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ON VARIETIES OF REGULAR $*$-SEMIGROUPS

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The aim of this paper is to describe all varieties of regular $*$-semigroups whose tolerance (congruence) lattices are modular, distributive or boolean, respectively.

1. PRELIMINARIES

By a regular $*$-semigroup we shall mean (see [1]) an algebra $\mathcal{S} = (S, \cdot, *)$ where $(S, \cdot)$ is a semigroup and $*$ is a unary operation on S satisfying the following

(1) $$(x^*)^* = x\,,$$

(2) $$x = xx^*x\,,$$

(3) $$(xy)^* = y^*x^*\,.$$

By $W\ (i = j)$ we denote the variety of all regular $*$-semigroups satisfying the identity $i = j$. Terminology and notation not defined here may be found in [2] and [3].

Lemma 1. $W(xx^* = yy^*) = W(xx^* = xyy^*x^*) \cap W(xx^* = x^*x)$.

Proof. It follows from (1), (2) and (3) that $W(xx^* = yy^*) \subseteq W(xx^* = (xy)(xy)^*) = W(xx^* = xyy^*x^*)$ and $W(xx^* = yy^*) \subseteq W(xx^* = x^*(x^*)^*) = W(xx^* = x^*x)$. Let $\mathcal{S} \in W(xx^* = xyy^*x^*) \cap W(xx^* = x^*x)$. According to (1), (2) and (3), in $\mathcal{S}$ we have $xx^* = xy^*yx^* = (xy^*)(xy^*)^* = (xy^*)^*(xy^*) = yx^*xy^* = yy^*$.

Lemma 2. $W(xx^* = xyx^*) = W(xx^* = xyy^*x^*) \cap W(x^2 = x)$.

Proof. It is clear that $W(xx^* = xyx^*) \subseteq W(xx^* = xyy^*x^*)$. By (2) we have $W(xx^* = xyx^*) \subseteq W(xx^* = xxx^*) \subseteq W(xx^*x = xxx^*x) = W(x = x^2)$. Let $\mathcal{S} \in W(xx^* = xyy^*x^*) \cap W(x^2 = x)$. According to (1), (2) and (3), in $\mathcal{S}$ we obtain $xx^* = x(xy^*)(yx^*)^*x^* = (xyx^*)(xy^*x^*) = (xyx^*)^2(xy^*x^*) = (xyx^*)x(yx^*xy^*)x^* = xyx^*xx^* = xyx^*$.

Lemma 3. *Let $\mathcal{S}_2 = (S_2, \cdot, *)$ be a two-element regular $*$-semigroup with the* *tables* $(S_2 = \{0, 1\})$

$\cdot$	1	0
1	1	0
0	0	0

$*$	
1	1
0	0

A variety V of regular $*$-semigroups does not contain $\mathcal{S}_2$ if and only if $V = W(xyy^*x^* = xx^*)$.

Proof. Clearly $\mathcal{S}_2 \notin W(xyy^*x^* = xx^*)$.

Suppose that $V \nsubseteq W(xyy^*x^* = xx^*)$. Then there exists a regular $*$-semigroup from V containing two elements u, v such that $uvv^*u^* \neq uu^*$. Put $a = uu^*$, $b = vv^*$. It follows from (1), (2) and (3) that $a = a^2 = a^*$, $b = b^2 = b^*$, $(ab)^* = ba$, $(ab)^2 = ab$, $(ba)^* = ab$, $(ba)^2 = ba$ and $a \neq aba$. Let $\mathcal{S} = (S, \cdot, *)$ be a regular $*$-semigroup generated by a and b. Clearly $\mathcal{S} \in V$. It is easy to show that $I = bS \cup Sb \cup SbS$ is an ideal of the semigroup $(S, \cdot)$, $S \setminus I = \{a\}$ and the Rees' factor semigroup $\mathcal{S}/I$ is isomorphic to $\mathcal{S}_2$. Therefore $\mathcal{S}_2 \in V$.

Lemma 4. *Let $\mathcal{S}_4 = (S_4, \cdot, *)$ be a four-element regular $*$-semigroup with the tables* $(S_4 = \{e, f, ef, fe\})$

$\cdot$	e	f	ef	fe
e	e	ef	ef	e
f	fe	f	f	fe
ef	e	ef	ef	e
fe	fe	f	f	fe

$*$	
e	f
f	e
ef	ef
fe	fe

A variety V of regular $*$-semigroups does not contain $\mathcal{S}_2$ and $\mathcal{S}_4$ if and only if $V = W(xx^* = yy^*)$.

Proof. Clearly $\mathcal{S}_2, \mathcal{S}_4 \notin W(xx^* = yy^*)$.

