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CONVEX DIRECTED SUBGROUPS OF RIGHT ORDERED TREE GROUPS

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A partially ordered set $(T, \leqq)$ is called a *tree* if

1. $\forall a,\ b \in T\ \exists c \in T;\ a, b \leqq c;$
2. $\exists a,\ b \in T;\ a \parallel b;$
3. $\forall a, x, y \in T;\ a \leqq x,\ y \Rightarrow x \leqq y$ or $y \leqq x$.

By a right partially ordered group we mean such a system $G = (G, \cdot, \leqq)$, where $(G, \cdot)$ is a group, $(G, \leqq)$ is a partially ordered set, and $a \leqq b$ implies $ac \leqq bc$ for all $a, b, c \in G$. As usual, $P(G) = \{x \in G;\ e \leqq x\}$ will denote the set of all positive elements of G.

If G is a right partially ordered group such that $(G, \leqq)$ is a tree, then G is called a *tr-group*. A *strong tr-group* (str-group) is any tr-group G such that $a \leqq b$ implies $ca \leqq cb$ for all $a, b \in G$ and $c \in P(G)$. A right partially ordered group G is called a *right o-group* (ro-group), if $(G, \leqq)$ is a linearly ordered set.

Remark. It is evident that a right partially ud-ordered group G is a tr-group if and only if there exist two non-comparable elements in G, and $P(G)$ is a chain.

Right o-group are studied e.g. in Kopytov's book [3], where one can find all necessary results from the theory of partially ordered groups.

In 1903, Frege (in the book [2]) asked a question which may be translated into modern terms as the problem whether there exists a tr-group not being an ro-group. In 1987, Adeleke, Dummett and Neumann (in the paper [1]) answered this question in the affirmative by giving a tr-order on a free group of rank 2 which is not an ro-order. Further, in [4], Varaksin proved that every free n-solvable group of rank $\geqq 2$, for any $n \geqq 2$, admits such a right partial order that the system obtained is a tr-group but not an ro-group. Moreover, right partial orders obtained in both papers are str-orders.

In this paper some structure properties of tr-groups and str-groups are studied.

Proposition 1. *Let A, B, C be partially ordered sets such that $C = A \times^{\rightarrow} B$ (i.e., C is a lexicographic product of A and B) and let A be a tree and B a linearly ordered set. Then C is a tree.*

Proof. 1. Let $(a_1, b_1), (a_2, b_2) \in C$, $(a_1, b_1) \parallel (a_2, b_2)$.

a) If $a_1 \parallel a_2$, then there exists $a_3 \in A$ with $a_1, a_2 < a_3$. It is clear that for each $b \in B$ we have $(a_1, b_1), (a_2, b_2) < (a_3, b)$.

b) If $a_1 = a_2$, then $b_1 \parallel b_2$, a contradiction.

2. Because there exist $a_1, a_2 \in A$ with $a_1 \parallel a_2$, we have $(a_1, b) \parallel (a_2, b)$ for each $b \in B$.

3. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in C$, $(a_1, b_1) < (a_2, b_2), (a_3, b_3)$.

a) Let $a_1 < a_2, a_3$. Then $a_2 \leqq a_3$ or $a_3 \leqq a_2$. In the case $a_2 < a_3$, we have $(a_2, b_2) < (a_3, b_3)$. Similarly for $a_3 < a_2$. Let $a_2 = a_3$. Then the linearity of B implies $(a_2, b_2) \leqq (a_3, b_3)$ or $(a_3, b_3) \leqq (a_2, b_2)$.

b) For $a_1 = a_3 < a_2$, we have $(a_3, b_3) < (a_2, b_2)$.

c) If $a_1 = a_2 = a_3$, then $b_1 < b_2, b_3$, and the assertion follows from the linearity of B. □

Proposition 2. *Let A, B be partially ordered sets. If $A \times^{\rightarrow} B$ is a tree, then either $|A| = 1$ and B is a tree or A is a tree and B is linearly ordered.*

Proof. Let a_1, a_2 be distinct elements of A. Since $A \times^{\rightarrow} B$ is a tree, there exist $a_3 \in A$, $b' \in B$ such that $(a_1, b) \leqq (a_3, b')$ and $(a_2, b) \leqq (a_3, b')$. Then $a_1, a_2 \leqq a_3$, so A satisfies the first of the axioms for a tree.

This implies that there exist $a_1, a_2 \in A$ with $a_1 < a_2$. If $b_0 \in B$, then $(a_1, b_0) < < (a_2, b)$ for all $b \in B$. Thus, by the third axiom for a tree, $\{a_2\} \times B$ is linearly ordered, and so the ordering of B is linear.

