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ON THE STRUCTURE OF SEMILATTICE SUMS

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1. INTRODUCTION

This paper investigates the structure of algebras of given type $\tau: \Omega \to \boldsymbol{N}$ in regular classes, and in particular in regular classes of modes (see Section 2, and [RS2]). Recall that an identity is *regular* if the sets of variables appearing on each side are equal ([Pł1], [RS2]). A class $\mathscr{V}$ of algebras of type $\tau: \Omega \to \boldsymbol{N}$ is *regular* if the only identities satisfied by all algebras in $\mathscr{V}$ are regular. Otherwise, $\mathscr{V}$ is *irregular*. The algebras considered in this paper are of type $\tau: \Omega \to \boldsymbol{Z}^+$ with $2 \in \tau(\Omega)$. The general references for algebras of given type are [Co] and [RS2].

Let $(S, +)$ be a (join) semilattice, and Ω a set of operation symbols. Then $(S, +)$ may be considered as an Ω-algebra on setting

$$x_1 \dots x_n \omega = x_1 + \dots + x_n$$

for each n-ary ω in Ω with $n \geqq 2$ and

$$x\omega = x$$

for unary ω in Ω. Such an algebra is called an Ω-*semilattice* [RS2, p. 31]. Conversely, given an Ω-semilattice (S, Ω) one may define a binary operation $+$ on S by

$$x + y = xy \dots y\omega$$

for each n-ary ω in Ω with $n \geqq 2$. The class of all Ω-semilattices forms a variety, called $\mathscr{S}\ell$. It is well-known (cf. [RS2, Proposition 235]) that $\mathscr{S}\ell$ satisfies exactly all regular identities between Ω-words, and a variety $\mathscr{V}$ of Ω-algebras is regular if and only if $\mathscr{V}$ contains the variety $\mathscr{S}\ell$.

If (A, Ω) is an Ω-algebra in a regular class $\mathscr{V}$ of Ω-algebras, then (A, Ω) has a universal homorphism onto an Ω-semilattice, the so-called Ω-samilattice replica of (A, Ω) (cf. [Mal, p. 234] or Section 2 below). In Section 2, an explicit method is given for constructing such a universal homomorphism. This is done by means of some special subalgebras of an Ω-algebra, namely walls, algebraically open subalgebras, and sinks. Their properties, meaning and relationship (for Ω-algebras, and especially for modes) are discussed. The main Decomposition Theorem 2.3 of this section reads that for every Ω-algebra, its semilattice replica is the semilattice of its principal walls, and the corresponding fibres (classes of decomposition) are algebraically open subalgebras.

Section 3 is devoted to construction techniques known generically as "semilattice sums". These methods allow one to reconstruct algebras in regular classes from their semilattice replicas and the corresponding fibres. We recall the definitions and basic properties of the Płonka sum [Pł2], [RS2] and the coherent Lallement sum [L], [RS2]. We give a new, simpler proof of the theorem saying that each Ω-algebra in a regular class is a coherent Lallement sum over an Ω-semilattice homomorphic image. Finally, we give a condition under which a coherent Lallement sum over a semilattice $(S, \cdot)$ can be embedded into a Płonka sum over the same semilattice $(S, \cdot)$.

The results of Section 2 and 3 are then applied in Sections 4 and 5. In Section 4 we investigate Ω-algebras in a regular class, with fibres belonging to an irregular variety $\mathscr{V}$. We give a necessary and sufficient condition (Theorem 4.1) for certain algebras to be subalgebras of a Płonka sum of $\mathscr{V}$-algebras, and discuss some consequences of this theorem. We expect this theorem to be useful in solving the problem of describing the structure of algebras in regularised varieties. Recall that the *regularisation* or *regularised variety* $\mathscr{V}^{\sim}$ of a variety $\mathscr{V}$ of Ω-algebras is the class of Ω-algebras satisfying all the regular identities that are satisfied in $\mathscr{V}$. The structure of $\mathscr{V}^{\sim}$-algebras is known in some special cases. If $\mathscr{V}$ is a so called *strongly irregular* variety [DG], i.e. $\mathscr{V}$ satisfies an identity

(1.1) $\qquad x \circ y = x$

for some binary Ω-word $x \circ y$ (containing both variables x and y), then $\mathscr{V}^{\sim}$ consists exactly of Płonka sums of algebra in $\mathscr{V}$. Note that such a strongly irregular variety has a basis for its identities consisting of some set of regular identitites and the unique irregular identity 1.1 (cf. [Me], [R2]). If $\mathscr{V}$ is an irregular variety of semigroups, then $\mathscr{V}^{\sim}$ consists of subalgebras of Płonka sums of $\mathscr{V}$-algebras (cf. [S1], [S2].) However, examples are known of varieties $\mathscr{V}^{\sim}$ of Ω-algebras that are not all necessarily Płonka sums or subalgebras of Płonka sums of $\mathscr{V}$-algebras.