Suppose that $V \nsubseteq W(xx^* = yy^*)$. It follows from Lemma 1 that $V \nsubseteq W(xx^* = xyy^*x^*)$ or $V \nsubseteq W(xx^* = x^*x)$. If $V \nsubseteq W(xx^* = xyy^*x^*)$, then by Lemma 3 we obtain $\mathcal{S}_2 \in V$. We can assume that $V \nsubseteq W(xx^* = x^*x)$ and $V \subseteq W(xx^* = xyy^*y^*)$. Then there exists a regular $*$-semigroup $\mathcal{S} = (S, \cdot, *)$ from V generated by element a such that $aa^* \neq a^*a$. We shall show that $\{aSa, aSa^*, a^*Sa, a^*Sa^*\}$ is a decomposition of S and so $\mathcal{S}_4$ is a homomorphic image of $\mathcal{S}$, hence we have $\mathcal{S}_4 \in V$. Assume by way of contradiction that $aS \cap a^*S \neq \emptyset$. Then $au = a^*v$ for some $u, v \in S$. By (3) and (1) we have $aa^* = auu^*a^* = au(au)^* = a^*v(a^*v)^* = a^*vv^*a = a^*a$, a contradiction. Therefore $aS \cap a^*S = \emptyset$ and dually we have $Sa \cap Sa^* = \emptyset$.

2. TOLERANCE AND CONGRUENCE LATTICES

For any regular $*$-semigroup $\mathcal{S} = (S, \cdot, *)$ by $\mathcal{S}^-$ we denote the semigroup $(S, \cdot)$. Recall that a tolerance on the semigroup $\mathcal{S}^-$ is a reflexive and symmetric subsemigroup of the direct product $\mathcal{S}^- \times \mathcal{S}^-$. By $\mathrm{Tol}(\mathcal{S}^-)$ we denote the lattice of all tolerances on $\mathscr{S}^-$ with respect to set inclusion (see [4] and [5]). Denote by $\vee$ or $\wedge$ the join or meet in Tol $(\mathscr{S}^-)$, respectively. The meet evidently coincides with the set intersection. For $M \subseteq S \times S$ we denote by $T_{\mathscr{S}^-}(M)$ (or simply $T(M)$) the least tolerance on $\mathscr{S}^-$ containing M. It is easy to show the following:

(4) $(x, y) \in T(M)$ if and only if $x = x_1 x_2 \dots x_m$ and $y = y_1 y_2 \dots y_m$ where either $(x_i, y_i) \in M$ or $(y_i, x_i) \in M$ or $x_i = y_i \in S$ for $i = 1, 2, \dots, m$.

(5) $A \vee B = T(A \cup B)$ for any $A, B \in \mathrm{Tol}(\mathscr{S}^-)$.

For $M \subseteq S \times S$ by M^* we denote the set $\{(x^*, y^*);\ (x, y) \in M\}$, where $\mathscr{S} = (S, \cdot, *)$ is a regular $*$-semigroup. Using (3) we obtain $A^* \in \mathrm{Tol}(\mathscr{S}^-)$ whenever $A \in \mathrm{Tol}(\mathscr{S}^-)$. It is easy to show that for any $A, B \in \mathrm{Tol}(\mathscr{S}^-)$ we have

(6) $(A^*)^* = A$,

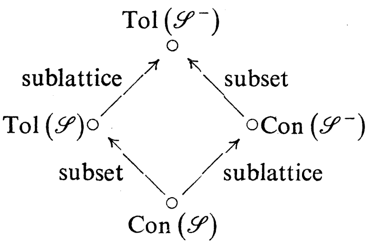
(7) $(A \wedge B)^* = A^* \wedge B^*$,

(8) $(A \vee B)^* = A^* \vee B^*$.

Evidently $A = A^*$ if and only if A is a tolerance on the regular $*$-semigroup $\mathscr{S}$. It follows from (6), (7) and (8) that $\mathrm{Tol}(\mathscr{S}) = \{A = A^*, A \in \mathrm{Tol}(\mathscr{S}^-)\}$ is a sublattice of the lattice Tol $(\mathscr{S}^-)$.

By Con $(\mathscr{S}^-)$ we denote the lattice of all congruences on a semigroup $\mathscr{S}^- = (S, \cdot)$. Clearly Con $(\mathscr{S}^-)$ is a subset of the lattice Tol $(\mathscr{S}^-)$, but it need not be a sublattice of Tol $(\mathscr{S}^-)$. For any regular $*$-semigroup $\mathscr{S} = (S, \cdot, *)$ $\mathrm{Con}(\mathscr{S}) = \mathrm{Con}(\mathscr{S}^-) \cap \mathrm{Tol}(\mathscr{S})$ is a sublattice of the lattice Con $(\mathscr{S}^-)$. Evidently Con $(\mathscr{S})$ is a lattice of all congruences on regular $*$-semigroup $\mathscr{S}$. We have the following diagram:

(9)



Theorem 1. *The following conditions for a variety V of regular $*$-semigroups are equivalent*:

1. $V \subseteq W(xx^* = yy^*)$.
2. Con $(\mathscr{S})$ = Tol $(\mathscr{S}^-)$ *for all* $\mathscr{S} \in V$.
3. Con $(\mathscr{S})$ *is a sublattice of* Tol $(\mathscr{S}^-)$ *for all* $\mathscr{S} \in V$.
4. Con $(\mathscr{S})$ *is a sublattice of* Tol $(\mathscr{S})$ *for all* $\mathscr{S} \in V$.
5. Con $(\mathscr{S}^-)$ *is a sublattice of* Tol $(\mathscr{S}^-)$ *for all* $\mathscr{S} \in V$.
6. Con $(\mathscr{S}^-)$ *is a sublattice of* Tol $(\mathscr{S})$ *for all* $\mathscr{S} \in V$.

