 = xy$. Last, S is said to be *separative* if (i) $x^2 = xy$ and $y^2 = yx$ imply $x = y$, and (ii) $x^2 = yx$ and $y^2 = xy$ imply $x = y$. If (i) holds on S, then S is called *left separative.*

Theorem 1. *Abian's relation $\leqq$ is a partial order on S if and only if S is left separative.*

Proof. The necessity is immediate, thus assume S is left separative. First, it will be shown that each member of S is idempotent. Let $a \in S$. Then $(a^2)^2 = a^2a^2 = a^2(aa) = (a^2a)(a^2a) = [(aa)a](a^2a) = (a^2a^2)(a^2a) = a^2(a^2a) = a^2[(aa)a] = a^2(a^2a^2) = a^2(a^2)^2$. Now, $[(a^2)^2]^2 = (a^2)^2(a^2)^2 = (a^2a^2)(a^2a^2) = (a^2a^2)a^2 = (a^2)^2a^2$. Thus, one has that $(a^2)^2 = (a^2)(a^2)^2$ and $[(a^2)^2]^2 = (a^2)^2a^2$. Hence, by the left separative condition, $(a^2)^2 = a^2$. Next, $(a^2)^2 = (aa)(aa) = a^2a$ and $a^2 = a^2a^2 = a(aa) = aa^2$. Hence $(a^2)^2 = a^2a$ and $a^2 = aa^2$ implies, by the left separative condition, that $a^2 = a$. Therefore each element of S is idempotent. Consequently, in a left separative distributive groupoid S, the left separative condition reduces to the statement $x = xy$ and $y = yx$ implies that $x = y$.

Next, assume $x \leqq y$ and $y \leqq z$. Then $x = xy$ and $y = yz$. Thus $x = xy = x(yz) = (xy)(xz) = x(xz)$. Therefore $x = x(xz)$. Now $xz = (xy)z = (xz)(yz)$. Thus $xz = (xz)y$ and this gives $xz = (xy)(zy) = x(zy) = (xz)(xy) = (xz)x$. Hence we have obtained that $x = x(xz)$, $xz = (xz)x$ and this implies, by the left separative condition, that $x = xz$. Therefore $x \leqq z$ and thus $\leqq$ is transitive.

Finally, $x \leqq y$ and $y \leqq x$ gives $x = xy$ and $y = yx$. Therefore, by left separativity, $x = y$ and so $\leqq$ is antisymmetric. Consequently, $\leqq$ is a partial ordering of S.

Theorem 2. *If S is left separative, then S is an idempotent partially ordered groupoid with respect to Abian's order relation $\leqq$.*

Proof. Since Theorem 1 implies S is idempotent and $\leqq$ is a partial ordering of S, all that is left to show is that the binary operation defined on S is compatible with $\leqq$. Thus, let each of $x, y, z \in S$ with $x \leqq y$. Then $x = xy$ and hence, by distributivity, $zx = z(xy) = (zx)(zy)$ and $xz = (xy)z = (xz)(yz)$. Therefore $zx \leqq zy$ and $xz \leqq yz$.

3. PSEUDO-COMPLEMENTATION

Let $(S, \cdot, \leqq)$ denote a partially ordered groupoid of the type described byTheorem 2 and suppose there exists a right zero in S, that is, an element $0 \in S$ such that $x0 = 0$ for each $x \in S$.

Definition. The statement that an element $a' \in S$ is a pseudo-complement of an element $a \in S$ means that (i) $aa' = 0$ and (ii) if $ax = 0$, for some $x \in S$, then $x \leqq a'$. If each element in S has a pseudo-complement then S will be called a *pseudo--complemented groupoid.*

It should be noted that if an element in S has a pseudo-complement then conditions (i) and (ii) imply that the pseudo-complement is uniquely determined. Also, it should be indicated at this point that every pseudo-complemented meet semilattice is an example of a pseudo-complemented left separative distributive groupoid with a right zero 0.

The following is an example of a pseudo-complemented groupoid of the type that is considered in this article which is not necessarily a pseudo-complemented meet semilattice.

Example. Let $(S, +, \cdot)$ denote a boolean lattice. Let $a \in S$ and define $f(x) = a + x$ for each $x \in S$. Define $x \circ y = f(x)\, y$ for each $x, y \in S$. Observing that $f[f(x)] = f(x)$, $f(xy) = f(x) f(y)$, and $f(x)\, x = x$ is true for each $x, y \in S$, it follows by direct calculation that $(S, \circ)$ is an idempotent distributive groupoid. This groupoid is left separative. For, let $x = x \circ y$ and $y = y \circ x$. Then $x = f(x)\, y$ and $y = f(y)\, x$. Thus $f(x) = f[f(x)\, y] = f[f(x)] f(y) = f(x) f(y)$ and similarly $f(y) = f(y) f(x)$. Consequently, $f(x) = f(y)$ and this gives that $x = f(x)\, y = f(y)\, y = y$. Also, the least element 0 in the boolean lattice $(S, +, \cdot)$ serves as a right zero in $(S, \circ)$ since $x \circ 0 = f(x)\, 0 = 0$. Note that $0 \circ x = f(0)\, x = (a + 0)\, x = ax$ and thus, in general, 0 is not necessarily a left zero. For pseudo-complementation in $(S, \circ)$, we proceed as follows. For each $b \in S$, define $b' = a^* b^*$, where $(*)$ denotes complementation in the boolean lattice $(S, +, \cdot)$. Then $b \circ b' = f(b)\, a^* b^* = (a + b)\, a^* b^* = 0$. Next, suppose $b \circ x = 0$ for some $x \in S$. Then $f(b)\, x = 0$ and this gives $(a + b)\, x = 0$ which implies $ax + bx = 0$. Thus $ax = 0$ and $bx = 0$. Hence $a^* + x^* = 0^* = 1$ and thus $xa^* = x$ which implies $xa^* b^* = xb^*$. Also, $b^* + x^* = 1$ and so $xb^* = x$. Therefore $xa^* b^* = x$. Hence $x \circ b' = f(x)\, b' = (a + x)\, b' = (a + x)\, a^* b^* = xa^* b^* = x$. Therefore $x \leqq b'$ where $\leqq$ denotes Abian's ordering of $(S, \circ)$. Consequently b' is a pseudo-complement of b.

It should be mentioned that the preceding system $(S, \circ)$ is an example of a distributive groupoid which is not intraregular as defined by K. Amma in [1]. It is an immediate consequence of intraregularity that if $ab = 0$, then $ba = 0$. For the above example, $a \circ 0 = 0$ for each $a \in S$ but $0 \circ a$ may not be equal to 0 for some $a \in S$ as was noted above.

4. A GLIVENKO-FRINK THEOREM

In the remaining part of this article, S will denote a pseudo-complemented left separative distributive groupoid. An element $x \in S$ will be said to be *closed* if $x'' = x$ and the set of all such closed elements will be denoted by $B(S)$. In addition, an element $x \in S$ such that $x' = 0$ will be said to be *dense*.

Lemma 3. *If $a, b \in S$ such that $a \leqq b$, then $b' \leqq a'$. Also, for each $b \in S$, $0b' = 0$.*

Proof. Suppose $a \leqq b$. Then $ab = a$. Thus $ab' = (ab)\, b' = (ab')(bb') = (ab')\, 0 = 0$. Hence $b' \leqq a'$. Next, $b0 = 0$ implies that $0 \leqq b'$ for each $b \in S$. Therefore $0b' = 0$.

Theorem 4. *If $a, b \in S$, then $(ab)'' = a''b''$.*

Proof. Let $x = (ab)'(a''b'')$. Then $bx = b[(ab)'(a''b'')]$ and thus $a(bx) = (ab)\{[a(ab)'][a(a''b'')]\} = \{[(ab)a]0\}\{(ab)[a(a''b'')]\} = 0\{(ab)[a(a''b'')]\} = [0(ab)]\{(0a)[(0a'')(0b'')]\} = [0(ab)][(0a)0] = [0(ab)]0 = 0$. Thus $bx \leqq a'$. Also, $a'(bx) = a'\{b[(ab)'(a''b'')]\} = (a'b)\{[a'(ab)'][(a'a'')(a'b'')]\} = (a'b)\{[a'(ab)'][0(a'b'')]\} = (a'b)\{[a'(ab)'][(0a')(0b'')]\} = (a'b)\{[a'(ab)']0\} = (a'b)0 = 0$. Therefore $bx \leqq a''$ and consequently $bx \leqq a'a'' = 0$ which implies that $bx = 0$ and thus $x \leqq b'$. Next, $b'x = b'[(ab)'(a''b'')] = [b'(ab)'][(b'a'')(b'b'')] = [b'(ab)'][(b'a'')0] = [b'(ab)']0 = 0$. Hence $x \leqq b''$. Therefore $x \leqq b'b'' = 0$ which gives that $x = 0$. Therefore $(ab)'(a''b'') = 0$ and this implies that $a''b'' \leqq (ab)''$. Now $(ab)b' = (ab')(bb') = (ab')0 = 0$ which gives that $b' \leqq (ab)'$ and thus, by Lemma 3, $(ab)'' \leqq b''$. Last, $(ab)a' = (aa')(ba') = 0(ba') = (0b)(0a') = (0b)0 = 0$. Thus $a' \leqq (ab)'$ and, again, by Lemma 3, $(ab)'' \leqq a''$. Hence $(ab)'' \leqq a''b''$. Therefore $a''b'' \leqq (ab)''$ and $(ab)'' \leqq a''b''$ implies that $(ab)'' = a''b''$.

To obtain the following Glivenko-Frink type theorem, we will make use of a result by Frink, see [3], who characterized a boolean lattice as a meet semilattice $(B, \cdot)$ containing a special element 0 along with a mapping $(*)$ from B into B possessing the property that $a \,.\, b^* = 0$ if and only if $a \,.\, b = a$. The least upper bound for a pair of elements $a, b \in B$ was shown to be $a \vee b = (a^* \,.\, b^*)^*$.

Theorem 5. *The set $B(S)$ of closed elements of S is a boolean lattice.*

Proof. First, it will be shown that $0 \in B(S)$. $0'0 = 0$ implies that $0 \leqq 0''$. Also, from Lemma 3, $00'' = 0$ which gives that $0'' \leqq 0'$. Hence $0''0'' \leqq 0'0'' = 0$. Thus $0'' \leqq 0$ and consequently $0'' = 0$. Therefore $0 \in B(S)$. Next, $B(S)$ is a subgroupoid of S since, by Theorem 4, the product of two closed elements is again closed. Let $x, y \in B(S)$. In order to show that $B(S)$ is a meet semilattice, we need to point out that xy is the greatest lower bound for x and y. Now $(xy)x' = (xx')(yx') = 0(yx') = (0y)(0x') = (0y)0 = 0$. Thus $x' \leqq (xy)'$ and, by Lemma 3, $(xy)'' \leqq x''$. Thus $xy \leqq x$. In a similar manner $(xy)y' = (xy')(yy') = (xy')0 = 0$ and thus $y' \leqq (xy)'$ which implies that $(xy)'' \leqq y''$. Hence $xy \leqq y$. Therefore xy is a lower bound for x and y. Finally, let $k \in B(S)$ such that $k \leqq x$ and $k \leqq y$. Then $k \leqq xk \leqq xy$. Consequently, xy is the greatest lower bound for x and y in $B(S)$. Therefore $B(S)$ is a meet semilattice. Last, one needs to show that the pseudo-complementation operation $(')$ has the property mentioned above for the unary operation $(*)$ in Frink's theorem. First, if $x \in B(S)$ then $x' \in B(S)$ since $(x')'' = (x'')' = x'$. Now let $a, b \in B(S)$ with $ab' = 0$. Then $b' \leqq a'$ and thus, by Lemma 3, $a'' \leqq b''$. Hence $a \leqq b$ and this gives $ab = a$. Next, suppose $ab = a$. Then $ab' = (ab)b' = (ab')(bb') = (ab')0 = 0$. Therefore $ab' = 0$ if and only if $ab = a$ and thus, by Frink's theorem, $B(S)$ is a boolean lattice.

For what follows, the greatest element in $B(S)$, namely $0'$, will be denoted by the symbol 1.

Lemma 6. *If $x \in S$, then $x' \leqq x'''$.*

Proof. By Lemma 3, $0x' = 0$. Thus $0 = 0x' = (x'x'')\,x' = x'(x''x')$. Hence $x''x' \leqq x''$ and thus $(x''x')\,x'' = x''x'$. Hence $x''x' = (x''x')\,x'' = x''(x'x'') = x''0 = 0$. Therefore $x' \leqq x'''$.

In the following theorem, a dual ideal K of S is defined to be a subgroupoid of S which has the property that if $x \in K$ and $y \in S$, with $x \leqq y$, then $y \in K$.

Theorem 7. *If D is the set of all dense elements of S, then D is a dual ideal of S.*

Proof. Let $d_1, d_2 \in D$. Then $d_1'' = 0' = 1$ and $d_2'' = 0' = 1$. Thus, by Theorem4, $(d_1 d_2)'' = d_1'' d_2'' = 11 = 1$. Hence $(d_1 d_2)''' = 1' = 0$. Now, by Lemma 6, $(d_1 d_2)' \leqq \leqq (d_1 d_2)''' = 0$ and thus $(d_1 d_2)' = 0$. Consequently, $d_1 d_2$ is dense and therefore D is a subgroupoid of S. Next, let $d \in D$ and $x \in S$ with $d \leqq x$. Then, by Lemma 3, $x' \leqq d'$ and this gives that $x' = 0$ since $d' = 0$ and thus $x \in D$. Therefore D is a dual ideal of S.

5. A STRUCTURE THEOREM

Theorem 8. *Suppose $(S, \cdot, \leqq)$ denotes a left separative pseudo-complemented distributive groupoid, where $\leqq$ is the associated partial order on $(S, \cdot)$. Let each of B and D denote the set of closed and dense elements of S respectively. If the members of D form an antichain with respect to $\leqq$, and, for each $x \in S$, there exists exactly one $d \in D$ such that $x = x''d$, then $(S, \cdot, \leqq)$ can be expressed as a cardinal sum of boolean lattices B_{d_i} where, for each $d_i \in D$, $1d_i$ is the greatest element of B_{d_i} and B_{d_i} is isomorphic to B.*

Proof. For each $d \in D$, let $B_d = \{xd \mid x \in B\}$. First, it should be noted that each $x \in S$ is a member of some B_d since, by hypothesis, $x = x''d_1$, where $x'' \in B$ and $d_1 \in D$ and thus $x \in B_{d_1}$. Let each of $d_1, d_2 \in D$ with $d_1 \neq d_2$. We want to show next that no member in B_{d_1} is comparable with a member in B_{d_2}. Suppose $x \in B_{d_1}$ and $y \in B_{d_2}$ and that $x \leqq y$. Now $x = x_1 d_1$ and $y = y_2 d_2$. By Theorem 4, $x'' = = x_1'' d_1'' = x_1'' 1 = x_1'' = x_1$ and similarly $y'' = y_2'' = y_2$. Thus $x = x'' d_1$ and $y = = y'' d_2$. From $x \leqq y$ we thus have $x'' d_1 \leqq y'' d_2$ which gives $(x'' d_1)(y'' d_2) = x'' d_1$. Applying Theorem 4 again, gives $[(x'')'' d_1''] [(y'')'' d_2''] = (x'')'' d_1''$ and consequently we have $x'' y'' = x''$. Returning again to $x'' d_1 \leqq y'' d_2$ and multiplying on the left by $x'' d_2$ gives $(x'' d_2)(x'' d_1) \leqq (x'' d_2)(y'' d_2)$ and thus $x''(d_2 d_1) \leqq (x'' y'')\, d_2 = x'' d_2$. Hence $x''(d_2 d_1) \leqq x'' d_2$ now implies that $[x''(d_2 d_1)]\,(x'' d_2) = x''(d_2 d_1)$ and thus $x''[(d_2 d_1)\, d_2] = x''(d_2 d_1)$. Therefore, from the uniqueness of the dense elements in the representation of each member of S, we have that $(d_2 d_1)\, d_2 = d_2 d_1$. Thus $d_2 d_1 \leqq d_2$ and this gives that $d_2 d_1 = d_2$ since the members of D form an antichain with respect to $\leqq$. Therefore $d_2 \leqq d_1$ which again implies that $d_2 = d_1$ and this is a contradiction since $d_2 \neq d_1$. Consequently no member in B_{d_1} is comparable with a member in B_{d_2}.

