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TERNARY STRUCTURES AND GROUPOIDS

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1. INTRODUCTION

We prove that the category of ternary structures with strong homomorphisms as morphisms is isomorphic with a particular category of groupoids. Objects of the latter category are groupoids whose carriers are power sets and operations are totally additive. Morphisms of this category are totally additive and atom-preserving homomorphisms of those groupoids. By means of the isomorphism of these categories, the problem of constructing all strong homomorphisms between two ternary structures is reduced to the problem of constructing all totally additive and atom-preserving homomorphisms between two groupoids. The properties of a ternary structure influence the properties of the groupoid corresponding to the structure with respect to the above mentioned isomorphism and vice versa. The relationship between properties of a ternary structure and properties of the corresponding groupoid is also studied here.

2. TOTALLY ADDITIVE AND ATOM-PRESERVING MAPPINGS

For any set A we denote by $\boldsymbol{P}(A)$ its power set, i.e., $\boldsymbol{P}(A) = \{X; X \subseteq A\}$.

Let A, A' be sets.

A mapping H of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$ is said to be *totally additive* if

$$H(\bigcup \{X_\iota;\ \iota \in I\}) = \bigcup \{H(X_\iota);\ \iota \in I\}$$

for any system $\{X_\iota;\ \iota \in I\}$ of subsets of the set A.

A mapping H of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$ is referred to as *atom-preserving* if for any $x \in A$ there exists $x' \in A'$ such that $H(\{x\}) = \{x'\}$.

If r is a relation from A to A', then for any $X \in \boldsymbol{P}(A)$ we put $\mathbf{P}[r](X) = \{x' \in A';$ there exists $x \in X$ with $(x, x') \in r\}$. Clearly, $\mathbf{P}[r]$ is a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$.

For any mapping H of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$, we set $\mathbf{Q}[H] = \{(x, x') \in A \times A';\ x' \in H(\{x\})\}$. Clearly, $\mathbf{Q}[H]$ is a relation from A to A'.

In what follows, we use the following results of [6].

Lemma 1. *If H is a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$ and if $\mathbf{Q}[H]$ is a mapping, then* *for any* $a \in A$ *and any* $a' \in A'$ *the conditions* $a' = \mathbf{Q}[H](a)$, $\{a'\} = H(\{a\})$ *are equivalent* ([6] Corollary 1). $\square$

Lemma 2. *Let* A, A' *be sets,* H *a mapping of* $\boldsymbol{P}(A)$ *into* $\boldsymbol{P}(A')$. *Then the following assertions are equivalent.*

(i) H *is totally additive.*

(ii) $H = (\mathbf{P} \circ \mathbf{Q})[H]$ ([6] Lemma 4). $\square$

Lemma 3. *Let* A, A' *be sets,* H *a mapping of* $\boldsymbol{P}(A)$ *into* $\boldsymbol{P}(A')$. *Then the following assertions are equivalent.*

(i) H *is totally additive and atom-preserving.*

(ii) $\mathbf{Q}[H]$ *is a mapping and* $H = (\mathbf{P} \circ \mathbf{Q})[H]$ *holds* ([6] Corollary 2). $\square$

Lemma 4. *If* A, A' *are sets and* h *a mapping of* A *into* A', *then* $(\mathbf{Q} \circ \mathbf{P})[h] = h$ ([6] Lemma 5). $\square$

3. CATEGORY **TER**

The instruments of the theory of categories needed in what follows may be easily found in [3].

Let A be a set and $t \subseteq A \times A \times A$. Then t is said to be a *ternary relation* on A and the ordered pair (A, t) is called a *ternary structure.*

If (A, t), (A', t') are ternary structures and h is a mapping of A into A', then h is said to be a *strong homomorphism* of the ternary structure (A, t) into (A', t') whenever the following holds: For any $x \in A$, $y \in A$, $z' \in A'$, the condition $(h(x), z', h(y)) \in t'$ is satisfied if and only if there exists $z \in A$ such that $(x, z, y) \in t$, $h(z) = z'$. (In [4] strong homomorphisms of ternary structures appear in a different meaning.)