Proof. $1 \Rightarrow 2 \Rightarrow 3, 4, 5, 6$. It is clear that $W(xx^* = yy^*)$ is the variety of all groups and it is well known that $\mathrm{Con}(\mathcal{S}) = \mathrm{Tol}(\mathcal{S}^-)$ for every group $\mathcal{S}$.

$4 \Rightarrow 3$, $5 \Rightarrow 3$ and $6 \Rightarrow 3$. Apply (9).

$3 \Rightarrow 1$. Suppose that $\mathrm{Con}(\mathcal{S})$ is a sublattice of $\mathrm{Tol}(\mathcal{S}^-)$ for every $\mathcal{S}$ from V. We shall prove that $\mathcal{S}_2, \mathcal{S}_4 \notin V$ (see Lemmas 3 and 4).

Assume by way of contradiction that $\mathcal{S}_2 \in V$. Then $\mathcal{S}_2 \times \mathcal{S}_2 \in V$ and so V contains a chain $\mathcal{C}$ of order 3. It is easy to verify that $\mathrm{Con}(\mathcal{C})$ is not sublattice of $\mathrm{Tol}(\mathcal{C}^-)$. Consequently $\mathcal{S}_2 \notin V$.

Now suppose that $\mathcal{S}_4 \in V$. Thus we have $\mathcal{S}_4 \times \mathcal{S}_4 \in V$. Put $A = \{((a, b), (a, v));\ a, b, v \in S_4\}$ and $B = \{((a, b), (u, b));\ a, b, u \in S_4\}$. It is clear that $A, B \in \mathrm{Con}(\mathcal{S}_4 \times \mathcal{S}_4)$. Let us put $Q = A \vee B$ in $\mathrm{Tol}(\mathcal{S}_4^- \times \mathcal{S}_4^-)$. By our assumption we have $Q \in \mathrm{Con}(\mathcal{S}_4 \times \mathcal{S}_4)$. Evidently $((ef, ef), (ef, fe)) \in A \subseteq Q$, $((ef, fe), (fe, fe)) \in B \subseteq Q$ and so $((ef, ef), (fe, fe)) \in Q$. According to (5) and (4) we have $((ef, ef), (fe, fe)) = \prod_{i=1}^{m} ((a_i, b_i), (u_i, v_i))$ where $((a_i, b_i), (u_i, v_i)) \in A \cup B$. Then $a_1 = u_1$ or $b_1 = v_1$. Consequently $fe \in eS_4$ or $ef \in fS_4$, which is impossible. Therefore $\mathcal{S}_4 \notin V$.

According to Lemma 4 we have $V \subseteq W(xx^* = yy^*)$.

Theorem 2. *The following conditions for a variety V of regular $*$-semigroups are equivalent*:

1. $V \subseteq W(xx^*x^*x = x^*xxx^*)$.
2. $\mathrm{Con}(\mathcal{S}) = \mathrm{Con}(\mathcal{S}^-)$ *for all* $\mathcal{S} \in V$.

Proof. $1 \Rightarrow 2$. Suppose that $V \subseteq W(xx^*x^*x = x^*xxx^*)$. Let $\mathcal{S} \in V$ and $A \in \mathrm{Con}(\mathcal{S}^-)$. First we shall show that

$$(10) \qquad (a^*, e) \in A \quad \text{whenever} \quad (a, e) \in A \quad \text{and} \quad e^2 = e.$$

Assume that $(a, e) \in A$ with $e^2 = e$. This implies $(a^2, e) \in A$ and $(a^2, a) \in A$. According to (2) and (1), we have $a^2 = (aa^*a)^2 = a(a^*aaa^*)\,a = a(aa^*a^*a)\,a$ and so $(a^2a^*a^*a^2, e) \in A$. Thus we obtain $(a^*aaa^*, e) = (aa^*a^*a, e) \in A$ and so $(a^*, e) = (a^*aa^*, e) \in A$.

Now we shall prove the following

$$(11) \qquad (a, b) \in A \quad \text{implies} \quad (a^*, b^*) \in A\,.$$

According to (2), we have $(bb^*)^2 = bb^*$ and so, by (10), (1) and (3), we obtain $(a, b) \in A \Rightarrow (ab^*, bb^*) \in A \Rightarrow (ba^*, bb^*) \in A \Rightarrow (ba^*, ab^*) \in A$. Analogously we can show that $(a, b) \in A$ implies $(a^*b, b^*a) \in A$. It follows from (2) that $(a, b) \in A \Rightarrow ((a^*b)\,a^*(ba^*), (b^*a)\,a^*(ab^*)) \in A \Rightarrow (a^*, b^*ab^*) = (a^*aa^*aa^*, b^*aa^*ab^*) \in A \Rightarrow (a^*, b^*) = (a^*, b^*bb^*) \in A$.