Last, we want to show that $(B, \leqq)$ is order-isomorphic to $(B_d, \leqq)$ for each $d \in D$.

Let $d \in D$ and, for each $x \in B$, define $f_d(x) = xd$. From the definition of B_d and f_d, it is immediate that f_d maps B onto B_d. Next, suppose that $f_d(x_1) = f_d(x_2)$. Then $x_1 d = x_2 d$, and by Theorem 4, $x_1'' d'' = x_2'' d''$ and thus $x_1'' = x_1'' 1 = x_2'' 1 = x_2''$. Hence $x_1 = x_2$ since each of $x_1, x_2 \in B$. Therefore f_d is a "one-to-one" mapping. Now suppose $x_1 \leqq x_2$. Then $x_1 d \leqq x_2 d$ and thus $f_d(x_1) \leqq f_d(x_2)$. Finally, suppose $f_d(x_1) \leqq \leqq f_d(x_2)$. Then $x_1 d \leqq x_2 d$ which implies that $(x_1 d)(x_2 d) = x_1 d$. Hence $(x_1 x_2) d = = x_1 d$. Applying Theorem 4, we obtain $(x_1 x_2)'' d'' = x_1'' d''$ and thus $x_1 x_2 = x_1$. Consequently, $x_1 \leqq x_2$. Therefore f_d is an order-isomorphic mapping from B to B_d. Since, by Theorem 5, B is a boolean lattice, we thus have that B_d is also a boolean lattice. Note that $x \leqq 1$, for each $x \in B$, implies that $f_d(x) = xd \leqq 1d$. Thus $1d$ is the greatest element for B_d.

Before proceeding to the last theorem, there are four observations that need to be made concerning the dense elements D given in Theorem 8.

Property (i). If $d \in D$, then $d1 = 1$. This follows from the fact that $0 \leqq 1$ implies that $d0 \leqq d1$ and thus $0 \leqq d1$. Now $10 \leqq 1(d1)$ and so $0 \leqq 1(d1)$. Since $d1$ is dense, $0 \in B_1$, $1(d1) \in B_{d1}$ and, by Theorem 8, no element in B_1 is comparable with an element in B_{d1}, we thus have that $d1 = 1$.

Property (ii). If $d_1, d_2 \in D$ and $d_1 d_2 = 1$, then $d_2 = 1$. Let each of 0_1 and 0_2 denote the least element in B_{d_1} and B_{d_2} respectively. Now, $f_{d_1}(0) = 0_1$ and $f_{d_2}(0) = 0_2$ and so $0d_1 = 0_1$ and $0d_2 = 0_2$. Thus $0_1 0_2 = (0d_1)(0d_2) = 0(d_1 d_2) = 01 = 0$. Hence $0_1 0_2 = 0$ implies that $0_2 \leqq 0_1'$. Now $0_1' \leqq 1$ since, $00_1' = 0$, by Lemma 3, and thus $0_1' \leqq 0'$. Consequently, $0_2 \leqq 1$, $1 \in B_1$, and $0_2 \in B_{d_2}$. Therefore, by Theorem 8, $d_2 = 1$.

Property (iii). If $d \in D$, then there exists exactly one $d_1 \in D$ such that $d = 1d_1$. This follows directly from the hypothesis of Theorem 8 and since $d'' = 1$.

Property (iv). If each of $d_1, d_2 \in D$ such that $d_1 d_2 = d_1$, then $d_1 = d_2$. Again, this follows from the fact that the members of D form an antichain with respect to Abian's order relation $\leqq$.