Clearly, $1_{(A,t)}$ is a strong homomorphism of (A, t) into (A, t). We prove that the composite of two strong homomorphisms is a strong homomorphism, too. Let h be a strong homomorphism of (A, t) into (A', t') and k a strong homomorphism of (A', t') into (A'', t''); suppose that $x \in A$, $y \in A$, $z'' \in A''$ are arbitrary.

If there exists $z \in A$ with $(x, z, y) \in t$, $z'' = k(h(z))$, we set $h(z) = z'$. Since h is a strong homomorphism, we obtain $(h(x), z', h(y)) \in t'$. As k is a strong homomorphism and $k(z') = z''$, we have $(k(h(x)), z'', k(h(y))) \in t''$.

On the other hand, if $(k(h(x)), z'', k(h(y))) \in t''$, then the fact that k is a strong homomorphism implies the existence of $z' \in A'$ with $(h(x), z', h(y)) \in t'$, $z'' = k(z')$, and the fact that h is a strong homomorphism entails the existence of $z \in A$ with $(x, z, y) \in t$, $z' = h(z)$. Thus $z'' = k(h(z))$.

This proves the existence of a category **TER** whose objects are ternary structures and whose morphisms are strong homomorphisms of these structures.

Example 1. Let (A, t), (A', t') be ternary structures. A mapping h of A into A' is said to be a *homomorphism* of (A, t) into (A', t') if $(x, z, y) \in t$ implies $(h(x), h(z),$

$h(y)) \in t'$. Clearly, any strong homomorphism is a homomorphism but the converse need not hold as is seen from the following. Let A, A' be non-empty sets, $t = \emptyset$, $t' = A' \times A' \times A'$. If h is an arbitrary mapping of A into A', then h is a homomorphism. For an arbitrary $a \in A$ we have $(h(a), h(a), h(a)) \in t'$, but for any $z \in A$ with $h(z) = h(a)$ we obtain $(a, z, a) \notin t$. Thus, h is no strong homomorphism of (A, t) into (A', t'). □

Example 2. Let t be a binary operation on the set A, i.e., t is a mapping of $A \times A$ into A. As usual, we denote by $t(a, b)$ the value assigned to the ordered pair $(a, b) \in \in A \times A$. A set provided with a binary operation is said to be a *groupoid* (see e.g., [1], [2]). A groupoid with the carrier A and with the operation t will be denoted by (A, t).

Let (A, t), (A', t') be groupoids, h a mapping of A into A'. Then h is said to be a *homomorphism* of the groupoid (A, t) into the groupoid (A', t') if and only if $h(t(a, b)) = t'(h(a), h(b))$ holds for any a, b in A.

A binary operation t on a set A may be considered to be a ternary relation: we put $(a, c, b) \in t'$ if and only if $c = t(a, b)$. In what follows, we shall not distinguish between t and t' and we shall write t for t'. Thus, a groupoid (A, t) can be also regarded as a ternary structure. Therefore, if (A, t), (A', t') are groupoids, we may consider groupoid-homomorphisms of (A, t) into (A', t') and strong homomorphisms of (A, t) into (A', t'). We prove that groupoid-homomorphisms coincide with strong homomorphisms.

Let (A, t), (A', t') be groupoids and h a mapping of A into A'.

Suppose that h is a groupoid-homomorphism and that $a, b \in A$, $c' \in A'$ are arbitrary. Then $(h(a), c', h(b)) \in t'$ means $c' = t'(h(a), h(b)) = h(t(a, b))$ which is equivalent to the existence of $c = t(a, b) \in A$ such that $h(c) = c'$, $(a, c, b) \in t$. Thus h is a strong homomorphism of the ternary structure (A, t) into (A', t').

On the other hand, if h is a strong homomorphism of the ternary structure (A, t) into (A', t') and if a, b are arbitrary elements in A, then $c = t(a, b)$ exists and $(a, c, b) \in t$ holds. Put $c' = h(c)$; then $(h(a), c', h(b)) \in t'$ holds which means $h(c) = = c' = t'(h(a), h(b))$, i.e., $h(t(a, b)) = t'(h(a), h(b))$. Thus, h is a groupoid-homomorphism of (A, t) into (A', t'). □

Let (A, t) be a ternary structure. The relation t and the structure (A, t) is said to be

(1) *symmetric* if and only if $(x, z, y) \in t$ implies $(y, z, x) \in t$ for any x, y, z in A;

(2) *asymmetric* if and only if $(x, z, y) \in t$ implies $(y, z, x) \notin t$ for any x, y, z in A;

(3) *cyclic*, if and only if $(x, z, y) \in t$ implies $(z, y, x) \in t$ for any x, y, z in A;

(4) *transitive* if and only if $(x, z, y) \in t$, $(y, z, u) \in t$ imply $(x, z, u) \in t$ for any x, y, z, u in A (cf. [4], [5]).

Theorem 1. *Let (A, t), (A', t') be ternary structures and h a strong homomorphism of (A, t) onto (A', t'). If (A, t) is symmetric (cyclic), then so is (A', t').*

Proof. Let us have $(x', z', y') \in t'$. We take $x, y \in A$ such that $h(x) = x'$, $h(y) = y'$, which is possible because h is surjective. Since $(h(x), z', h(y)) \in t'$ and since h is a strong homomorphism, there exists $z \in A$ such that $h(z) = z'$ and $(x, z, y) \in t$. If t is symmetric, then $(y, z, x) \in t$ which implies that $(y', z', x') = (h(y), z', h(x)) \in t'$ and t' is symmetric. If t is cyclic, then $(z, y, x) \in t$ which implies that $(z', y', x') = = (h(z), y', h(x)) \in t'$ and t' is cyclic. □

Similar results for asymmetry and transitivity do not hold as is seen from the following.

Example 3. Suppose that $A = \{a, b, c, d\}$, $A' = \{a', b', c'\}$, $t = \{(a, b, c), (c, d, a)\}$, $t' = \{(a', b', c'), (c', b', a')\}$ and that $h(a) = a'$, $h(b) = b' = h(d)$, $h(c) = c'$. Then (A, t) is asymmetric and transitive while (A', t') is neither asymmetric nor transitive ($(a', b', c') \in t'$, $(c', b', a') \in t'$ would imply $(a', b', a') \in t'$ for a transitive t'). We prove that h is a strong homomorphism of (A, t) onto (A', t'). Indeed, if $(x, z, y) \in t$, then $(h(x), h(z), h(y)) \in t'$. On the other hand, if $x \in A$, $y \in A$, $z' \in A'$ are such that $(h(x), z', h(y)) \in t'$, then either $(h(x), z', h(y)) = (a', b', c')$ or $(h(x), z', h(y)) = = (c', b', a')$. In the first case we have $x = a$, $y = c$ and choose $z = b$; in the latter we have $x = c$, $y = a$ and choose $z = d$. Thus $(x, z, y) \in t$, $h(z) = z'$. □

4. TOTALLY ADDITIVE OPERATIONS

Let (A, t) be a ternary structure. For any $X \in \boldsymbol{P}(A)$, $Y \in \boldsymbol{P}(A)$ we set

$$\mathbf{R}[t](X, Y) = \\ = \{z \in A;\ \text{there exist } x \in X \text{ and } y \in Y \text{ such that } (x, z, y) \in t\}.$$

Clearly, $(\boldsymbol{P}(A), \mathbf{R}[t])$ is a groupoid.

Theorem 2. *Let (A, t), (A', t') be ternary structures and h a mapping of A into A'. Then the following assertions are equivalent.*

(i) *h is a strong homomorphism of (A, t) into (A', t').*

(ii) *$\mathbf{P}[h]$ is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), \mathbf{R}[t])$ into $(\boldsymbol{P}(A'), \mathbf{R}[t'])$.*

Proof. If (i) holds and X, $Y \in \boldsymbol{P}(A)$ are arbitrary, then $\mathbf{P}[h](\mathbf{R}[t](X, Y)) = = \mathbf{P}[h](\{z \in A;$ there exist $x \in X$, $y \in Y$ with $(x, z, y) \in t\}) = \{h(z);$ there exist $x \in X$, $y \in Y$ with $(x, z, y) \in t\}$. Similarly, $\mathbf{R}[t'](\mathbf{P}[h](X), \mathbf{P}[h](Y)) = \mathbf{R}[t'](\{h(x);$ $x \in X\}, \{h(y); y \in Y\}) = \{z' \in A';$ there exist $x \in X$, $y \in Y$ with $(h(x), z', h(y)) \in t'\}$. The fact that h is a strong homomorphism implies that $\mathbf{P}[h](\mathbf{R}[t](X, Y)) = = \mathbf{R}[t'](\mathbf{P}[h](X), \mathbf{P}[h](Y))$ for any $X \in \boldsymbol{P}(A)$ and any $Y \in \boldsymbol{P}(A)$ which means that $\mathbf{P}[h]$ is a homomorphism of the groupoid $(\boldsymbol{P}(A), \mathbf{R}[t])$ into $(\boldsymbol{P}(A'), \mathbf{R}[t'])$.

