Peter J. de Paepe; Johannes Joseph Oscar Odili Wiegerinck

A note on pervasive function algebras

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 1, 61–63

Persistent URL: http://dml.cz/dmlcz/102433

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

A NOTE ON PERVASIVE FUNCTION ALGEBRAS

P. J. DE PAEPE and J. J. O. O. WIEGERINCK[1]), Amsterdam

(Received December 19, 1989)

In [1] Čerych poses the following problem. Are any two nonzero annihilating measures for a pervasive function algebra absolutely continuous with respect to each other? In this note we give examples showing the answer is negative.

For a compact Hausdorff space X, $C(X)$ will denote the Banach algebra of continuous complex-valued functions on X (provided with the supremum norm $\|\cdot\|_X$). A function algebra on X is a closed subalgebra of $C(X)$ containing the constant functions and separating the points of X. Such an algebra is called pervasive (on X) if for any proper compact subset Y of X, the restrictions of the elements of A to Y are dense in $C(Y)$. For more information on function algebras we refer to [2].

The standard example of a pervasive function algebra is the disk algebra, considered as an algebra of functions on the rim of the disk: the disk algebra consists of all continuous functions on $D = \{|z| \leqq 1\}$ which are analytic on $\{|z| < 1\}$. By the maximum principle for analytic functions, A can be regarded as a function algebra on $\Gamma = \{|z| = 1\}$. It is easily seen that A is pervasive on Γ. Čerych's question is motivated by this example. Indeed, a theorem by F. and M. Riesz [3] states that any nonzero measure on the unit circle, annihilating the polynomials, and Lebesgue measure on the circle are absolutely continuous with respect to each other. We give some examples showing that such a phenomenon does not hold in general for pervasive function algebras. We will not go into details in the first two easy examples but give proofs for the more complicated third example.

Example 1. Let A be the disk algebra and $B = \{f \in A: f(0) = f(1)\}$, considered as an algebra on Γ. Then B is pervasive on Γ. Let ε_1 be point mass at the point 1, then the measures $\mu_1 = \mathrm{d}\theta/2\pi - \varepsilon_1$ and $\mu_2 = \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta$ both annihilate B and μ_2 is absolutely continuous with respect to μ_1, but μ_1 is not absolutely continuous with respect to μ_2.

Example 2. A, in a sense, more symmetric situation, is obtained by considering the algebra B on Γ consisting of all elements of the disk algebra which identify the points $1/2$ and 1 and the points $-1/2$ and -1. The algebra B is pervasive on Γ. Now for every point of the disk there is a (unique) representing measure on Γ with

[1]) Second author supported by a fellowship of the Royal Netherlands' Academy of Arts and Sciences.

respect to the disk algebra, i.e. a positive measure m with $\int f\,\mathrm{d}m = f(z)$ for all $f \in A$. For a point z of Γ this simply is point mass ε_z at z and for an interior point z of the disk, this measure is $\mathrm{d}m_z = P_z\,\mathrm{d}\theta/2\pi$, where P_z is the Poisson kernel for z. Now $\mu_1 = m_{1/2} - \varepsilon_1$ and $\mu_2 = m_{-1/2} - \varepsilon_{-1}$ both annihilate the elements of B, μ_1 is not absolutely continuous with respect to μ_2 and μ_2 is not absolutely continuous with respect to μ_1.

Example 3. We use the same ingredients as in the previous examples (considering functions which identify certain points of the disk and measures which are the difference of two representing measures) to obtain an example of a pervasive function algebra B and an infinite sequence $\mu_1, \mu_2, \ldots$ of measures which annihilate the elements of B and such that μ_n is not absolutely continuous with respect to μ_k if n and k are different.

For convenience we consider the disk $D = \{|z - 1/2| \leqq 1/2\}$ and denote its boundary by Γ. A will be the disk algebra associated with this disk.

Let $a_1, a_2, \ldots$, respectively $b_1, b_2, \ldots$ be sequences in $D \setminus \Gamma$, respectively $\Gamma \setminus \{1\}$, converging sufficiently fast to 1 so that the corresponding Blaschke products converge.

Let $B = \{f \in A : f(a_k) = f(b_k) \text{ for } k = 1, 2, \ldots\}$.

For every positive integer N, let g_N be the Blaschke product associated with the sequence $a_N, a_{N+1}, \ldots$:

$$g_N(z) = \prod_{k \geqq N} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \bar{a}_k z}, \quad |z| < 1.$$

Similarly one defines h_N for the sequence $b_N, b_{N+1}, \ldots$.

Let $f_N = g_N h_N (z - 1)\, p_N$, considered as a function on Γ (or D) where p_N is of the form $\exp q_N$ with q_N a polynomial chosen in such a way that $f_N(a_k) = f_N(b_k)$, $k = 1, \ldots, N - 1$ (for $N = 1$, p_N is chosen identically 1).

Note that the functions $zf_1, f_1, f_2, \ldots$ belong to B and separate the points of Γ. So B is a function algebra on Γ. We now show that B is pervasive on Γ. So let K be a non-trivial compact subset of Γ, F an element of $C(K)$ and $\varepsilon > 0$. Without loss of generality $b_1, b_2, \ldots$ and hence also the point 1 belong to K. We may also assume that F vanishes on a neighborhood U of the point 1 (relative with respect to K).

For some N, $b_N, b_{N+1}, \ldots \in U$, so since A is pervasive on Γ, there is a polynomial p with

$$\left\| \frac{F}{f_N} - p \right\|_K < \frac{\varepsilon}{\|f_N\|_\Gamma}$$

Since $K \cup \{a_1, \ldots, a_{N-1}\}$ is polynomially convex, there is a polynomial q with

$$\|q\|_K < \frac{\varepsilon}{\|f_N\|_\Gamma},$$

$q(b_K) = 0$ and $q(a_k) = p(b_k) - p(a_k)$ for $k = 1, \ldots, N - 1$. It follows that

$$\|F - (p + q) f_N\|_K < 2\varepsilon$$

and since $(p + q) f_N \in B$ we are done.

Finally, let m_n be the unique representing measure on Γ for the point a_n with respect to the disk algebra and let ε_n be point mass at b_n. Then $\mu_n = m_n - \varepsilon_n$ annihilates the algebra B, and for all n different from k the measure μ_n is not absolutely continuous with respect to μ_k.

References

[1] *J. Čerych:* A remark on linear codimensions of function algebras, Czech. Math. J. *36* (1986), 511–515.

[2] *G. Leibowitz:* Lectures on complex function algebras, Scott, Foresman and Company, Glenview, Illinois, 1970.

[3] *F.* and *M. Riesz:* Über die Randwerte einer analytischen Funktion, Quatrième Congrès des Mathématiciens Scandinaves, Stockholm 1916, Almquist and Wiksells, Uppsala 1920, 27–44.

Authors' addresses: Universiteit van Amsterdam, Faculteit Wiskunde en Informatica, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands.