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SELFDUALITY OF THE SYSTEM OF INTERVALS OF A PARTIALLY ORDERED SET

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1. INTRODUCTION

For a partially ordered set P we denote by Int P the system of all intervals $[a, b] = \{x \in P: a \leqq x \leqq b\}$, where $a, b \in P$ and $a \leqq b$, including the empty set. The system Int P is partially ordered by the set-theoretical inclusion.

If P is a lattice, then Int P is a lattice as well. In general, Int P need not be a lattice.

In [1], the following theorem was presented:

(A) Let L be a finite lattice. Then Int L is selfdual if and only if either (i) card $L \leqq 2$, or (ii) card $L = 4$ and L has two atoms.

Next, in [1] the author proposed the problem whether there exists an infinite lattice L such that Int L is selfdual.

In the present paper it will be shown that the answer to this problem is negative. Namely, the following result will be proved:

(B) Let P be a partially ordered set with card $P > 4$. Then the partially ordered system Int P is not selfdual.

Some questions concerning Int L (where L is a lattice) have been studied in the papers [2]–[9].

2. PROOF OF (*B*)

If Q is a partially ordered set and a, b, c are elements of Q, then by writing $a \vee b = c$ we express the fact that c is the least upper bound of the set $\{a, b\}$ in Q; the meaning of $a \wedge b = c$ is the dual one. If a and b are incomparable, then we write $a \parallel b$; the fact that a is covered by b will be expressed by writing $a \prec b$.

Q is said to be *selfdual* if there exists a dual automorphism of Q. If f is a dual automorphism of Q and $a, b, c \in Q$, then

$$a \vee b = c \Leftrightarrow f(a) \wedge f(b) = f(c),$$

and dually.

In what follows, P denotes a partially ordered set. Let $X \in$ Int P.

X is an atom of Int P if and only if there is $a \in P$ with $X = \{a\}$.

Let $X = [a, b]$. Then X is a dual atom of Int P if and only if there is $[u, v] \in$ Int P such that $[u, v] = P$, and either (i) $a = u$ and $b \prec v$, or (ii) $u \prec a$ and $b = v$.

Let $X = [a_1, a_2]$ and $Y = [b_1, b_2]$ belong to Int P. Then $X \wedge Y$ does exist in Int P if and only if either $X \cap Y = \emptyset$ (and in this case $X \wedge Y = \emptyset$), or both $a_1 \vee b_1$ and $a_2 \wedge b_2$ exist in P (and then $X \wedge Y = [a_1 \vee b_1, a_2 \wedge b_2]$).

Similarly, $X \vee Y$ exists in Int P if and only if both $a_1 \wedge b_1$ and $a_2 \vee b_2$ exist in P; in such a case $X \vee Y = [a_1 \wedge b_1, a_2 \vee b_2]$.

In particular, if $[a, b] \in$ Int P, then

$$[a, b] = \{a\} \vee \{b\}.$$

Next, if $a, b, c \in P$ and $a < b < c$, then

$$[a, b] \wedge [b, c] = \{b\}.$$

2.1. Lemma. *Assume that the system* Int P *is selfdual. Then there are* $u, v \in L$ *with* $a \leqq v$ *such that* $P = [u, v]$.

Proof. There exists a dual automorphism f of Int P. Since $\emptyset$ is the least element of Int P, $f(\emptyset)$ must be the largest element of Int P. Clearly $f(\emptyset) \neq \emptyset$ and hence there is $[u. v] \in$ Int P such that $[u\ v] = P$.

In proving (B) we proceed by way of contradiction; suppose that card $P > 4$ and that the partially ordered system Int P is selfdual. Let f be a fixed dual automorphism of Int P. In view of 2.1 there are $u, v \in P$ such that $P = [u, v]$.

2.2. Lemma. *Let* $[a, b] \in$ Int P, $[a, b] \neq P$. *Then there are dual atoms* X_1 *and* X_2 *in* Int P *such that* $X_1 \wedge X_2 = [a, b]$.

Proof. There is $[a_1, b_1] \in$ Int P such that $f([a_1, b_1]) = [a, b]$. Put $f(\{a_1\}) = X_1$ and $f(\{b_1\}) = X_2$. Since $\{a_1\}$ nad $\{b_1\}$ are atoms of Int P, both X_1 and X_2 are dual atoms of Int P. Next, $\{a_1\} \vee \{b_1\} = [a_1, b_1]$. By applying the dual automorphism f we obtain that $X_1 \wedge X_2 = [a, b]$.

2.3. Lemma. *Let* $[a, b] \in$ Int P, $a \neq b$, $a \neq u$, $b \neq v$. *Then* $u \prec a$ *and* $b \prec v$.

Proof. Let X_1 and X_2 be as in 2.2. There are $c_i, d_i \in P$ $(i = 1, 2)$ with $X_1 = [c_1, d_1]$ and $X_2 = [c_2, d_2]$. If $c_1 = c_2 = u$, then $a = u$, which is a contradiction. Thus, without loss of generality we can suppose that $c_1 = u$ and $c_2 \neq u$. Then $d_2 = v$, $d_1 \prec v$ and $u \prec c_2$. Since $X_1 \cap X_2 = [a, b]$, we infer that $a = c_2$ and $b = d_1$.