By definition, $\mathbf{P}[h]$ is totally additive. By Lemma 4, we have $(\mathbf{Q} \circ \mathbf{P})[h] = h$. Thus, $\mathbf{Q}[\mathbf{P}[h]]$ is a mapping and $(\mathbf{P} \circ \mathbf{Q})[\mathbf{P}[h]] = (\mathbf{P} \circ \mathbf{Q} \circ \mathbf{P})[h] = \mathbf{P}[h]$ and, therefore, $\mathbf{P}[h]$ is totally additive and atom preserving by Lemma 3. Thus (ii) holds.

Suppose that (ii) is satisfied. If $(h(x), z', h(y)) \in t'$ holds for $x \in A$, $y \in A$, $z' \in A'$, then $z' \in \mathbf{R}[t'](\{h(x)\}, \{h(y)\}) = \mathbf{R}[t'](\mathbf{P}[h](\{x\}), \mathbf{P}[h](\{y\})) =$ $= \mathbf{P}[h](\mathbf{R}[t](\{x\}, \{y\})) = \mathbf{P}[h](\{z; (x, z, y) \in t\}) = \{h(z); (x, z, y) \in t\}$ which implies the existence of $z \in A$ with $(x, z, y) \in t$, $h(z) = z'$. On the other hand, if $z \in A$, $(x, z, y) \in t$, $h(z) = z'$ hold, then $z \in \mathbf{R}[t](\{x\}, \{y\})$ and, therefore, $z' \in$ $\in \mathbf{P}[h](\mathbf{R}[t](\{x\}, \{y\}) = \mathbf{R}[t'](\mathbf{P}[h](\{x\}), \mathbf{P}[h](\{y\})) = \mathbf{R}[t'](\{h(x)\}, \{h(y)\})$, which means $(h(x), z', h(y)) \in t'$. Hence, h is a strong homomorphism and (i) holds. □

Let A be a set, N a binary operation on $\boldsymbol{P}(A)$. The operation N is said to be *totally additive* if the following conditions are satisfied.

(ta_1) For any system $\{X_\iota; \iota \in I\}$ of subsets of A and for any subset Y of A the equation $N(\bigcup \{X_\iota; \iota \in I\}, Y) = \bigcup \{N(X_\iota, Y); \iota \in I\}$ holds.

(ta_2) For any system $\{Y_\varkappa; \varkappa \in K\}$ of subsets of A and for any subset X of A the equation $N(X, \bigcup \{Y_\varkappa; \varkappa \in K\}) = \bigcup \{N(X, Y_\varkappa;); \varkappa \in K\}$ holds.

Lemma 5. *If (A, t) is a ternary structure, then $\mathbf{R}[t]$ is a totally additive operation.*

Proof. If $\{X_\iota; \iota \in I\}$ is a system of subsets of A and Y is a subset of A, then $\mathbf{R}[t](\bigcup \{X_\iota; \iota \in I\}, Y) = \{z \in A;$ there exist $x \in \bigcup \{X_\iota; \iota \in I\}$ and $y \in Y$ with $(x, z, y) \in$ $\in t\} = \{z \in A;$ there exist $\iota \in I$, $x \in X_\iota$, $y \in Y$ with $(x, z, y) \in t\} = \bigcup \{\{z \in A;$ there exist $x \in X_\iota$, $y \in Y$ with $(x, z, y) \in t\}, \iota \in I\} = \bigcup \{R(X_\iota, Y); \iota \in I\}$, which is (ta_1). Similarly, we prove that (ta_2) holds. □

If A is a set and N a binary operation on $\boldsymbol{P}(A)$, then we put

$$\mathbf{S}[N] = \{(x, z, y) \in A \times A \times A;\ z \in N(\{x\}, \{y\})\}\,.$$