It follows from (11) that $A = A^*$ and so $\mathrm{Con}(\mathcal{S}^-) \subseteq \mathrm{Con}(\mathcal{S})$. According to (9), we get $\mathrm{Con}(\mathcal{S}^-) = \mathrm{Con}(\mathcal{S})$.

$2 \Rightarrow 1$. Suppose that $\mathrm{Con}(\mathcal{S}) = \mathrm{Con}(\mathcal{S}^-)$ for all $\mathcal{S} \in V$. Assume by way of contradiction that there is a regular $*$-semigroup from V such that $(aa^*)(a^*a) \neq (a^*a)(aa^*)$ for some its element a. Let us put $e = aa^*$ and $f = a^*a$. It follows from (2) and (3) that $e = e^2 = e^*$, $f = f^2 = f^*$, $ef = ef(ef)^* ef = ef(fe) ef = (ef)^2$, $fe = (fe)^2$ and $ef \neq fe$. By $\mathcal{S} = (S, \cdot, *)$ we denote the regular $*$-semigroup generated by e and f. Clearly $\mathcal{S} \in V$.

Now, we shall show that $eS \cap fS = \emptyset$. Assume by way of contradiction that $eS \cap fS \neq \emptyset$. Then we have $b = eu = fv$ for some $u, v \in S$. Hence $b = eb = fb$ and so $bb^* = ebb^* = fbb^*$. By (3) and (1) we have $bb^* = (bb^*)^* = bb^*e = bb^*f$. Therefore $bb^* = cbb^* = bb^*c$ for every $c \in S$. Then we have $ef = (ef)^n \in Sbb^*S = \{bb^*\}$ for some positive integer n and so $ef = bb^* = (bb^*)^* = fe$, a contradiction.

We have $eS \cap fS = \emptyset$. Let us put $(u, v) \in A$ if and only if either $u, v \in eS$ or $u, v \in fS$. It is clear that $A \in \mathrm{Con}(\mathcal{S}^-)$. By our assumption we have $A \in \mathrm{Con}(\mathcal{S})$ and so $(e, ef) \in A$ implies $(e, fe) = (e^*, (ef)^*) \in A$, which is a contradiction.

Hence we get $V \subseteq W(xx^*x^*x = x^*xxx^*)$.

Theorem 3. *The following conditions for a variety V of regular $*$-semigroups are equivalent:*

1. $V \subseteq W(xx^* = yy^*)$ *or* $V \subseteq W(x^* = x^n)$ *for some positive integer n.*
2. $\mathrm{Tol}(\mathcal{S}) = \mathrm{Tol}(\mathcal{S}^-)$ *for all* $\mathcal{S} \in V$.

Proof. $1 \Rightarrow 2$. Apply Theorem 1.

$2 \Rightarrow 1$. Suppose that $\mathrm{Tol}(\mathcal{S}) = \mathrm{Tol}(\mathcal{S}^-)$ for all $\mathcal{S} \in V$. Then clearly $\mathrm{Con}(\mathcal{S}) = \mathrm{Con}(\mathcal{S}^-)$ for all $\mathcal{S} \in V$. According to Theorem 2, we have

(12) $$V \subseteq W(xx^*x^*x = x^*xxx^*).$$

Assume by way of contradiction that $V \nsubseteq W(xx^* = yy^*)$ and $V \nsubseteq W(x^* = x^n)$ for all positive integer n. It follows from Lemma 4 that either $\mathcal{S}_2 \in V$ or $\mathcal{S}_4 \in V$. Clearly $\mathcal{S}_4 \notin W(xx^*x^*x = x^*xxx^*)$ and so, by (12), we have $\mathcal{S}_2 \in V$. Therefore $\mathcal{S}_2 \times \mathcal{S}_2 \in V$ and so $\mathcal{S}_3 \in V$, where $\mathcal{S}_3 = (S_3, \cdot, *)$ is a three-element regular $*$-semigroup with the tables ($S_3 = \{e, f, 0\}$)