If $\mathscr{V}$ is a regular variety of Ω-algebras, then the class of all fibres of $\mathscr{V}$-algebras does not necessarily form a subvariety. This is the case for barycentric algebras [RS1], [RS2], [RS3] and commutative binary modes [JK2], [RS2]. In both these classes, the fibres belong to the quasivariety of cancellative $\mathscr{V}$-algebras. Both barycentric algebras and commutative binary modes are modes in the sense of [RS2]. The main Theorem 5.5 of Section 5 states that each coherent Lallement sum of cancellative modes is a subalgebra of a Płonka sum of cancellative modes. In particular, the theorem holds for barycentric algebras (cf. [RS3]) and for commutative binary modes as discussed in Section 6. Commutative binary modes in regularised varieties $\mathscr{V}^{\sim}$ are just Płonka sums of $\mathscr{V}$-algebras (Theorem 6.2). The final Theorem 6.4 shows that each commutative binary mode is a subalgebra of a Płonka sum of commutative quasigroup modes.

2. DECOMPOSITION OF AN Ω-ALGEBRA OVER ITS SEMILATTICE REPLICA

In this section, the Decomposition Theorem 2.3 is formulated and proved. As its name implies, it describes how an Ω-algebra breaks up into smaller pieces. These pieces are indexed by a semillatice associated with the Ω-algebra. This semilattice is the $(\Omega -)$ *semilattice replica* [RS2, p. 17] (AR, Ω) of the Ω-algebra (A, Ω). The semilattice (AR, Ω) is the quotient (A^{ϱ}, Ω) of (A, Ω) by the *semilattice replica congruence* ϱ such that any Ω-homomorphism $f: (A, \Omega) \to (S, \Omega)$ from (A, Ω) to an Ω-semilattice (S, Ω) factors as $f = (\text{nat}\ \varrho) f'$ through a unique homomorphism $f' = (A^{\varrho}, \Omega) \to (S, \Omega)$. Thus the congruence ϱ identifies precisely those elements of A which are identified in all Ω-homomorphisms from (A, Ω) to Ω-semilattices.

The semilattice replica of an Ω-algebra may be given an explicit description in terms of "walls" of the Ω-algebra. Recall that a subset X of an Ω-algebra (A, Ω) is a subalgebra of (A, Ω) iff

$$(2.1) \qquad \forall \omega \in \Omega\,, \quad \forall x_1, \ldots, x_n \in A\,, \quad (x_1, \ldots, x_n \in X) \Rightarrow (x_1 \ldots x_n\omega \in X)\,.$$

A subset W of A is said to be a *wall* [RS, p. 61] of (A, Ω) iff

$$(2.2) \qquad \forall \omega \in \Omega\,, \quad \forall x_1, \ldots, x_n \in A\,, \quad (x_1, \ldots, x_n \in X) \Leftrightarrow (x_1 \ldots x_n\omega \in X)\,.$$

Thus walls are special subalgebras. Each Ω-algebra has the improper wall (A, Ω). If (A, Ω) is a convex set considered as a barycentric algebra [RS2, RS3], then the walls in the sense of (2.2) are just the walls in the geometric sense [Mi, p. 8]. (Some authors also use the term "face" here, although the word "face" may be used with different meanings. Compare also the concept of "filter" as used in semigroup theory [Pe, I.8.2].) The set of walls of an Ω-algebra (A, Ω) is partially ordered by inclusion. From (2.2), the intersection of a family of walls is again a wall. For a subset X of A, let $[X]$ denote the intersection of all walls of (A, Ω) containing X. Then the set of walls of (A, Ω) forms a join semilattice under the operation $W + W' = [W \cup W']$. For a singleton $X = \{x\}$, write $[\{x\}] = [x]$. Such walls are called *principal walls*.

Lemma 2.1. *For any $x_1, \ldots, x_n$ in A and n-ary ω in Ω, one has* $[x_1] + \ldots + [x_n] = [x_1 \ldots x_n\omega]$.