Lemma 6. *For any ternary structure (A, t) the condition $(\mathbf{S} \circ \mathbf{R})[t] = t$ holds.*

Proof. Clearly, $(x, z, y) \in t$ is equivalent to $z \in \mathbf{R}[t](\{x\}, \{y\})$, which means $(x, z, y) \in \mathbf{S}[\mathbf{R}[t]]$. Thus, $t = \mathbf{S}[\mathbf{R}[t]] = (\mathbf{S} \circ \mathbf{R})[t]$. □

Lemma 7. *If A is a set and N a totally additive operation on $\boldsymbol{P}(A)$, then $N(X, Y) =$ $= \bigcup \{N(\{x\}, \{y\}); (x, y) \in X \times Y\}$ holds for arbitrary $X \in \boldsymbol{P}(A)$ and $Y \in \boldsymbol{P}(A)$.*

Proof. $N(X, Y) = N(\bigcup \{\{x\}; x \in X\}, Y) = \bigcup \{N(\{x\}, Y); x \in X\} = \bigcup \{N(\{x\},$ $\bigcup \{\{y\}; y \in Y\}); x \in X\} = \bigcup \{\bigcup \{N(\{x\}, \{y\}); y \in Y\}, x \in X\} = \bigcup \{N(\{x\}, \{y\});$ $(x, y) \in X \times Y\}$. □

Lemma 8. *If A is a set and N a totally additive operation on $\boldsymbol{P}(A)$, then $N =$ $= (\mathbf{R} \circ \mathbf{S})[N]$.*

Proof. For arbitrary $X \in \boldsymbol{P}(A)$, $Y \in \boldsymbol{P}(A)$ the condition $z \in N(X, Y)$ is equivalent to the existence of $(x, y) \in X \times Y$ such that $z \in N(\{x\}, \{y\})$ by Lemma 7. By definition of $\mathbf{S}$, this means $(x, z, y) \in \mathbf{S}[N]$ for some $x \in X$ and some $y \in Y$ which may be written as $z \in \mathbf{R}[\mathbf{S}[N]](X, Y)$. Thus, we have $N = \mathbf{R}[\mathbf{S}[N]] = (\mathbf{R} \circ \mathbf{S})[N]$. □

Theorem 3. *Let A, A' be sets $(\boldsymbol{P}(A), N), (\boldsymbol{P}(A'), N')$ groupoids with totally additive operations, H a totally additive mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$. Then the following conditions are equivalent.*

(i) $\mathbf{Q}[H]$ *is a strong homomorphism of the ternary structure* $(A, \mathrm{S}[N])$ *into* $(A', \mathrm{S}[N'])$.

(ii) H *is a totally additive atom-preserving homomorphism of the groupoid* $(\boldsymbol{P}(A), N)$ *into* $(\boldsymbol{P}(A'), N')$.

Proof. Put $h = \mathbf{Q}[H]$, $t = \mathrm{S}[N]$, $t' = \mathrm{S}[N']$. We obtain $\mathbf{P}[h] = H$ by Lemma 2, $\mathbf{R}[t] = N$, $\mathbf{R}[t'] = N'$ by Lemma 8. Then (i) coincides with (i) of Theorem 2 and (ii) is identical with (ii) of the same theorem, i.e., they are equivalent. □

5. CATEGORY **PGR**

We now introduce the category **PGR** (of Power-set GRoupoids). Objects of this category are groupoids of the form $(\boldsymbol{P}(A), N)$ where A is a set and N is a totally additive operation on $\boldsymbol{P}(A)$. By a morphism of the object $(\boldsymbol{P}(A), N)$ into the object $(\boldsymbol{P}(A'), N')$ in **PGR** we mean a totally additive atom-preserving homomorphism of the groupoid $(\boldsymbol{P}(A), N)$ into $(\boldsymbol{P}(A'), N')$.

Since $1_{(\boldsymbol{P}(A),N)}$ is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), N)$ into itself and since the composite of two totally additive atom-preserving homomorphisms is a totally additive atom-preserving homomorphism, **PGR** is a category.

6. ISOMORPHISM OF CATEGORIES **TER** AND **PGR**

We now introduce two functors. F is a functor of the category **TER** into **PGR** and G is a functor of the category **PGR** into **TER**. These functors will be defined by presenting object mappings Fo, Go and morphism mappings Fm, Gm.

If (A, t) is an object in the category **TER** and h a morphism in this category, we put

$$Fo(A, t) = (\boldsymbol{P}(A), \mathbf{R}[t]) , \quad Fm(h) = \mathbf{P}[h] .$$

If $(\boldsymbol{P}(A), N)$ is an object in **PGR** and H is a morphism in this category, we put

$$Go(\boldsymbol{P}(A), N) = (A, \mathrm{S}[N]) , \quad Gm(H) = \mathbf{Q}[H] .$$

Main Theorem. *F is a functor of the category* **TER** *into* **PGR** *and G is a functor of the category* **PGR** *into* **TER** *such that both $F \circ G$ and $G \circ F$ are identity functors.*

Proof. (1) By Lemma 5, $Fo(A, t)$ is an object in **PGR** for any object (A, t) in **TER**. Furthermore, $Fm(h)$ is a morphism in **PGR** for any morphism h in **TER** by Theorem 2. Clearly, $Fm(1_{(A,t)}) = 1_{Fo(A,t)}$ for any object (A, t) in **TER**. Finally, if h is a morphism of (A, t) into (A', t') in **TER** and h' is a morphism of (A', t') into (A'', t'') in **TER**, we obtain $(Fm(h') \circ Fm(h))(X) = \{(h' \circ h)(x);\ x \in X\} = (Fm(h' \circ h))(X)$ for any $X \in \boldsymbol{P}(A)$, which means $Fm(h') \circ Fm(h) = Fm(h' \circ h)$.

This implies that F is a functor.

(2) Similarly, $Go(\boldsymbol{P}(A), N) = (A, \mathrm{S}[N])$ is an object in **TER** for any object $(\boldsymbol{P}(A), N)$ in **PGR**. Furthermore, $Gm(H)$ is a morphism in **TER** for any morphism H

in **PGR** by Theorem 3. Clearly, $Gm(1_{(\boldsymbol{P}(A),N)}) = \{(x, y) \in A \times A; y \in 1_{(\boldsymbol{P}(A),N)}(\{x\})\} = 1_A = 1_{Go(\boldsymbol{P}(A),N)}$.

If H is a morphism of $(\boldsymbol{P}(A), N)$ into $(\boldsymbol{P}(A'), N')$ in **PGR** and H' is a morphism of $(\boldsymbol{P}(A'), N')$ into $(\boldsymbol{P}(A''), N'')$ in **PGR**, then $\mathbf{Q}[H]$, $\mathbf{Q}[H']$, $\mathbf{Q}[H' \circ H]$ are morphisms in **TER**, i.e., mappings. Let $x \in A$, $x'' \in A''$ be arbitrary. By Lemma 1, the condition $x'' = \mathbf{Q}[H' \circ H](x)$ is equivalent to $\{x''\} = H'(H(\{x\}))$. Since H, H' are atom-preserving, we obtain $H(\{x\}) = \{x'\}, \{x''\} = H'(\{x'\})$ for some $x' \in A'$. By Lemma 1, we obtain $x' = \mathbf{Q}[H](x)$, $x'' = \mathbf{Q}[H'](x')$, i.e. $x'' = (\mathbf{Q}[H'] \circ \mathbf{Q}[H])(x)$, which means $\mathbf{Q}[H' \circ H] = \mathbf{Q}[H'] \circ \mathbf{Q}[H]$, i.e. $Gm(H' \circ H) = Gm(H') \circ Gm(H)$.

It follows that G is a functor.

(3) If (A, t) is an object in **TER**, then $Fo(A, t) = (\boldsymbol{P}(A), \mathbf{R}[t])$ and $Go(Fo(A, t)) = (A, \mathbf{S}[\mathbf{R}[t]]) = (A, t)$ by Lemma 6. Similarly, if $(\boldsymbol{P}(A), N)$ is an object in **PGR**, we obtain $Go(\boldsymbol{P}(A), N) = (A, \mathbf{S}[N])$ and $Fo(Go(\boldsymbol{P}(A), N)) = (\boldsymbol{P}(A), \mathbf{R}[\mathbf{S}[N]]) = (\boldsymbol{P}(A), N)$ by Lemma 8. Thus, $Go \circ Fo$ is the identity on the class of all objects in **TER** and $Fo \circ Go$ is the identity on the class of all objects in **PGR**.

If h is a morphism in **TER**, we have $Fm(h) = \mathbf{P}[h]$ and $Gm(Fm(h)) = \mathbf{Q}[\mathbf{P}[h]] = h$ by Lemma 4. Finally, if H is a morphism in **PGR**, we obtain $Gm(H) = \mathbf{Q}[H]$, $Fm(Gm(H)) = \mathbf{P}[\mathbf{Q}[H]] = H$ by Lemma 2. Hence, $Gm \circ Fm$ is the identity on the class of all morphisms in **TER** and $Fm \circ Gm$ is the identity on the class of all morphisms in **PGR**.

We have proved that $F \circ G$ and $G \circ F$ are identity functors. □

Corollary 1. *The functor F is an isomorphism of the category* **TER** *onto* **PGR** *and the functor G is an isomorphism of the category* **PGR** *onto* **TER**. □

Corollary 2. *Let (A, t), (A', t') be ternary structures.*

(i) *For any strong homomorphism h of the structure (A, t) into (A', t') there exists a totally additive atom-preserving homomorphism H of the groupoid $(\boldsymbol{P}(A), \mathbf{R}[t])$ into $(\boldsymbol{P}(A'), \mathbf{R}[t'])$ such that $h = \mathbf{Q}[H]$.*

(ii) *If H is an arbitrary totally additive atom-preserving homomorphism of the groupoid $(\boldsymbol{P}(A), \mathbf{R}[t])$ into $(\boldsymbol{P}(A'), \mathbf{R}[t'])$, then $\mathbf{Q}[H]$ is a strong homomorphism of the structure (A, t) into (A', t').* □

7. POWER SET GROUPOIDS OF PARTICULAR STRUCTURES

We now investigate the question whether a particular property (symmetry, asymmetry, cyclicity, transitivity) of a ternary structure (A, t) influences its power set groupoid $(\boldsymbol{P}(A), \mathbf{R}[t])$.

Theorem 4. *Let (A, t) be a ternary structure. Then the following assertions are equivalent.*

(i) *(A, t) is symmetric.*

(ii) *For any $X, Y \in \boldsymbol{P}(A)$ the condition $\mathbf{R}[t](X, Y) = \mathbf{R}[t](Y, X)$ holds.*

Proof. Suppose that (i) holds and that $z \in \mathbf{R}[t](X, Y)$ is arbitrary. Then there exist $x \in X$ and $y \in Y$ such that $(x, z, y) \in t$ which implies that $(y, z, x) \in t$ and, therefore, $z \in \mathbf{R}[t](Y, X)$ holds. Thus, $\mathbf{R}[t](X, Y) \subseteq \mathbf{R}[t](Y, X)$ which implies (ii).

If (ii) holds and $(x, z, y) \in t$ it satisfied, then $z \in \mathbf{R}[t](\{x\}, \{y\}) = \mathbf{R}[t](\{y\}, \{x\})$ which implies that $(y, z, x) \in t$. We have proved (i). □

Theorem 5. *Let (A, t) be a ternary structure. Then the following assertions are equivalent.*

(i) *(A, t) is transitive.*

(ii) *For any $X \in \boldsymbol{P}(A)$, $U \in \boldsymbol{P}(A)$, $z \in A$ the condition $\mathbf{R}[t](X, \{z\}) \cap \cap \mathbf{R}[t](\{z\}, U) \subseteq \mathbf{R}[t](X, U)$ is satisfied.*

Proof. If (i) holds and $X \in \boldsymbol{P}(A)$, $U \in \boldsymbol{P}(A)$, $z \in A$ are arbitrary, then for any $y \in \mathbf{R}[t](X, \{z\}) \cap \mathbf{R}[t](\{z\}, U)$ there are $x \in X$ and $u \in U$ such that $(x, y, z) \in t$, $(z, y, u) \in t$. The transitivity of t implies that $(x, y, u) \in t$, which means $y \in \mathbf{R}[t](X, U)$. Thus, (ii) holds.