$\cdot$	e	f	0
e	e	0	0
f	0	f	0
0	0	0	0

$*$	
e	e
f	f
0	0

For any positive integer n there exists a regular $*$-semigroup $\mathcal{P}_n = (P_n, \cdot, *)$ such that $\mathcal{P}_n \in V$ and $\mathcal{P}_n \notin W(x^* = x^n)$. Therefore $a_n^* \neq a_n^n$ for some element $a_n \in P_n$. It is easy to show that the direct product $\mathcal{P} = \bigtimes_{n=1}^{\infty} \mathcal{P}_n$ belongs to V and $a^* \neq a^n$ for all positive integer n, where $a = (a_1, a_2, \ldots, a_n, \ldots)$. Let $A = T((a, e), (a^*, f))$ be the tolerance on $\mathcal{P}^- \times \mathcal{S}_3^-$ generated by $((a, e), (a^*, f))$. Evidently $\mathcal{P} \times \mathcal{S}_3 \in V$ and so by our assumption we have $A \in \mathrm{Tol}(\mathcal{P}^- \times \mathcal{S}_3^-) = \mathrm{Tol}(\mathcal{P} \times \mathcal{S}_3)$. Hence $A^* = A$ and (1), (4) imply $((a^*, e), (a, f)) = ((a, e), (a^*, f))^* = ((a, e), (a^*, f))^m$ for some positive integer m. Consequently $a^* = a^m$, which is a contradiction.

Theorem 4. *The following conditions for a variety V of regular $*$-semigroups are equivalent*:

1. $V \subseteq W(xyy^*x^* = xx^*)$.
2. *The lattice* $\mathrm{Tol}(\mathcal{S}^-)$ *is modular for all* $\mathcal{S} \in V$.
3. *The lattice* $\mathrm{Tol}(\mathcal{S})$ *is modular for all* $\mathcal{S} \in v$.

Before the proof we formulate two lemmas. Recall that an idempotent e of a regular $*$-semigroup $\mathcal{S}$ is said to be a *projection* if $e^* = e$. It follows from (1), (2) and (3) that xx^* is a projection for every element x of $\mathcal{S}$.

Lemma 5. *Let $\mathcal{S} \in W(xyy^*x^* = xx^*)$. Then for every element x of $\mathcal{S}$ and every projection e of $\mathcal{S}$ we have*

$$xex^* = xx^* .$$

Lemma 6. *Let $\mathcal{S} \in W(xyy^*x^* = xx^*)$ and $A, B \in \mathrm{Tol}(\mathcal{S}^-)$. Then for every projection e of $\mathcal{S}$ we have*

(i) $AB = A(e, e) B$,

(ii) $(e, e) A(e, e) = (e, e) A^*(e, e)$,

(iii) $(e, e) AB(e, e) = (e, e) BA(e, e)$.

Proof. (i) Assume that $(a, c) \in A$ and $(b, d) \in B$. Then by (1), (2) and Lemma 5 we have $(a, c)(b, d) = (a, c)(bb^*c^*c, bb^*c^*c)(e, e)(c^*c, c^*c)(b, d) \in A(e, e) B$. Therefore $AB \subseteq A(e, e) B \subseteq AB$.

(ii) and (iii). First we shall show the following

$$(13) \qquad (e, e) AB(e, e) = (e, e) B^*A^*(e, e) .$$

Suppose that $(a, c) \in A$ and $(b, d) \in B$. According to (1), (2) and Lemma 5, we obtain $(e, e)(a, c)(b, d)(e, e) = (e, e)(ecde, ecde)(d^*, b^*)(c^*, a^*)(eabe, eabe) . . (e, e) \in (e, e) B^*A^*(e, e)$. Thus we have $(e, e) AB(e, e) \subseteq (e, e) B^*A^*(e, e)$. Analogously we can show that $(e, e) B^*A^*(e, e) \subseteq (e, e) AB(e, e)$.

If we put $B = \mathrm{id} = B^*$ then (13) yields $(e, e) A(e, e) \subseteq (e, e) AB(e, e) = = (e, e) B^*A^*(e, e) \subseteq (e, e) A^*(e, e)$. Analogously we can get $(e, e) A^*(e, e) \subseteq \subseteq (e, e) A(e, e)$.

Finally, using (13) and (i) and (ii) of Lemma 6 we have $(e, e) AB(e, e) = (e, e) . . B^*(e, e) A^*(e, e) = (e, e) B(e, e) A(e, e) = (e, e) BA(e, e)$.

Proof of Theorem 4. $1 \Rightarrow 2$. Suppose that $\mathcal{S} \in W(xyy^*x^* = xx^*)$, $A, B, C \in \in \mathrm{Tol}(\mathcal{S}^-)$ and $A \subseteq C$.

First, we shall show that

$$(14) \qquad ABAB \subseteq AB .$$

Indeed, by Lemma 6, we have $ABAB = A(e, e) BA(e, e) B = A(e, e) AB(e, e) B \subseteq \subseteq AB$.

Now, we shall prove the following inclusions:

(15) $AB \cap C \subseteq A(B \cap C)$,

(16) $BA \cap C \subseteq (B \cap C) A$,

(17) $ABA \cap C \subseteq A(B \cap C) A$ and

(18) $BAB \cap C \subseteq (B \cap C) A(B \cap C)$.

Inclusion (15). Let $(x, y) \in AB \cap C$. Then by Lemma 6, we have $(x, y) = (a, c)(eb, ed)$, where $(a, c) \in A$, $(eb, ed) \in B$ and e is a projection of $\mathcal{S}$. It follows from (1), (2), Lemma 5 and Lemma 6 that $(eb, ed) = (ea^*e, ec^*e)(x, y) \in (e, e) \cdot A^*(e, e) C = (e, e) A(e, e) C \subseteq C$.

Inclusion (16), This is dual to (15).

Inclusion (17). Let $(x, y) \in ABA \cap C$. According to Lemma 6, we obtain $(x, y) = (ue, ve)(a, c)$, where $(ue, ve) \in AB$, $(a, c) \in A$ and e is a projection of $\mathcal{S}$. It follows from (1), (2), Lemma 5 and Lemma 6 that $(ue, ve) = (x, y)(ea^*e, ec^*e) \in C(e, e) \cdot A^*(e, e) = C(e, e) A(e, e) \subseteq C$. From (15) we have $(ue, ve) \in A(B \cap C)$ and so $(x, y) \in A(B \cap C) A$.

Inclusion (18). Let $(x, y) \in BAB \cap C$. Then, by Lemma 6, we have $(xx^*e, yy^*e) \in CC^*(e, e) = C(e, e) C^*(e, e) = C(e, e) C(e, e) \subseteq C$. Further we obtain $(x, y) \in (b, d) AB$, where $(b, d) \in B$ and so, by (3), Lemma 5 and Lemma 6, we get $(xx^*e, yy^*e) = (bb^*e, dd^*e) \in BB^*(e, e) \subseteq B$. According to Lemma 5, (1) and (14), we have $(x, y) = (xx^*e, yy^*e)(e, e)(x, y) \in (xx^*e, yy^*e) ABAB \subseteq (xx^*e, yy^*e) AB$. Consequently $(x, y) = (xx^*e, yy^*e)(eu, ev)$, where $(eu, ev) \in AB$. It follows from Lemma 5 that $(eu, ev) = (ex, ey) \in C$ and so, by (15), we get $(eu, ev) \in A(B \cap C)$. Therefore $(x, y) = (xx^*e, yy^*e)(eu, ev) \in (B \cap C) A(B \cap C)$.

Finally, it follows from (4), (5), (14), (15), (16), (17) and (18) that $(A \vee B) \wedge C = (A \cup B \cup AB \cup BA \cup ABA \cup BAB) \cap C \subseteq A \cup (B \cap C) \cup A(B \cap C) \cup (B \cap C) A \cup A(B \cap C) A \cup (B \cap C) A(B \cap C) = A \vee (B \wedge C) \subseteq (A \vee B) \wedge C$.

Therefore the lattice Tol $(\mathcal{S}^-)$ is modular.

$2 \Rightarrow 3$. This follows from (9).

$3 \Rightarrow 1$. Suppose that Tol $(\mathcal{S})$ is modular for all $\mathcal{S} \in V$. We shall show that $\mathcal{S}_2 \notin V$ (see Lemma 3). It is easy to show that Tol $(\mathcal{S}_2 \times \mathcal{S}_2)$ is not modular (see Corollary 1.1 of [6]). Consequently $\mathcal{S}_2 \notin V$ and so, by Lemma 3, we have $V \subseteq W(xx^* = xyy^*x^*)$.

Theorem 5. *The following conditions for a variety V of regular $*$-semigroups are equivalent*:

1. $V \subseteq W(xx^* = yy^*)$.
2. *The lattice* Con $(\mathcal{S}^-)$ *is modular for all* $\mathcal{S} \in V$.
3. *The lattice* Con $(\mathcal{S})$ *is modular for all* $\mathcal{S} \in V$.

Proof. $1 \Rightarrow 2$. It is well known.

$2 \Rightarrow 3$. This follows from (9).

$3 \Rightarrow 1$. Suppose that $\mathrm{Con}(\mathscr{S})$ is modular for all $\mathscr{S} \in V$. We shall show that $\mathscr{S}_2, \mathscr{S}_4 \notin V$ (see Lemmas 3 and 4). It is easy to show that $\mathrm{Con}(\mathscr{S}_2 \times \mathscr{S}_2)$ is not modular (see Theorem 6 of [7]). Consequently $\mathscr{S}_2 \notin V$.

Now, we shall prove that $\mathrm{Con}(\mathscr{S}_4 \times \mathscr{S}_4)$ is not modular. By A we denote the congruence on $\mathscr{S}_4 \times \mathscr{S}_4$ which is associated with the following partition of $S_4 \times S_4$

$$\{(e, fe), (ef, fe), (fe, fe), (f, fe)\},$$
$$\{(e, e), (ef, e)\}, \{(fe, e), (f, e)\},$$
$$\{(e, f), (fe, f)\}, \{(ef, f), (f, f)\},$$
$$\{(e, ef)\}, \{(ef, ef)\}, \{(f, ef)\}, \{(fe, ef)\}.$$

Let us put $B = \{((a, b), (a, c));\ a, b, c \in S_4\}$ and $C = \{((a, b), (c, b));\ a, b, c \in S_4\}$. It is clear that $A, B, C \in \mathrm{Con}(\mathscr{S}_4 \times \mathscr{S}_4)$, $A \subseteq C$ and $B \wedge C = \mathrm{id}$. We have $((e, e), (f, e)) \notin A = A \vee (B \wedge C)$ and $((e, e), (f, e)) \in C$. Evidently

$$((e, e), (e, f)) \in B,$$
$$((e, f), (fe, f)) \in A,$$
$$((fe, f), (fe, e)) \in B,$$
$$((fe, e), (f, e)) \in A$$

and so $((e, e), (f, e)) \in A \vee B$. Therefore $((e, e), (f, e)) \in (A \vee B) \wedge C$. We have $A \vee (B \wedge C) \neq (A \vee B) \wedge C$ and so $\mathrm{Con}(\mathscr{S}_4 \times \mathscr{S}_4)$ is not modular. Consequently $\mathscr{S}_4 \notin V$.

It follows from Lemma 4 that $V \subseteq W(xx^* = yy^*)$.

4. DISTRIBUTIVITY

Theorem 6. *The following conditions for a variety V of regular $*$-semigroups are equivalent*:

1. $V \subseteq W(xyx^* = xx^*)$.
2. *The lattice* $\mathrm{Tol}(\mathscr{S}^-)$ *is distibutive for all* $\mathscr{S} \in V$.
3. *The lattice* $\mathrm{Tol}(\mathscr{S})$ *is distributive for all* $\mathscr{S} \in V$.
4. *The lattice* $\mathrm{Tol}(\mathscr{S}^-)$ *is boolean for all* $\mathscr{S} \in V$.
5. *The lattice* $\mathrm{Tol}(\mathscr{S})$ *is boolean for all* $\mathscr{S} \in V$.

Before the proof we formulate two lemmas.

Lemma 7. *Let $\mathscr{S} \in W(xyx^* = xx^*)$. Then for all elements u, v, w of $\mathscr{S}$ and every projection e of $\mathscr{S}$ we have*

(i) $u = ueu$,

(ii) $uvw = uew$.

Proof. Suppose that $\mathscr{S} \in W(xyx^* = xx^*)$. Then we have $eye = e$ for every element y of $\mathscr{S}$ and for projection e of $\mathscr{S}$.

(i) It follows from (1), (2) and Lemma 5 that $ueu = ueu^*eu = uu^*u = u$.

(ii) We have $uvw = ueuvwew = uew$.

Lemma 8. *Let $\mathcal{S} \in W(xyx^* = xx^*)$ and $A, B, C \in \mathrm{Tol}(\mathcal{S}^-)$. Then we have*

(i) $ABC = AC$,

(ii) $AB \cap C = (A \cap C)(B \cap C)$.

Proof. (i) According to Lemma 6 and Lemma 7 we have $ABC = A(e, e) B(e, e) C = A(e, e) C = AC$ for some projection e of $\mathcal{S}$.

(ii) Assume that $(u, v) \in AB \cap C$. Then by Lemma 6 we obtain $(u, v) = (a, c) . (e, e)(b, d)$ where $(a, c) \in A$ and $(b, d) \in B$. We have $(ae, ce) = (aebe, cede) = (ue, ve) \in A \cap C$ and analogously $(eb, ed) = (eu, ev) \in B \cap C$. Therefore $(u, v) = (ae, ce)(eb, ed) \in (A \cap C)(B \cap C)$. Consequently $AB \cap C \subseteq (A \cap C)(B \cap C) \subseteq AB \cap C$.

Proof of Theorem 6. 1 $\Rightarrow$ 4. Suppose that $\mathcal{S} \in W(xyx^* = xx^*)$. First, we shall show that the lattice $\mathrm{Tol}(\mathcal{S}^-)$ is distributive.

Let $A, B, C \in \mathrm{Tol}(\mathcal{S}^-)$. According to Lemma 8 and (5) we get $(A \vee B) \wedge C = (A \cup B \cup AB \cup BA) \cap C = (A \cap C) \cup (B \cap C) \cup (A \cap C)(B \cap C) \cup (B \cap C)(A \cap C) = (A \wedge C) \vee (B \wedge C)$.

Now we shal show that $\mathrm{Tol}(\mathcal{S}^-)$ is boolean. Let $A \in \mathrm{Tol}(\mathcal{S}^-)$. Choose a projection e of $\mathcal{S} = (S, \cdot, *)$ and put $B = T((Se \times Se) \cup (eS \times eS) \setminus A)$.

Let $u, v \in S$. It follows from Lemma 7 that $(u, v) = (ue, ve)(eu, ev)$. Clearly $(ue, ve), (eu, ev) \in A \cup B$. According to (4) and (5) we get $(u, v) \in A \vee B$. Therefore $A \vee B = S \times S$.