Now, let $(a_1, b_1) \parallel (a_2, b_2)$. Then we can have none of $a_1 < a_2$, $a_2 < a_1$, $a_1 = a_2$, and so $a_1 \parallel a_2$. Thus A satisfies the second axiom.

Finally, if $a_1, a_2, a_3 \in A$ and $a_1 \leqq a_2, a_3$, then, for any $b \in B$, we have $(a_1, b) \leqq \leqq (a_2, b), (a_3, b)$. Thus $(a_2. b) \leqq (a_3, b)$ or $(a_3, b) \leqq (a_2, b)$, and so $a_2 \leqq a_3$ or $a_3 \leqq a_2$.

Hence A is a tree. □

Let $G = (G, \cdot, \leqq)$ be a right partially ordered group, N a normal convex subgroup of G. We can define a partial order "$\leqq$" on G/N as:

$$\forall x, y \in G ; \quad Nx \leqq Ny \Leftrightarrow_{\text{df}} \exists a \in N ; \quad x \leqq ay .$$

Let us verify that the relation "$\leqq$" is a partial order on G/N. The reflexivity is evident. Further, let $x, y \in G$ and let $Nx \leqq Ny$, $Ny \leqq Nx$, i.e. there exist $a_1, a_2 \in N$ such that $x \leqq a_1 y$, $y \leqq a_2 x$. We have $a_2 x = x a_3$, where $a_3 \in N$, hence $y a_3^{-1} \leqq x$. From this we obtain $y a_3^{-1} \leqq x \leqq a_1 y$, hence $a_4 y \leqq x \leqq a_1 y$, where $a_4 \in N$. Therefore $a_4 \leqq \leqq x y^{-1} \leqq a_1$, and since N is convex, $x y^{-1} \in N$, and so $Nx = Ny$. Hence, "$\leqq$" is antisymmetric. To prove the transitivity suppose that $x, y, z \in G$ abd $Nx \leqq Ny$, $Ny \leqq Nz$. Then there exist $a_1, a_2 \in N$ such that $x \leqq a_1 y$, $y \leqq a_2 z$. Let $a_1 y = y a_3$, where $a_3 \in N$. Then $x a_3^{-1} \leqq y$ and $y \leqq a_2 z = z a_4$, where $a_4 \in N$. Hence $x \leqq z a_4 a_3$, and so $Nx \leqq Nz$.

Now, it is evident that G/N with the partial order "$\leqq$" is a right partially ordered group.

If for each $g \in G$, $Ng > N$ implies $ag > e$ for all $a \in N$, then G is called a *lex-extension* of the right partially ordered group N by means of the right partially ordered group $\bar{G} = G/N$.

Since the lex-extension G of a right partially ordered group N by means of $\bar{G}$ is (as a partially ordered set) isomorphic to the lexicographic product of the partially ordered sets $\bar{G}$ and N, the following theorem is true.

Theorem 3. *If G is a right partially ordered group which is the lex-extension of a right partially ordered group N by means of a right partially ordered group $\bar{G}$, then G is a tr-group if and only if N is an ro-group and $\bar{G}$ is a tr-group.* □

A subgroup H of a right partially ordered group G is called a *ud-subgroup* of G, if H is up-directed (i.e. if $\forall a, b \in H\ \exists c \in H;\ a, b \leqq c$). Note that, contrary to (two-sided) partially ordered groups, a ud-subgroup need not be down-directed. A convex ud-subgroup of G will be called a *cud-subgroup* of G.

Lemma 4. *Let H be a subgroup of a tr-group G and let there exist $g \in G$ such that $ag > e$ for each $a \in H$. Then H is an ro-subgroup of G.*

Proof. If $ag > e$ for each $a \in H$, then $a > g^{-1}$ for each $a \in H$, and this means H is a chain. □

Lemma 5. *Let H be a normal ud-subgroup of a tr-group G, let $g \in G$ and let $g > b$ for each $b \in P(H)$. Then $ag > e$ for each $a \in H$.*

Proof. Let $g > b$ for each $b \in P(H)$. Since H is a ud-subgroup, for any $a \in H$ there exists $b \in P(H)$ such that $a \leqq b$. Hence $g > a$ for each $a \in H$. But this means $ga > e$ for each $a \in H$. From the normality of H we obtain $ag > e$ for each $a \in H$. □

Theorem 6. *Let H be a normal cud-subgroup of a tr-group G, and let there exist $g \in G$, $g < e$, such that $g \notin P(H)^{-1}$. Then H is an ro-subgroup of G.*

Proof. Let $g < e$, $g \notin P(H)^{-1}$. Then $g^{-1} > e$, and since H is convex, $g^{-1} > b$ for each $b \in P(H)$. By Lemmas 4 and 5, we obtain that H is an ro-subgroup of G. □

In [1] it is shown that every tr-group G is generated by its subset of positive elements $P(G)$. Moreover, $G = P(G)^{-1} . P(G)$. Because $P(G)$ is a chain, the set of all normal cud-subgroups of G is linearly ordered by inclusion. And, since every of these subgroups is an ro-subgroup, all subgroups belong to just one chain in G.

