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SPANNING TREES OF INFINITE GRAPHS

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INTRODUCTION

In [13] Zelinka conjectured that *if G is a connected infinite locally finite graph, and τ an end of G then*:

Conjecture 1. *If $m(\tau)$ is the maximum number of pairwise disjoint rays in τ, then, for any cardinal k with $1 \leqq k \leqq m(\tau)$, there is a spanning tree of G having exactly k ends included in τ_0.*

Conjecture 2. *Any spanning tree T of G contains a ray which belongs to the end τ.*

He proved Conjecture 2 in the case where τ is a free end, i.e. τ can be separated from any other end by a finite set of vertices; and Conjecture 1 in the case where τ is also free with $m(\tau)$ finite. Actually Zelinka used the concept of *degree* of an end τ rather than that of $m(\tau)$, but it turns out that these two notions coincide when the graph is locally finite.

In this paper we prove these conjectures, and even improve the first by replacing the local finiteness of G by the assumption that G has a coterminal spanning tree, i.e. a spanning tree having exactly one end included in each end of G; this condition is always satisfied by locally finite graphs. We recall different partial results about the existence of coterminal spanning trees, a problem which is still far to be entirely solved, and we give a new one by showing that: *a connected graph having exactly one end has a coterminal spanning tree if the set of vertices, which cannot be separated from this end by a countable set of vertices, is countable.* For infinitely connected graphs, this condition turns out to be equivalent to a recent one given by Seymour and Thomas [12]. Finally we characterize some classes of connected infinite graphs such that if G is one of them and if, for every end τ of G, $k(\tau)$ is a fixed cardinal $\leqq m(\tau)$, then there is a spanning tree of G having, for any end τ, exactly $k(\tau)$ disjoint rays belonging to τ.

To prove these results we almost essentially use the concepts and results of [11]. So the terminology and notation will be for the most part that used in that paper. Besides, since most of the results of the different papers [6] to [10] are recapitulated in [11], we will, for simplicity, only refer to [11] when possible.

1. PRELIMINARIES

1.1. Each ordinal α is defined as the set of ordinals less than α. If X is a set we denote by $|X|$ its cardinality and, for a cardinal n, by $[X]^n$ (resp. $[X]^{<n}$, $[X]^{\leqq n}$) the set of its subsets of cardinality n (resp. $<n$, $\leqq n$).

1.2. A *graph* G is a set $V(G)$ (*vertex set*) together with a set $E(G) \subseteq [V(G)]^2$ (*edge set*). For $x \in V(G)$ the set $V(x; G) := \{y \in V(G): \{x, y\} \in E(G)\}$ is the *neighborhood* of x, and its cardinality is the *degree* of x. A graph is *locally finite* if all its vertices have finite degrees. H is a *subgraph* of G if $V(H)$ and $E(H)$ are subsets of $V(G)$ and $E(G)$, respectively. H is an *induced subgraph* if H is a subgraph such that $E(H) = [V(H)]^2 \cap E(G)$. For $A \subseteq V(G)$ we denote by $G - A$ the subgtaph of G induced by $V(G) - A$; and if H is a subgraph of G, then we set $G - H := G - V(H)$. For $B \subseteq E(G)$ we denote by $G \setminus B$ the smallest subgraph of G with $E(G \setminus B) = E(G) - B$. The *union* of a family $(G_i)_{i \in I}$ of graphs is the graph $\bigcup_{i \in I} G_i$ given by $V(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} V(G_i)$ and $E(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} E(G_i)$. The *intersection* is defined similarly. If H is a subgraph of G, and X a subgraph of $G - H$, the *boundary of H with X* is the set $\mathfrak{B}(H, X) := \{x \in V(H): V(x; G) \cap V(X) \neq \emptyset\}$. The set of components of G is denoted by $\mathfrak{C}_G$, and if x is a vertex, then $\mathfrak{C}_G(x)$ is the component of G containing x. A *path* $W := \langle x_0, \ldots, x_n \rangle$ is a graph with $V(W) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(W) = \{\{x_i, x_{i+1}\}: 0 \leqq i < n\}$. A *ray* or *one-way infinite path* $R := \langle x_0, x_1, \ldots \rangle$ is defined similarly. A path $\langle x_0, \ldots, x_n \rangle$ is called an *x_0x_n-path*. For $A, B \subseteq V(G)$, an *AB-path* of G is an xy-path of G whose only vertices in $A \cup B$ are x et y, with $x \in A$ and $y \in B$.