Proof. On the one hand $x_1 \ldots x_n\omega \in [x_1 \ldots x_n\omega]$ implies $x_1, \ldots, x_n \in [x_1 \ldots x_n\omega]$ by (2.2), so $[x_1], \ldots, [x_n] \subseteq [x_1 \ldots x_n\omega]$ and $[x_1] + \ldots + [x_n] \leqq_+ [x_1 \ldots x_n\omega]$. On the other hand, since $x_1, \ldots, x_n$ are contained in the subalgebra $[x_1] + \ldots + [x_n]$, one has $x_1 \ldots x_n\omega \in [x_1] + \ldots + [x_n]$, whence $[x_1 \ldots x_n\omega] \leqq_+ [x_1] + \ldots + [x_n]$. ■

In particular, it follows that for x and y in A and n-ary ω in Ω, one has $[x] + [y] = [x] + [y] + \ldots + [y] = [xy \ldots y\omega]$. This shows that the principal walls of the Ω-algebra (A, Ω) form a subsemilattice $(W, +)$ of the join semilattice of all walls. Moreover, it shows that the mapping

$$(2.3) \qquad A \to W\,; \quad x \mapsto [x]$$

is an Ω-homomorphism from the Ω-algebra (A, Ω) onto the Ω-semilattice (W, Ω) of principal walls.

If (A, Ω) is a barycentric algebra, then walls also have a topological significance. If the affine hull of a convex set is a finite-dimensional Euclidean space, then the convex set is open in its affine hull iff it has no proper non-empty walls (see [RS2, Exercise 386], cf. also [N, 4.4]). This motivates the following. An Ω-algebra is said to be *algebraically open* iff it has no proper non-empty walls.

Proposition 2.2. *Let (A, Ω) be an Ω-algebra. Then the following conditions are equivalent*:

(i) *(A, Ω) is algebraically open*;

(ii) *(A, Ω) has no proper non-empty walls*;

(iii) *there is no Ω-epimorphism $(A, \Omega) \to (\{0, 1\}, \Omega)$ onto the two-element join semilattice $\{0 \leqq_+ 1\}$.*

Proof. If $|A| \leqq 1$, the equivalence is clear, so assume $|A| > 1$. The equivalence of (i) and (ii) is by definition. Now (ii) implies (iii) since a proper non-empty wall W furnishes an Ω-epimorphism $(W \to \{0\}) \cup ((A - W) \to \{1\})$. Conversely, (iii) implies (ii) since the preimage of 0 under an Ω-epimorphism $(A, \Omega) \to (\{0 \leqq_+ 1\}, \Omega)$ is a proper non-empty wall of (A, Ω). ■

Theorem 2.3. (Decomposition Theorem). *The semilattice replica of an Ω-algebra (A, Ω) is its semilattice of principal walls. The fibres over the semilattice are algebraically open subalgebra of (A, Ω).*

Proof. Let (A, Ω) be an Ω-algebra. Let σ be the kernel of the homomorphism (2.3), so that $x \, \sigma \, y$ iff $[x] = [y]$. Then σ contains the semillatice replica congruence ϱ. The ϱ-classes are subalgebras of (A, Ω), and have no non-trivial semilattice quotients. Thus they are algebraically open by Proposition 2.2. It remains to show that σ actually coincides with ϱ.

Now for b in A, one has

$$(2.4) \qquad \forall a \in b^\varrho, \quad [a] = [b^\varrho];$$

certainly $a \in b^\varrho$ implies $[a] \leqq [b^\varrho]$. Conversely, note that $[a] \cap b^\varrho$ is a non-empty wall of b^ϱ, since $x_1 \dots x_n\omega \in [a] \cap b^\varrho$ with $x_1, \dots, x_n \in b^\varrho$ implies $x_1, \dots, x_n \in [a]$, so $x_1, \dots, x_n \in [a] \cap b^\varrho$. But b^ϱ, being algebraically open, has no proper non-empty walls. Thus $[a] \cap b^\varrho = b^\varrho$, whence $b^\varrho \subseteqq [a]$ and $[b^\varrho] \leqq [a]$, completing the verification of (2.4).