Corollary 1. *Every tr-group contains a greatest proper normal cud-subgroup* (which is an ro-group).

Proof. Let us denote by H the union of all proper normal cud-subgroups of G. It is evident that H is a convex ro-subgroup of G, hence $H \neq G$. □

Proposition 7. *If an ro-group G is an str-group, then G is a linearly ordered group* (o-group).

Proof. Let $a, b \in G$, $a \leqq b$, $x \in P(G)^{-1}$. Let $xa > xb$. Since $x^{-1} \in P(G)$, we have $x^{-1}xa > x^{-1}xb$, a contradiction. Therefore $xa \leqq xb$. □

As a consequence we obtain the following theorem.

Theorem 8. *If G is an str-group and H is a normal cud-subgroup of G, and if there exists $g \in G$, $g < e$, such that $g \notin P(H)^{-1}$, then H is an o-subgroup of G.* □

Corollary 2. *The greatest proper normal cud-subgroup of every str-group is an o-subgroup.* □

Theorem 9. *Let G be a tr-group, N a normal cud-subgroup of G. Then G is the lex-extension of N by means of G/N.*

Proof. Let $x \in G$, $xN > N$. Then there exists $c \in N$ such that $xc > e$, i.e. $x > c^{-1}$. From the u-directedness of N we obtain the existence of $b \in P(N)$ such that $c^{-1} \leqq b$. Since x and b are comparable, we have $x < b$ or $b < x$. In the first case, $x \in N$, a contradiction. Hence $b < x$, and since N is convex, $x > a$ for each $a \in N$. □

Let G be a group. A system $S(G)$ of subgroups of G which is linearly ordered by inclusion is called *full*, if e, $G \in S(G)$, and if $S(G)$ contains the union and the intersection of every set of subgroups of $S(G)$. A jump $A \prec B$ in a full system $S(G)$ is any pair $A, B \in S(G)$ such that $A \subset B$ and $A \subseteq C \subseteq B$ imply $A = C$ or $B = C$ for each $C \in S(G)$. If $g \in G$, $g \neq e$, then g defines a jump $A \prec B$, where A is the union of all subgroups of $S(G)$ not containing g and B is the intersection of all subgroups of $S(G)$ containing g.

A system $S(G)$ is called *subnormal*, if for each $g \in G$, $g \neq e$, in the jump $A \prec B$ defined by g, A is a normal subgroup of B. A system $S(G)$ is called *normal*, if all subgroups from $S(G)$ are normal in G. A subnormal system $S(G)$ is called *solvable*, if the factor group B/A is abelian for every jump $A \prec B$.

Let now G be a tr-group. We will denote the system of all normal cud-subgroups of G by $\bar{C}(G)$. By Theorem 6 it is clear that $\bar{C}(G)$ is a full system of subgroups of G.

Theorem 10. *If G is a tr-group such that the normal system $\bar{C}(G)$ is solvable, then G is an ro-group.*

Proof. Let H be the greatest proper normal cud-subgroup of G. By the assumption, G/H is abelian, and by Theorem 9, G is the lex-extension of H by means of G/H. This means, by Theorem 3, that G/H is a tr-group. But G/H is abelian and so it is an o-group. So we have that G is the lex-extension of an ro-group by means of an o-group, hence G is an ro-group. □

Corollary 3. *If the assumptions of Theorem* 10 *are satisfied, then the convex subgroups of G form a full system of subgroups of G.* □

Theorem 11. *If G is an str-group such that the system $\bar{C}(G)$ is solvable, then G is an o-group.* □

Note. V. M. Kopytov has informed the author that N. L. Petrova showed that any tr-group is a torsion-free group. But this fact is not proved directly, and her proof uses a representation of a tr-group in terms of automorphisms of the group.

Here we will show that this proposition can be proved directly from the definition of a tr-group. Namely, let G be a tr-group and $x \in G$. Suppose that x has finite order n. Since $\langle x \rangle$ is finite, there exists $y \in G$ such that $y \geqq x^i$ for all i and, multiplying on the right by suitable powers of x, we have $yx^i \geqq e$ for all i. Therefore $\{y, yx, \ldots, yx^{n-1}\}$ is linearly ordered. The map $yx^i \mapsto yx^{i+1}$ is an order automorphism and therefore it must be trivial. Thus $x = e$ and G is torsion-free.

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