1.3. The *ends* of a graph G (this concept was introduced by Freudenthal [1] and independently by Halin [2]) are the classes of the equivalence relation $\sim_G$ defined on the set of all rays of G by: $R \sim_G R'$ if and only if there is a ray R'' whose intersections with R and R' are infinite; or equivalently if and only if $\mathfrak{C}_{G-S}(R) = \mathfrak{C}_{G-S}(R')$ for any $S \in [V(G)]^{<\omega}$ (where $\mathfrak{C}_{G-S}(R)$ denotes the component of $G - S$ containing a subray of R). We will denote by $[R]_G$ the class of a ray R of G modulo $\sim_G$, by $\mathfrak{C}_{G-S}([R]_G)$ the component $\mathfrak{C}_{G-S}(R)$, and by $\mathfrak{T}(G)$ the set of all ends of G. Notice that if G is a tree, then two rays of G are equivalent modulo $\sim_G$ if and only if they have a common subray; hence two disjoint rays of a tree correspond to different ends of this tree.

A subgraph H of G is *terminally faithful* (resp. *terminally full*, *coterminal*) if the map $\varepsilon_{HG}: \mathfrak{T}(H) \to \mathfrak{T}(G)$ given by $\varepsilon_{HG}([R]_H) = [R]_G$ for every ray R of H, is injective (resp. surjective, bijective). We denote by $\mathfrak{T}_H(G)$ the image of ε_{HG}, i.e. the set of ends of G having rays of H as elements.

1.4. An infinite subset S of $V(G)$ is *concentrated* in G if it has the following equivalent properties [11, Theorem 1.4]:

(i) there is an end τ such that $S - V(\mathfrak{C}_{G-F}(\tau))$ is finite for any $F \in [V(G)]^{<\omega}$ (S is said to be "*concentrated in* τ");

(ii) for all infinite subsets T and U of S there is an infinite family of pairwise disjoint TU-paths in G.

1.5. A set S of vertices of G is *dispersed* if it has the following equivalent properties [11, 2.5]:

(i) for every $\tau \in \mathfrak{T}(G)$ there is an $F \in [V(G)]^{<\omega}$ such that $S \cap V(\mathfrak{C}_{G-F}(\tau)) = \emptyset$;

(ii) S has no concentrated subset.

1.6. For $\Omega \subseteq \mathfrak{T}(G)$ let $m(\Omega) := \sup\{|\mathfrak{R}|: \mathfrak{R}$ is a set of pairwise disjoin elements of $\bigcup\Omega\}$. If $\Omega = \{\tau\}$, we write $m(\tau)$ for $m(\{\tau\})$, and we call it the *multiplicity* of τ. By [10, 11.5] the supremum is attained, i.e. there is a set of pairwise disjoint rays in $\bigcup\Omega$ of cardinality $m(\Omega)$. This was already proved by Halin [3, Satz 1] and [4, Satz 1] when $\Omega = \mathfrak{T}(G)$ and $|\Omega| = 1$, respectively.

For a subgraph H and an end τ of G, we will set $m_H(\tau) := m(\varepsilon_{HG^{-1}}(\tau))$. By the remark in 1.3 about ends of trees, notice that if H is a tree, then $m_H(\tau) \leqq |\varepsilon_{HG^{-1}}(\tau)|$, with the equality in particular if $m_H(\tau)$ is finite.

In [13] Zelinka defined the degree of an end τ of a locally finite graph G as follows. For a non-empty finite subset A of $V(G)$, let $c(A, \tau) := \min\{|S|: S \in [V(G) - A]^{<\omega}$ and $A \cap V(\mathfrak{C}_{G-S}(\tau)) = \emptyset\}$. Then the *degree* of τ is $d(\tau) := \sup\{c(A, \tau): A \in [V(G)]^{<\omega}\}$.

A particular case of the Mengerian theorem [11, Theorem 1.9] states that: *For any non-empty subset A of $V(G)$ and any end τ of G, $c(A, \tau)$ is equal to the maximum number of raysin τ originating in A and having at most their endpoints in common.* With this result we easily see that, *for a locally finite graph, the multiplicity and the degree of an end coincide.*

1.7. A vertex x is a *neighbor* of an end τ if $x \in V(\mathfrak{C}_{G-S}(\tau))$ for any $S \in [V(G) - \{x\}]^{<\omega}$. We denote by V_τ the *neighborhood* of τ. The cardinal $v(\tau) := |V_\tau|$ is called the *valency* of τ.

1.8. A *multi-ending* of a graph G is an inducted subgraph M of G satisfying:

M1. M is connected;

M2. the boundary of M with every component of $G - M$ is finite;

M3. any infinite subset of $V(M)$ which is concentrated in G is also concentrated in M;

M4. M contains a ray;

M5. $V_{[R]_M} = V_{[R]_G}$ for any ray R of M;

M6. for any family $(R_i)_{i\in I}$ of pairwise disjoint rays of G such that $\{[R_i]_G: i \in I\} \subseteq$

$\subseteq \mathfrak{T}_M(G)$, there is a family $(R_i')_{i \in I}$ of pairwise disjoint rays of M such that $R_i \cap R_i'$ is infinite for every $i \in I$.

A multi-ending M is an *ending* if $|\mathfrak{T}(M)| = 1$; it is a *discrete* multi-ending if each end τ of M is *free*, i.e. there is a finite subset S of $V(M)$ such that $\mathfrak{C}_{G-S}(\tau) \neq \mathfrak{C}_{G-S}(\tau')$ for any end $\tau' \neq \tau$ of M. By M3 a multi-ending is terminally faithful; and by M6 $m([R]_M) = m([R]_G)$ fro any ray R of M.

By [11, Theorem 2.1] *for every end τ of G there is an ending M of G such that $\mathfrak{T}_M(G) = \{\tau\}$*. By [8, 1.4.2] *for every discrete multi-ending M of G, there is a dispersed set S of G such that $\mathfrak{C}_{G-S}(\tau) \neq \mathfrak{C}_{G-S}(\tau')$ for distinct $\tau, \tau' \in \mathfrak{T}_M(G)$.*

1.9. Let G be a connected graph having no subdivision of the dyadic tree (3-regular infinite tree) as a terminally faithful subgraph. By [11, Theorem 2.6] there is a sequence $(G_n)_{n \geqq 0}$ of multi-endings of G, called a *terminal expansion* of G, satisfying the following conditions: for every $n \geqq 0$,

E1. G_n is an induced subgraph of G_{n+1};

E2. any component of $G_n - G_{n-1}$ (with the convention $G_{-1} := \emptyset$) is a discrete multi-ending of $G - G_{n-1}$;

E3. $G = \bigcup_{n \geqq 0} G_n$ and $\mathfrak{T}(G) = \bigcup_{n \geqq 0} \mathfrak{T}_{G_n}(G)$.

2. SPANNING TREES

2.1 Theorem. *Any spanning tree of a connected infinite locally finite graph is terminally full.*

Proof. Let G be a connected infinite locally finite graph and T a spanning tree of G. Let τ be an end of G. We have to prove that $\tau \in \mathfrak{T}_T(G)$, i.e. that T contains a ray $R \in \tau$. By 1.8 there is an ending M of G such that $\mathfrak{T}_M(G) = \{\tau\}$. Let T' be the smallest subtree of T containing $T \cap M$. Since M is infinite, T' is then an infinite locally finite tree. Thus it contains a ray R. For any component X of $G - M$, $R \cap X$ is finite by the definition of T'; hence $\mathfrak{C}_{G-B}(R) \neq X$ where $B := \mathfrak{B}(M, X)$; notice that B is finite by 1.7.M2. Therefore $[R]_G \in \mathfrak{T}_M(G) = \{\tau\}$. □

This result generalizes Lemma 1 of [13], and thus proves Conjecture 2 of this same paper.

2.2 Lemma. [8, 3.1]. *Let G be a connected graph, T a spanning tree of G, T_0 any tree of G, and a a vertex of T_0. Then*

$$T_1 := T_0 \cup (T \setminus \{\{x, y\} \in E(T) \colon y \in V(T_0) \quad and \quad x \leqq_a y\})$$

where $\leqq_a$ is the natural partial order on $V(G)$ induced by T in which a is the least element, is a spanning tree of G.

2.3. Lemma. *Let T be a spanning tree of a connected infinite graph G. Let τ_0*

be an end of G, and k a cardinal such that $m_T(\tau_0) \leqq k \leqq m(\tau_0)$. Then G has a spanning tree T_0 such that $m_{T_0}(\tau_0) = k$ and $m_{T_0}(\tau) = m_T(\tau)$ for every end $\tau \neq \tau_0$.

Proof. We can suppose that $k > m_T(\tau_0)$, otherwise T would have the required properties. For simplication we will still denote by k the function which maps every end to a cardinal such that $k(\tau_0) = k$ and $k(\tau) = m_T(\tau)$ if $\tau \neq \tau_0$.

Let $\mathfrak{R}$ be a set of pairwise disjoint rays in τ_0 such that $m_{T \cup \bigcup \mathfrak{R}}(\tau_0) = k(\tau_0)$. This is possible since $k(\tau_0) \leqq m(\tau_0)$ and since, by 1.6, the supremum $m(\tau_0)$ is attained. And let $\mathfrak{R}'$ be a set of cardinality $k(\tau_0)$ of pairwise disjoint rays of $T \cup \bigcup \mathfrak{R}$ belonging to the end τ_0. Denote by T' a tree of G containing $\bigcup \mathfrak{R}'$, and which is minimal with respect to inclusion. By the minimality of T', any ray of T_0 belongs to τ_0 since all element of $\mathfrak{R}'$ belong to τ_0, and furthermore $T' - \bigcup \mathfrak{R}'$ is finite if so is $k(\tau_0)$. Hence, in both cases, $m_{T'}(\tau_0) = k(\tau_0)$. Besides $m_{T \cup T'}(\tau_0) = k(\tau_0)$. Indeed, this is obvious if $k(\tau_0)$ is infinite, and when it is finite this is a consequence of the fact that $m_{T \cup \bigcup \mathfrak{R}'}(\tau_0) = k(\tau_0)$ and that $T' - \bigcup \mathfrak{R}'$ is finite. On the other hand, for any end $\tau \neq \tau_0$ of G, $m_{T \cup T'}(\tau) = k(\tau) = 1$ since, for any ray R of $T \cup T'$ which belongs to τ, there is a finite set S of vertices such that $\mathfrak{C}_{G-S}(\mathfrak{R}) = \mathfrak{C}_{G-S}(\tau) \neq \mathfrak{C}_{G-S}(\tau_0)$, hence $\mathfrak{C}_{G-S}(\mathfrak{R}) \cap T'$ is finite by the minimality of T'; this proves that R has a subray in $T - T'$, hence that $m_{T \cup T'}(\tau) = m_T(\tau)$.

Then, by Lemma 2, for $a \in V(T')$ the tree

$$T_0 := T' \cup (T \setminus \{\{x, y\} \in E(T) \colon y \in V(T') \quad \text{and} \quad x \leqq_a y\})$$

is a spanning tree of G such that $m_{T_0}(\tau) = k(\tau)$ for any end τ of G, since $k(\tau) = m_{T'}(\tau) \leqq m_{T_0}(\tau) \leqq m_{T \cup T'}(\tau) = k(\tau)$. Consequently T_0 has the required properties. □

We get immediately:

2.4. Theorem. *Let G be a connected infinite graph having a coterminal spanning tree. Let τ_0 be an end of G, and k a cardinal such that $1 \leqq k \leqq m(\tau_0)$. Then G has a spanning tree T such that $m_T(\tau_0) = k$ and $m_T(\tau) = 1$ for every end $\tau \neq \tau_0$.*

2.5. Remarks. The problem of determing which infinite connected graphs have a coterminal spanning tree is still unsolved. Recently Seymour and Thomas [12] proved that there is a one-ended connected graph without coterminal spanning tree; more precisely they showed that: *There is an infinitely connected graph G of cardinality ω_1 such that every spanning tree contains a subdivision of the ω_1-regular tree as a subtree.* We recall the following partial results:

A connected infinite graph G has a coterminal spanning tree if: (i) *G is countable* (Halin [2, Satz 3]); (ii) *G contains no subdivided infinite complete graph as a subgraph* (Halin [5, Theorem 10.1]); (iii) *G is infinitely connected and contains no subdivision of the ω_1-regular tree as a subgraph* (Seymour and Thomas [12, (1.7)]); (iv) *G is one-ended and its end has countable multiplicity or valency* (Polat [11, Theorem 2.11]).