Also, considering the semilattice replica (A^ϱ, Ω) as a join semilattice $(A^\varrho, \leqq_+)$, one has

$$(2.5) \qquad [b^\varrho] = \bigcup \{c^\varrho \mid c^\varrho \leqq_+ b^\varrho\}$$

for each b in A. Certainly $c^\varrho \leqq_+ b^\varrho$ implies $cb \dots b\omega \in c^\varrho + b^\varrho = b^\varrho \subseteqq [b^\varrho]$, whence $c \in [b^\varrho]$, so that $[b^\varrho] \supseteqq \bigcup \{c^\varrho \mid c^\varrho \leqq_+ b^\varrho\}$. On the other hand, the right hand side of (2.5) is the preimage in (A, Ω) under nat ϱ of the principal wall $\{c^\varrho \mid c^\varrho \leqq_+ b^\varrho\}$ of the semilattice (A^{ϱ}, Ω) generated by b^{ϱ}. Now preimages of walls under epimorphisms are walls, so that $[b^{\varrho}]$ is contained in the right hand side of (2.5).

To complete the proof of Theorem 2.3, assume $x\ \sigma\ y$ in A. Then by (2.4), one has $[x^{\varrho}] = [x] = [y] = [y^{\varrho}]$. The expression (2.5) then shows $x^{\varrho} \leqq_+ y^{\varrho}$ and $y^{\varrho} \leqq_+ x^{\varrho}$, whence $x^{\varrho} = y^{\varrho}$ or $x\ \varrho\ y$. Thus σ is also contained in ϱ, so that ϱ and σ do indeed coincide. ■

We conclude this section with some consideration of the possibility of replacing the concept of "wall" by the concept of "sink" in the Decomposition Theorem. A *sink* [RS2, p. 73] S in an Ω-algebra (A, Ω) is a subset S of A satisfying

(2.6) $$\forall \omega \in \Omega\,, \quad \forall x_1, \ldots, x_n \in A\,, \quad (\exists 1 \leqq i \leqq n.\ x_i \in S) \Rightarrow (x_1 \ldots x_n\omega \in S)\,.$$

In semigroup theory and binary mode theory, sinks have been called "ideals" (see [Pe, p. 4] and [JK2, p. 12]). Poyatos [Po] uses the term "trunk" rather than "ideal" or "sink". Sinks are automatically subalgebras. The sink (A, Ω) of an algebra (A, Ω) is described as *improper*, and all other sinks are called *proper*. An algebra is called *impermeable* [RS2, p. 74] if it has no proper non-empty sinks. Now from the defining property of sinks, it follows immediately that for each set X of sinks of the algebra (A, Ω), the union $\bigcup \{S \mid S \in X\}$ and intersection $\bigcap \{S \mid S \in X\}$ are also sinks of (A, Ω) [Po, Prop. 1]. For a given element a of (A, Ω), let (a) denote the *principal sink* generated by a, the intersection $\bigcap \{S \mid a \in S\}$ of the sinks S of (A, Ω) containing $\{a\}$.

Note that, in general, the complement of a wall in an Ω-algebra is a sink. Conversely, although the complement of a sink need not be a subalgebra, the complement is a wall if it is a subalgebra. Sinks satisfying the latter property are called *prime*. Equivalently, a prime sink may be described as a sink satisfying the following property:

(2.7) $$\forall \omega \in \Omega\,, \quad \forall x_1, \ldots, x_n \in A\,, \quad (x_1 \ldots x_n\omega \in S) \Rightarrow (\exists 1 \leqq i \leqq n.\ x_i \in S)\,.$$

Theorem 2.3 can be formulated using prime sinks rather than principal walls.

Now let (A, Ω) be a mode as defined in [RS2], i.e. (A, Ω) is *idempotent* (each singleton subset $\{a\}$ of an algebra (A, Ω) is a subalgebra of (A, Ω)) and *entropic*, (the mapping $\omega' \colon (A^m, \Omega) \to (A, \Omega)$ is an Ω-homomorphism, or equivalently the identity

(2.8) $$(x_{11} \ldots x_{1m}\omega') \ldots (x_{n1} \ldots x_{nm}\omega')\,\omega = (x_{11} \ldots x_{n1}\omega) \ldots (x_{1m} \ldots x_{nm}\omega)\,\omega'$$

is satisfied in (A, Ω) for each n-ary ω in Ω).

If (A, Ω) is a mode, then in the Decomposition Theorem 2.3 one may replace the semilattice of principal walls by the semilattice of principal sinks. To prove this, let us first note that the following lemma, an analogue of Lemma 2.1 for walls, follows by Proposition 364 and Corollary 366 in [RS2].