2.4. Lemma. *Let* C *be a chain in* P. *Then* card $C \leqq 4$.

Proof. This is an immediate consequence of 2.3.

Denote $f(\{u\}) = X$ and $f(\{v\}) = Y$. Then

$$\{u\} \vee \{v\} = P, \quad \{u\} \wedge \{v\} = \emptyset,$$

whence

(1) $$X \cap Y = \emptyset,$$

(2) $$X \vee Y = P.$$

There are x_i and y_i in P $(i = 1, 2)$ such that $X = [x_1, x_2]$ and $Y = [y_1, y_2]$. From the fact that X and Y are dual atoms of Int P and from (1), (2) we infer that some of the following conditions is valid:

$$(\alpha) \quad x_1 = u\,, \quad x_2 \prec v\,, \quad u \prec y_1\,, \quad y_2 = v\,, \quad x_2 \parallel y_1\,;$$

$$(\beta) \quad y_1 = u\,, \quad y_2 \prec v\,, \quad u \prec x_1\,, \quad x_2 = v\,, \quad y_2 \parallel x_1\,.$$

Next, let $z \in L$, $f(\{z\}) = [t_1, t_2]$. We have either

$$(\alpha_1) \quad t_1 = u \quad \text{and} \quad t_2 \prec v\,,$$

or

$$(\beta_1) \quad u \prec t_1 \quad \text{and} \quad t_2 = v\,.$$

From the relation $[u, z] \cap [z, v] = \{z\}$ we obtain

$$f([u, z]) \vee f([z, v]) = f(\{z\})\,.$$

Because of $[u, z] = \{u\} \vee \{z\}$ and the analogous relation for $[z, v]$, we get

(3) $$(f(\{u\}) \wedge f(\{z\})) \vee (f(\{z\}) \wedge f(\{v\})) = f(\{z\})\,.$$

2.5. Lemma. *Assume that* (α) *and* (α_1) *yre valid. Let* $u \neq z$. *Then* $y_1 \leqq t_2$.

Proof. In view of (3) we have

(4) $$([u, x_2] \wedge [u, t_2]) \vee ([u, t_2] \wedge [y_1, v]) = [u, t_2]\,.$$

Then

$$[u, x_2] \wedge [u, t_2] = [u, x_2 \wedge t_2]\,.$$

Next,

$$[u, t_2] \wedge [y_1, v] = [y_1, t_2] \quad \text{if} \quad y_1 \leqq t_2\,.$$

and $[u, t_2] \wedge [y_1, v] = \emptyset$ otherwise.

First we consider the case when $y_1 \nleqq t_2$. Then (4) yields

$$[u, x_2 \wedge t_2] = [u, t_2]\,,$$

whence $t_2 \leqq x_2$. The case $t_2 < x_2$ is impossible, since both t_2 and x_2 are covered by v. If $t_2 = x_2$, then $z = u$, which is a contradiction. Hence $y_1 \leqq t_2$.

2.6. Lemma. *Assume that* (α) *is valid. Let* $t \in P$, $t \prec v$, $t \neq x_2$. *Then* $y_1 \leqq t$.

Proof. Since $[u, t]$ is a dual atom in Int P, there is $z \in P$ such that $f(\{z\}) = [u, t]$. From $t \neq x_2$ we infer that $z \neq u$. Therefore according to 2.5 the relation $y_1 \leqq t$ is valid.

2.7. Lemma. *Assume that* (α) *and* (β_1) *hold. Let* $z \neq v$. *Then* $t_1 \leqq x_2$.

Proof. By virtue of (3), the relation

(5) $$([u, x_2] \wedge [t_1, v]) \vee ([t_1, v] \wedge [y_1, v]) = [t_1, v]$$

is valid. We have

$$[u, x_2] \wedge [t_1, v] = [t_1, x_2] \quad \text{if} \quad t_1 \leqq x_2\,, \quad \text{and}$$

$$[u, x_2] \wedge [t_1, v] = \emptyset \quad \text{otherwise}\,.$$

Next, $[t_1, v] \wedge [y_1, v] = [t_1 \vee y_1, v]$.

If $t_1 \nleqq x_2$, then (5) implies that

$$[t_1 \vee y_1, v] = [t_1, v]$$

is valid, whence $y_1 \leqq t_1$. The case $y_1 < t_1$ cannot occur, since $u \prec y_1$ and $u \prec t_1$. If $y_1 = t_1$, then $z = v$, which is a contradiction. Therfeore $t_1 \leqq x_2$.

2.8. Lemma. *Assume that* (α) *is valid. Let* $t \in P$, $u \prec t$, $t \neq y_1$. *Then* $t \leqq x_2$.

The proof is analogous to that of 2.6 with the distinction that we apply 2.7 instead of 2.5.

2.9. Lemma. *Assume that* (α) *is valid. Let* t *be an element of* P *which does not belong to the set* $\{u, v, x_2, y_1\}$. *Then either* $u \prec t \prec x_2$ *or* $y_1 \prec t \prec v$.

Proof. In view of 2.4 we have either $u \prec t$ or $t \prec v$. Now it suffices to apply 2.6 and 2.8.

Under the assumption that (α) holds we denote

$$A = \{t \in P: u \prec t \prec x_2\}, \quad B = \{t \in P: y_1 \prec t \prec v\}.$$

2.10. Corollary. *Assume that* (α) *is valid. Then* $A \cap B \neq \emptyset$.

This is a consequence of 2.9 and of the fact that card $P > 4$.

The result of the above corollary can be sharpened by the following consideration.

2.11. Lemma. *Let* (α) *be valid and let* $b \in B$. *Then there is* $a \in A$ *such that* $a < b$.

Proof. In view of 2.2 there are dual atoms $[z_1, z_2]$ and $[z_3, z_4]$ of Int P such that $[b, v] = [z_1, z_2] \wedge [z_3, z_4]$. Since $[b, v]$ is not a dual atom of Int P we infer that $[z_1, z_2] \neq [z_3, z_4]$. Hence $z_1 = z_4 = v$ and $z_1 \neq z_3$. Next, z_1 and z_3 must belong to the set $A \cup \{y_1\}$. Thus either z_1 or z_3 belongs to A. Clearly $z_1 < b$ and $z_3 < b$.

2.12. Lemma. *Let* (α) *be valid and let* $a \in A$. *Then there is* $b \in B$ *such that* $a < b$.

The proof is analogous to that of 2.11.

2.13. Lemma. *Let* (α) *be valid. Then* $A \neq \emptyset$ *and* $B \neq \emptyset$.

Proof. This is a consequence of 2.10, 2.11 and 2.12.

2.14. Lemma. *The condition* (α) *cannot hold.*

Proof. By way of contradiction, suppose that (α) is valid. Then we have $\{u\} < < [u, x_2]$, whence $f([u, x_2]) < f(\{u\}) = [u, x_2]$. Since $[u, x_2]$ is a dual atom of Int P, $f([u, x_2])$ must be an atom of Int P. Thus we have three possibilities:

(a) $f([u, x_2]) = \{u\}$;

(b) there is $a_1 \in A$ such that $f([u, x_2]) = \{a_1\}$;

(c) $f([u, x_2]) = \{x_2\}$.

Next, the relation

$$(6) \qquad f([u, x_2]) = f(\{u\} \vee \{x_2\}) = f(\{u\}) \wedge f(\{x_2\})$$

is valid.

First, suppose that (a) holds. Then in view of (6), $u \in f(\{x_2\})$. Because $f(\{x_2\})$ is a dual atom of Int L and since $f(\{x_2\}) \neq f(\{u\}) = [u, x_2]$, there is $b_1 \in B$ such that $f(\{x_2\}) = [u, b_1]$. Thus (6) yields

$$\{u\} = [u, x_2] \wedge [u, b_1] .$$

Hence no element of A is less than b_1, contradicting 2.11.

Next, assume that (b) is valid. In view of (6) we infer

$$(7) \qquad \{a_1\} = [u, x_2] \wedge f(\{x_2\}) .$$

Thus $a_1 \in f(\{x_2\})$. Since $f(\{x_2\})$ is a dual atom of Int L distinct from $[u, x_2]$, we have either

$$(8) \qquad f(\{x_2\}) = [a_1, v] ,$$

or there is $b_1 \in B$ with $a_1 < b_1$ such that

$$(9) \qquad f(\{x_2\}) = [u, b_1] .$$

If (8) were valid we would have

$$[u, x_2] \wedge f(\{x_2\}) = [u, x_2] \wedge [a_1, v] \supset \{x_2\} ,$$

contradicting (7). If (9) holds, then $a \in [u, x_2] \wedge f(\{x_2\})$ and in view of (7) we arrive at a contradiction.

At last let us consider the case (c). Thus, according to (6),

$$(10) \qquad \{x_2\} = [u, x_2] \wedge f(\{x_2\}) .$$

Therefore $x_2 \in f(\{x_2\}) \neq [u, x_2]$. Since $f(\{x_2\})$ is a dual atom of Int L, there exists $a_1 \in A$ such that $f(\{x_2\}) = [a_1, v]$. Then

$$[u, x_2] \wedge f(\{x_2\}) = [u, x_2] \wedge [a_1, v] \supset \{a_1\} ;$$

in view of (10) we arrive at a contradiction.

2.15. Lemma. *The condition* (β) *cannot hold.*

The proof requires steps analogous to those which were applied in 2.5.–2.14. The details are omitted.

In view of 2.14 and 2.15 the proof of (B) is complete.

The following assertion is obvious.

2.16. Lemma. *Let P be a partially ordered set having the least and the largest element, and let* card $P \leqq 4$. *Then P is a lattice.*

Theorems (A), (B) and Lemmas 2.1, 2.16 yield:

(C) Let P be a partially ordered set. Then the following conditions are equivalent:

(i) The partially ordered set Int P is selfdual.

(ii) P is a lattice such that either card $P \leqq 2$, or card $P = 4$ and P has two atoms.

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