See [11, Section 2.10] for some extensions of this last result. Any connected infinite locally finite graph has a fortiori a coterminal spanning tree. Hence 2.4 proves and even improves Conjecture 1 of [13]. We will give another condition for the existence of a coterminal spanning tree in a one-ended graph which will generalize the above mentioned two. First we recall three results.

2.6. [11, Theorem 2.10] *Let G be a one-ended connected graph, and let τ be its only end. Then G has a rayless spanning tree if and only if $v(\tau) \neq 0$ and G has a coterminal spanning tree.*

2.7 [11, Corollaries 2.1 and 2.2] *If $m(\mathfrak{T}(G))$ is countable, and if the neighborhood of every end is not empty, then G has a rayless spanning tree.*

2.8 [10, Theoreme 12.3] *Let G be a connected graph, and let $\Omega \subseteq \mathfrak{T}(G)$ be such that $m(\Omega) \geqq \varkappa$ for some regular uncountable cardinal $\varkappa$. Then the set*

$$V_\Omega^\kappa := \{x \in V(G)\colon \mathfrak{C}_{G-S}(x) \text{ contains a ray belonging to } \bigcup\Omega$$
$$\text{for any } S \in [V(G-x)]^{<\kappa}\}$$

is non-empty. Besides, on the one hand either $|V_\Omega^\kappa| \geqq \varkappa$ or the set Γ of components of $G - V_\Omega^\kappa$ containing an element of $\bigcup\Omega$ is of cardinality $\geqq \varkappa$; and on the other hand $m(\{[R]_X\colon R \text{ is a ray of } X \text{ belonging to } \bigcup\Omega\}) < \varkappa$ for every $X \in \Gamma$.

2.9. Theorem. *Let G be a one-ended connected graph with $\mathfrak{T}(G) = \{\tau\}$. If $V_\tau^{\omega_1}$ is countable, then G has a coterminal spanning tree.*

Proof. This is a consequence of 2.6 if $m(\tau) \leqq \omega$. Suppose $m(\tau) > \omega$. $V_\tau^{\omega_1}$ is countably infinite. Indeed, suppose that $V_\tau^{\omega_1}$ is finite, then, since G is one-ended, there is just one component of $G - V_\tau^{\omega_1}$ containing a ray, but this is a contradiction with 2.8.

(a) First we show that $V_\tau^{\omega_1}$ is contained in a ray of G. Let $V_\tau^{\omega_1} = \{x_n\colon n < \omega\}$. We define by induction a sequence $(W_n)_{n<\omega}$ of finite paths of G such that $x_n \in V(W_n)$ and $W_n \subseteq W_{n+1}$. Let $W_0 := \langle x_0 \rangle$, and $n \geqq 0$. Suppose that $W_n = \langle y_0, \ldots, y_k \rangle$ is defined such that $y_0 = x_0$, $y_k = x_i$ for some $i \leqq n$, and $x_n = y_j$ for some $j \leqq k$. If $x_{n+1} \in V(W_n)$, then $W_{n+1} := W_n$. Assume $x_{n+1} \notin V(W_n)$. The vertices y_k and x_{n+1} belong to the same component of $G - (W_n - y_k)$, since W_n is finite and $y_k, x_{n+1} \in V_\tau^{\omega_1}$. Thus there is a (y_k, x_{n+1})-path P having only y_k in common with W_n. Then define $W_{n+1} := W_n \cup P$. Finally, $\bigcup_{n<\omega} W_n$ is a ray which contains $V_\tau^{\omega_1}$.

(b) Let X be a component of $G - W$. By 2.8, $m(\mathfrak{T}(X)) \leqq \omega$. Let $a \notin V(G)$, and let

$$X^+ := X \cup \bigcup \{\langle a, v, x \rangle\colon (v, x) \in V(W) \times V(X) \quad \text{and} \quad \{v, x\} \in E(G)\}\,.$$

If $\mathfrak{T}(X) = \emptyset$, then denote by T_X any spanning tree of X^+. Suppose $\mathfrak{T}(X) \neq \emptyset$. Then $m(\mathfrak{T}(X^+))$ is countable, and a is a neighbor of every end of X^+. Hence, by 2.7, X^+ has a rayless tree T_X.