Suppose that $A \wedge B \neq \mathrm{id}$. Then there exist $u, v \in S$ such that $(u, v) \in A \cap B$ and $u \neq v$. By (4), (5) and Lemma 7 we get $(u, v) = (a, c)(e, e)(b, d)$, where either $a = c$ or $(a, c) \in (Se \times Se) \cup (eS \times eS) \setminus A$ and either $b = d$ or $(b, d) \in (Se \times Se) \cup (eS \times eS) \setminus A$. If $(a, c) \in (Se \times Se) \setminus A$, then by our assumption we obtain $(a, c) = (ae, ce) = (aeb, ced)(e, e) = (u, v)(e, e) \in A$, which is a contradiction. Therefore $(a, c) \notin (Se \times Se) \setminus A$. Dually we obtain that $(b, d) \notin (eS \times eS) \setminus A$. Consequently we have the following possibilities:

Case 1. $a = c$. Then $b \neq d$ and so $(b, d) \in (Se \times Se) \setminus A$. Hence by our assumption we have $(u, v) = (aebe, aede) = (ae, ae)$, a contradiction.

Case 2. $b = d$. Then dually we obtain a contradiction.

Case 3. $a \neq c$ and $b \neq d$. Then $(a, c) \in eS \times eS$ and $(b, d) \in Se \times Se$. According to our assumption we get $u = aeb = e = ced = v$, a contradiction.

Therefore $A \wedge B = \mathrm{id}$. Consequently, the lattice $\mathrm{Tol}(\mathcal{S}^-)$ is boolean.

4 $\Rightarrow$ 2 and 5 $\Rightarrow$ 3. Trivially.

2 $\Rightarrow$ 3. This follows from (9).

4 $\Rightarrow$ 5. According to (9), (7) and (8), $\mathrm{Tol}(\mathcal{S})$ is a boolean subalgebra of $\mathrm{Tol}(\mathcal{S}^-)$ for every $\mathcal{S} \in V$.

3 $\Rightarrow$ 1. Suppose that $\mathrm{Tol}(\mathcal{S})$ is distributive for all $\mathcal{S} \in V$. It follows from Theorem 4 that

$$V \subseteq W(xyy^*x^* = xx^*) . \tag{19}$$

First we shall show that

$$(20) \qquad V \cap W(xx^* = x^*x) \subseteq W(x = xx^*) .$$

Assume by way of contradiction that there is a regular $*$-semigroup from V such that $aa^* = a^*a$, $a \neq aa^*$ for some its element a. Therefore V contains a non-trivial group and so $\mathscr{R} \in V$, where $\mathscr{R}$ is a finite cyclic group of a prime order. Clearly $\mathscr{R} \times \mathscr{R} \in V$ and so, by Theorem 1, the lattice $\text{Tol}(\mathscr{R} \times \mathscr{R}) = \text{Con}(\mathscr{R} \times \mathscr{R})$ is distributive. By Ore's Theorem [8] the group $\mathscr{R} \times \mathscr{R}$ is locally cyclic. Since $\mathscr{R} \times \mathscr{R}$ is finite, we obtain that $\mathscr{R} \times \mathscr{R}$ is cyclic, which is a contradiction.

Now we shall prove that

$$(21) \qquad V \subseteq W(x = x^2) .$$

Assume by way of contradiction that there is a regular $*$-semigroup $\mathscr{S}$ from V containing non-idempotent element a. Let us put $b = a^2a^*$. According to (19), (1), (2) and (3), we have $bb^* = a^2a^*a(a^*)^2 = a(aa^*)a^* = aa^* = (aa^*)(aa^*) = (aa^*)a^*a(aa^*)^* = a(a^*)^2 a^2a^* = b^*b$. It follows from (20) that $b = bb^*$. This and (19), (1), (2) and (3) imply $a^2a^* = a^2a^*(a^2a^*)^* = a^2a^*a(a^*)^2 = a^2(a^*)^2$ and so $a^2 = aa^*(a^2a^*)a = aa^*(a^2a^*a^*)a = aa^*a = a$.

From (19), (21) and Lemma 2 it follows that $V \subseteq W(xyx^* = xx^*)$.

Theorem 7. *The following conditions for a variety V of regular $*$-semigroups are equivalent*:

1. *V is trivial.*
2. *The lattice* $\text{Con}(\mathscr{S}^-)$ *is distributive for all* $\mathscr{S} \in V$.
3. *The lattice* $\text{Con}(\mathscr{S})$ *is distributive for all* $\mathscr{S} \in V$.

Proof. $1 \Rightarrow 2$. Trivially.

$2 \Rightarrow 3$. This follows from (9).

$3 \Rightarrow 1$. Suppose that $\text{Con}(\mathscr{S})$ is distributive for all $\mathscr{S} \in V$. It follows from Theorem 5 that $V \subseteq W(xx^* = yy^*)$. Theorem 1 and Theorem 6 imply $V \subseteq W(xyx^* = xx^*)$. According to (2) and (1), we get $x = xx^*x = xxx^* = xx^* = yy^* = yyy^* = yy^*y = y$.

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