Theorem 9. *If B is a boolean lattice and D is an idempotent distributive groupoid containing an element 1 satisfying properties* (i), (ii), (iii), *and* (iv), *then there exists a pseudo-complemented left separative distributive groupoid $(S, \circ)$ such that*

1. *the set of closed elements B_1 in S is order-isomorphic to B relative to Abian's order relation $\leqq$,*
2. *the set of dense elements D' in S is isomorphic to D and no two distinct members in D' are comparable with respect to $\leqq$, and*
3. *if $x \in S$, then $x = x'' \circ d$ for exactly one $d \in D'$, where $(')$ denotes the pseudo-complement operation in $(S, \circ)$.*

Proof. Without loss of generality, we may assume that $B \cap D = \emptyset$. From the sets B and D and noting property (iii), one can construct, for each $d_i \in D$, a set B_{d_i}

such that card B_{d_i} = card B, $B_{d_i} \cap D = \{1d_i\}$ and, if $d_i \neq d_j$, then $B_{d_i} \cap B_{d_j} = \emptyset$. Next, for each $d_i \in D$, partially order B_{d_i} so that B_{d_i} is order-isomorphic to B in such a way that $1d_i$ is the greatest element in B_{d_i}. For each $d_i \in D$, with $d_i \neq 1$, let f_{d_i} denote an order-isomorphic mapping from B_1 to B_{d_i} and let f_1 denote the identity mapping from B_1 to B_1. Let $S = \bigcup_{d_i \in D} B_{d_i}$. Let $x, y \in S$ and suppose $x \in B_{d_i}$ and $y \in B_{d_j}$. Define $x \circ y = f_{d_i d_j}[f_{d_i}^{-1}(x) \cdot f_{d_j}^{-1}(y)]$, where "$\cdot$" denotes the greatest lower bound operation in B_1.

By direct computation, $(S, \circ)$ is an idempotent distributive groupoid. Next, we want to show that $(S, \circ)$ is left separative. Suppose $x, y \in S$ such that $x = x \circ y$ and $y = y \circ x$. Then, for exactly one $d_1, d_2 \in D$, $x \in B_{d_1}$ and $y \in B_{d_2}$. Thus $x = x \circ y = f_{d_1 d_2}[f_{d_1}^{-1}(x) \cdot f_{d_2}^{-1}(y)]$ and $y = y \circ x = f_{d_2 d_1}[f_{d_2}^{-1}(y) \cdot f_{d_1}^{-1}(x)]$. Hence $f_{d_1 d_2}^{-1}(x) = f_{d_1}^{-1}(x) \cdot f_{d_2}^{-1}(y)$ and $f_{d_2 d_1}^{-1}(y) = f_{d_2}^{-1}(y) \cdot f_{d_1}^{-1}(x)$. Now, each of $f_{d_1}^{-1}(x), f_{d_2}^{-1}(y)$ is a member of the boolean lattice B_1 and, since "$\cdot$" denotes the greatest lower bound operation in B_1, we thus have that $f_{d_1 d_2}^{-1}(x) = f_{d_2 d_1}^{-1}(y)$. Hence, for some $u \in B_1$, $f_{d_1 d_2}(u) = x$ and $f_{d_2 d_1}(u) = y$. Hence $x \in B_{d_1 d_2}$ and $y \in B_{d_2 d_1}$. Thus $x \in B_{d_1} \cap B_{d_1 d_2}$ and $y \in B_{d_2} \cap B_{d_2 d_1}$. Since the sets B_{d_i} are disjoint, we obtain that $d_1 = d_1 d_2$ and $d_2 = d_2 d_1$. Therefore, by property (iv), $d_1 = d_2$ and this implies that $x = f_{d_1}(u) = f_{d_2}(u) = y$. Consequently, $(S, \circ)$ is left separative and, by Theorems 1 and 2, Abian's order relation is a compatible partial ordering of S.

In $(S, \circ)$, a pseudo-complementation operation $(')$ is constructed in the following manner. Let $x \in S$. Then $x \in B_{d_1}$ for exactly one $d_1 \in D$. Define $x' = [f_{d_1}^{-1}(x)]^*$, where "$*$" denotes the complement in the boolean lattice B_1. Also, let 0 denote the least element in B_1. Then $x \circ x' = f_{d_1 1}\{f_{d_1}^{-1}(x) \cdot f_1^{-1}[(f_{d_1}^{-1}(x))^*]\} = f_1[f_{d_1}^{-1}(x) \cdot (f_{d_1}^{-1}(x))^*] = f_{d_1}^{-1}(x) \cdot [f_{d_1}^{-1}(x)]^* = 0$, by property (i). Now, suppose $x \circ y = 0$ for some $y \in S$. Then $y \in B_{d_2}$ for exactly one $d_2 \in D$ and $x \circ y \in B_{d_1 d_2}$. Hence $0 \in B_1 \cap B_{d_1 d_2}$ which implies that $d_1 d_2 = 1$. From property (ii), we thus have that $d_2 = 1$. Therefore $y \in B_1$. Thus $0 = x \circ y = f_{d_1 1}[f_{d_1}^{-1}(x) . f_1^{-1}(y)] = f_1[f_{d_1}^{-1}(x) . f_1^{-1}(y)] = f_1[f_{d_1}^{-1}(x) . y] = f_{d_1}^{-1}(x) . y = f_{d_1}^{-1}(x) \circ y$ by property (i) and since "$\circ$" and "$\cdot$" coincide on B_1. Consequently, by Abian's order relation and since "$\circ$" and "$\cdot$" coincide on B_1, $y \leqq x'$. As a consequence of the preceding statements, the set of closed elements in S is exactly the elements of B_1 and, as a result of the construction of B_1 from above, the set of closed elements, namely B_1, is order-isomorphic to B.

Next, suppose $x \in D'$, where D' denotes the dense elements of S. Then $0 = x' = [f_{d_1}^{-1}(x)]^*$ implies that $f_{d_1}^{-1}(x) = 0^* = 1$. Thus $x = f_{d_1}(1) = 1d_1$ from the definition of f_{d_1}, and this implies that $x \in D$. Hence $D' \subseteq D$. Now, let $d \in D$. Then, by property (iii), there exists exactly one $d_2 \in D$ such that $d = 1d_2$. Hence $d' = [f_{d_2}^{-1}(d)]^* = 1^* = 0$ and this implies that $d \in D'$. Thus $D \subseteq D'$ and, therefore $D' = D$. Last, it will be shown that if $d_1, d_2 \in D'$, then $d_1 \circ d_2 = d_1 d_2$. By property (iii), there exists exactly one $d_3, d_4 \in D$ such that $1d_3 = d_1$ and $1d_4 = d_2$. Hence $d_1 \circ d_2 = f_{d_3 d_4}[f_{d_3}^{-1}(d_1) . f_{d_4}^{-1}(d_2)] = f_{d_3 d_4}[1 . 1] = f_{d_3 d_4}(1) = 1(d_3 d_4) = (1d_3)(1d_4) = d_1 d_2$. Consequently, $(D', \circ)$ coincides with the given groupoid D whose binary operation was indicated by juxtaposition. Consequently, by property (iv), no two distinct members of D' are comparable with respect to Abian's relation $\leqq$.

Finally, for part (3) of the proof, we proceed as follows. Let $x \in S$. Then $x \in B_{d_1}$ for exactly one $d_1 \in D$. Also, by property (iii), there exists exactly one $d_3 \in D$ such that $1d_3 = d_1$. From the definition of the pseudo-complement, $x'' = f_{d_1}^{-1}(x)$. Thus $x'' \circ d_1 = f_{1d_3}\{[f_1^{-1}(f_{d_1}^{-1}(x))] \cdot f_{d_3}^{-1}(d_1)\} = f_{d_1}[f_{d_1}^{-1}(x) \cdot 1] = f_{d_1}[f_{d_1}^{-1}(x)] = x$. Now, suppose $x'' \circ d_2 = x$ for some $d_2 \in D$. Then, by property (iii), there exists exactly one $d_4 \in D$ such that $1d_4 = d_2$. Thus $x = x'' \circ d_2 = f_{1d_4}\{[f_1^{-1}(f_{d_1}^{-1}(x))] \cdot f_{d_4}^{-1}(d_2)\} = f_{d_2}[f_{d_1}^{-1}(x) \cdot 1] = f_{d_2}[f_{d_1}^{-1}(x)]$. Hence $f_{d_2}^{-1}(x) = f_{d_1}^{-1}(x)$. This implies that $f_{d_2}(u) = x = f_{d_1}(u)$ for some $u \in B_1$. Therefore $x \in B_{d_1}$ and $x \in B_{d_2}$ which thus implies that $d_1 = d_2$ since $B_{d_1} \cap B_{d_2} = \emptyset$ for $d_1 \neq d_2$.

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