Thus clearly $T := W \cup \bigcup \{T_X - a\colon X \in \mathfrak{C}_{G-W}\}$ is a spanning tree of G which is also coterminal since any ray of T has a subray in W. □

This condition of existence of a coterminal spanning tree is stictly weaker than condition 2.5 (iv), since if $m(\tau)$ or $v(\tau)$ are countable then so is $V_\tau^{\omega_1}$, but the converse is false in general. Furthermore, Theorem 2.9 with next result show that condition 2.5 (iii) given by Seymour and Thomas for infinitely connected graphs only, also holds if this restriction is replaced by the weaker one of one-ended graphs.

2.10. Proposition. *Let G be a one-ended graph, with $\mathfrak{T}(G) = \{\tau\}$. Then $V_\tau^{\omega_1}$ is countable if and only if G has no subdivision of the ω_1-regular tree T_{ω_1} as a subgraph.*

We need the following lemma.

2.11. Lemma. *Let G be a one-ended graph, with $\mathfrak{T}(G) = \{\tau\}$, and $V_\tau^{\omega_1} \neq \emptyset$. Then $(V\mathfrak{C}_{G-S}(x)) \cap (V_\tau^{\omega_1} - \{x\}) \neq \emptyset$ for any $x \in V_\tau^{\omega_1}$, and any $S \in [V(G - x)]^{\leqq \omega}$.*

Proof. Let $X := \mathfrak{C}_{G-S}(x)$. Suppose that $m(\mathfrak{T}(X))$ is countable. Let Δ be a set of pairwise disjoint rays of X, which is maximal with respect to inclusion, and such that $x \notin A := V(\bigcup \Delta)$. Then $\mathfrak{C}_{G-(S \cup A)}(x)$ is rayless, with $S \cup A$ countable; a contradiction with $x \in V_\tau^{\omega_1}$. Hence $m(\mathfrak{T}(X))$ is uncountable, thus $V_{\mathfrak{T}(X)}^{\omega_1} \neq \emptyset$ by 2.8, and this proves the result since $V_{\mathfrak{T}(X)}^{\omega_1} \subseteq V_\tau^{\omega_1}$. □

2.12. Proof of Proposition 2.10.

(a) If G has a subdivision T of T_{ω_1} as a subgraph, then every vertex of T whose degree in T is > 2, hence equal to ω_1, clearly belongs to $V_\tau^{\omega_1}$. Therefore $V_\tau^{\omega_1}$ is uncountable.

(b) Assume now that $V_\tau^{\omega_1}$ is uncountable. We define by induction the sequence $(T_\alpha)_{\alpha<\omega_1}$ of countable trees of G, such that T_α is a subtree of T_β if $\alpha < \beta$, and the sequence $(x_\alpha)_{\alpha<\omega_1}$ of pairwise distinct elements of $V_\tau^{\omega_1}$ such that $x_\beta \in V(T_\alpha)$ if and only if $\beta < 2^\alpha$ (ordinal exponentiation). Let x_0 be any element of $V_\tau^{\omega_1}$, and $T_0 := \langle x_0 \rangle$, and let $\alpha < \omega_1$. Assume that T_β and x_γ are defined for every $\beta < \alpha$ and $\gamma < 2^\beta$.

If α is a limit ordinal, then $T_\alpha := \bigcup_{\beta<\alpha} T_\beta$. Suppose that $\alpha = \beta + 1$. We define by induction on γ, with $-1 \leqq \gamma < \alpha$, the countable tree A_γ and the vertex $x_{\alpha+\gamma} \in V(A_\gamma) \cap V_\tau^{\omega_1}$. Let $A_{-1} := T_\beta$ and $x_{\alpha-1} = x_\beta$. Let γ be an ordinal $<\alpha$. Suppose that A_δ and $x_{\alpha+\delta}$ are defined for all δ, $-1 \leqq \delta < \gamma$. By the hypothesis x_γ is a vertex of $A_{<\gamma} := \bigcup_{\delta<\gamma} A_\delta$. Since $A_{<\gamma}$ is countable, there is, by Lemma 2.11, an element y of $V_\tau^{\omega_1}$ distinct from x_γ and belonging to the component of $G - (A_{<\gamma} - x_\gamma)$ containing x_γ. Denote by W an $x_\gamma y$-path of this component, and define $x_{\alpha+\gamma} := y$ and $A_\gamma := A_{<\gamma} \cup W$. Finally let $T_\alpha := \bigcup_{-1\leqq\gamma<\alpha} A_\gamma$.

Then, by the construction, the tree $T := \bigcup_{\alpha<\omega_1} T_\alpha$ is a subdivision of the ω_1-regular tree; the vertices x_α being the vertices of T of degree ω_1. □

We conclude this paper with a result which partially extends Theorem 2.4.

2.13. Theorem. *Let G be a connected infinite graph having no subdivision of the dyadic tree as a terminally faithful subgraph, and such that each of its endings has a coterminal spanning tree. Let k be a function which maps every end τ of G*

to a cardinal $k(\tau) \leqq m(\tau)$ with $k(\tau) > 0$ if $v(\tau) = 0$. Then G has a spanning tree T such that $m_T(\tau) = k(\tau)$ for every end τ of G.

Proof. (a) Suppose that all ends of G are free. By 1.8 there is a dispersed set S of G such that $\mathfrak{C}_{G-S}(\tau) \neq \mathfrak{C}_{G-S}(\tau')$ if $\tau \neq \tau'$. Since S is dispersed, for every end τ, there is a finite set A_τ of vertices of G such that $S \cap V(\mathfrak{C}_{G-A_\tau}(\tau)) = \emptyset$. The set $S' := S \cup \cup \bigcup_{\tau \in \mathfrak{T}(G)} A_\tau$ is obviously dispersed. Then, by the connectivity of G, there is a tree T_S of G containing S' and every rayless component of $G - S'$, and such that $T_S \cap \cap \mathfrak{C}_{G-A_\tau}(\tau)$ is finite for every end τ. $V(T_S)$ is then dispersed, and the boundary of T_S with every component of $G - T_S$ is finite. Let X be such a component, and $B_X := := \mathfrak{B}(T_S, X)$. Then the subgraph M_X of G induced by $V(X) \cup B_X$ is an ending of G. Let τ_X be its only end. By the axioms M5 and M6 of multi-endings, τ_X and the corresponding end of G have the same multiplicity and the same valency. Thus, for simplicity, we will still denote by τ_X this end of G. By the hypothesis M_X has a coterminal spanning tree. Thus, by 2.6 if $k(\tau_X) = 0$, and by 2.4 if $k(\tau_X) > 0$, M_X has a spanning tree T_X such that $m_{T_X}(\tau_X) = k(\tau_X)$. Now denote by E_X a subset of the set of of edges of T_X which are incident with both B_X and $V(X)$, so that, for each component C of $T_X - B_X$, there is exactly one edge in E_X which is incident with C. And let F_X be the spanning forest of M_X whose set of edges is $E(T_X - B_X) \cup E_X$, Then clearly

$$T := T_S \cup \bigcup \{F_X : X \in \mathfrak{C}_{G-T_S}\}$$

is a spanning tree of G such that $m_T(\tau) = k(\tau)$ for every end τ of G.

(b) Suppose now that some end of G is not free, and let $(G_n)_{n \geqq 0}$ be a terminal expansion of G (see 1.9). By 1.9 E2, each component X of $G_n - G_{n-1}$ $(G_{-1} := \emptyset)$ is a discrete multi-ending of $G - G_{n-1}$. As in (a) denote by B_X the boundary of G_{n-1} with X. This is a finite set, thus the subgraph M_X of G induced by $V(X) \cup B_X$ is a discrete multi-ending of G. Then all ends of M_X are free and have, by the axioms of multi-endigs, the same valencies and the same multiplicities as the corresponding end of G. Thus, by (a), M_X has a spanning tree T_X such that $m_{T_X}(\tau) = k(\tau)$ for every end $\tau \in \mathfrak{T}_{M_X}(G)$. Finally denote by E_X a subset of the set of edges of T_X which are incident with both B_X and $V_i(X)$, so that, for each component C of $T_X - B_X$, there is exactly one edge in E_X which is incident with C. And let F_X be the spanning forest of M_X whose set of edges is $E(T_X - B_X) \cup E_X$. Then clearly

$$T := T_{G_0} \cup \bigcup \{F_X : X \in \mathfrak{C}_{G_n - G_{n-1}}\}$$

is a spanning tree of G such that $m_T(\tau) = k(\tau)$ for every end τ of G. □

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