If (ii) holds, then, in particular, for any x, y, z, u in A the condition $(x, z, y) \in t$, $(y, z, u) \in t$ means $z \in \mathbf{R}[t](\{x\}, \{y\}) \cap \mathbf{R}[t](\{y\}, \{u\})$. By (ii), we obtain $z \in \in \mathbf{R}[t](\{x\}, \{u\})$, i.e. $(x, z, u) \in t$. Thus, (i) holds. □

The last result provokes the question of the characteristic property of (A, t) such that $\mathbf{R}[t](X, Z) \cap \mathbf{R}[t](Z, U) \subseteq \mathbf{R}[t](X, U)$ for any X, Z, U in $\boldsymbol{P}(A)$.

The structure (A, t) and the ternary relation t will be said to be *storngly transitive* if for any x, y, z, u, v in A the following condition is satisfied: $(x, z, y) \in t$, $(v, z, u) \in t$ imply $(x, z, u) \in t$.

Theorem 6. *Let (A, t) be a ternary structure. Then the following assertions are equivalent.*

(i) *(A, t) is strongly transitive.*

(ii) *For any X, Z, U in $\boldsymbol{P}(A)$ the condition $\mathbf{R}[t](X, Z) \cap \mathbf{R}[t](Z, U) \subseteq \subseteq \mathbf{R}[t](X, U)$ is satisfied.*

Proof. If (i) holds and X, Z, U in $\boldsymbol{P}(A)$ are arbitrary, then for any $y \in \mathbf{R}[t](X, Z) \cap \cap \mathbf{R}[t](Z, U)$ there exist $x \in X$, $z_1 \in Z$, $z_2 \in Z$, and $u \in U$ such that $(x, y, z_1) \in t$, $(z_2, y, u) \in t$. Since t is strongly transitive, we obtain $(x, y, u) \in t$ which implies that $y \in \mathbf{R}[t](X, U)$. Thus, (ii) holds.

If (ii) holds and $(x, z, y) \in t$, $(v, z, u) \in t$ are satisfied, then $z \in \mathbf{R}[t](\{x\}, \{y, v\}) \cap \cap \mathbf{R}[t](\{y, v\}, \{u\}) \subseteq \mathbf{R}[t](\{x\}, \{u\})$ and, thus, $(x, z, u) \in t$ which means that (i) holds. □

Theorem 7. *Let (A, t) be a ternary structure. Then the following assertions are equivalent.*

(i) *(A, t) is cyclic.*

(ii) *For any x, y, z in A the following condition is satisfied: $z \in \mathbf{R}[t](\{x\}, \{y\})$ if and only if $y \in \mathbf{R}[t](\{z\}, \{x\})$.*

Proof. If (i) holds and $z \in \mathbf{R}[t](\{x\}, \{y\})$, then $(x, z, y) \in t$ which implies that $(z, y, x) \in t$ and, thus, $y \in \mathbf{R}[t](\{z\}, \{x\})$. Similarly, $y \in \mathbf{R}[t](\{z\}, \{x\})$ implies $z \in \mathbf{R}[t](\{x\}, \{y\})$. Thus (ii) holds.

If (ii) holds and $(x, z, y) \in t$, then $z \in \mathbf{R}[t](\{x\}, \{y\})$ which implies that $y \in \mathbf{R}[t](\{z\}, \{x\})$. Hence $(z, y, x) \in t$ and (i) holds. □

Theorem 8. *Let (A, t) be a ternary structure. Then the following assertions are equivalent.*

(i) (A, t) *is asymmetric.*

(ii) *For any x, y in A the following condition is satisfied*: $\mathbf{R}[t](\{x\}, \{y\}) \cap \mathbf{R}[t](\{y\}, \{x\}) = \emptyset$.

Proof. If (i) holds and $z \in \mathbf{R}[t](\{x\}, \{y\}) \cap \mathbf{R}[t](\{y\}, \{x\})$, then $(x, z, y) \in t$, $(y, z, x) \in t$ which is a contradiction. Hence, (i) implies (ii).

If (ii) holds and $(x, z, y) \in t$, then $z \in \mathbf{R}[t](\{x\}, \{y\})$ which implies that $z \notin \mathbf{R}[t](\{y\}, \{x\})$ and, therefore, $(y, z, x) \notin t$. Thus, (ii) implies (i). □

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