Lemma 2.4. *For any $x_1, \ldots, x_n$ in A and n-ary ω in Ω, one has* $(x_1) \cap \ldots \cap (x_n) = (x_1 \ldots x_n\omega)$. ■

In particular, it follows that for x and y in A and n-ary ω in Ω, one has $(x) \cap (y) = (x) \cap (y) \cap \ldots \cap (y) = (xy \ldots y\omega)$. This shows that the principal sinks of the mode (A, Ω) form a subsemilattice $(S, \cap)$ of the meet semilattice of all sinks. Moreover, it shows that the mapping

$$A \to S\,; \quad x \mapsto (x) \tag{2.9}$$

is an Ω-homomorphism from the mode (A, Ω) onto the Ω-semilattice (S, Ω) of principal sinks.

If (A, Ω) is a mode, then Proposition 2.2 can be rewritten as follows.

Proposition 2.5. *Let (A, Ω) be a mode. Then the following conditions are equivalent*:

(i) *(A, Ω) is algebraically open*;
(ii) *(A, Ω) has no proper non-empty walls*;
(iii) *(A, Ω) has no proper non-empty sinks*;
(iv) *(A, Ω) is impermeable*;
(v) *there is no Ω-epimorphism $(A, \Omega) \to (\{0, 1\}, \Omega)$ onto the two-element join semilattice $\{0 \leqq_+ 1\}$.*

Proof. We need only prove that (iii) and (v) are equivalent. Now (v) implies (iii), since a proper non-empty sink S contains a proper principal sink (s) for s in S, whence the homomorphism (2.9) maps (A, Ω) onto a non-trivial semilattice that always has a homomorphism onto the two element semilattice. Conversely (iii) implies (v), since the preimage of 1 under an Ω-epimorphism $(A, \Omega) \to (\{0 \leqq_+ 1\}, \Omega)$ is a proper non-empty sink of (A, Ω). ∎

By Proposition 2.5, it follows that for a mode (A, Ω) the semilattices of principal walls and principal sinks are isomorphic. We do not know whether this true in general. Consider, for example, algebras in the Mal'cev product [Ma2] of an irregular variety $\mathscr{V}$ of Ω-algebras and the variety $\mathscr{S}\ell$ of Ω-semilattices. Denote the Mal'cev product of $\mathscr{V}$ and $\mathscr{S}\ell$ by $\mathscr{V} \circ \mathscr{S}\ell$, and recall that this is the class of all Ω-algebras (A, Ω) having a homomorphism $h\colon (A, \Omega) \to (S, \Omega)$ onto an Ω-semilattice (S, Ω) such that the corresponding fibres $h^{-1}(s)$, for s in S, belong to the variety $\mathscr{V}$. No such fibre has an Ω-epimorphism onto the two-element semilattice (cf. [RS2, Proposition 235]). So the condition (v) of Proposition 2.5 is satisfied. But we do not know if this is equivalent to (iii). If Ω-algebras in the Decomposition Theorem 2.3 belong to the class $\mathscr{V} \circ \mathscr{S}\ell$, the words "principal walls" may be replaced by the words "sinks generated by the fibres". However we do not know if sinks generated by fibres are principal in general.

If $\mathscr{V}$ is a strongly irregular variety, the following holds.

Proposition 2.6. [R3] *Let $\mathscr{V}$ be a strongly irregular variety of Ω-algebras. Let $h\colon (A, \Omega) \to (S, \Omega)$ be a homomorphism of (A, Ω) onto its Ω-semilattice replica (S, Ω) with the corresponding fibres $A_s = h^{-1}(s)$, s in S, in the variety $\mathscr{V}$. Then* *a subset B of A is a sink in (A, Ω) if and only if $B = \bigcup (A_t \mid t \in T)$, where T is a sink in (S, Ω).* ■

Corollary 2.7. *For any $x_1, \ldots, x_n$ in A and n-ary ω in Ω, one has $(x_1) \cap \ldots \cap (x_n) = (x_1 \ldots x_n \omega)$.*

Proof. It is enough to notice that for x_i in A_{k_i}, where $i = 1, \ldots, n$ and $k = k_1 + \ldots + k_n$, $(x_1) \cap \ldots \cap (x_n) = (\bigcap (A_j \mid j \geqq k_1)) \cap \ldots \cap (\bigcap (A_j \mid j \geqq k_n)) = (A_j \mid j \geqq k)$. ■

It follows that Proposition 2.5 holds for (A, Ω) as described in Proposition 2.6.

3. SEMILATTICE SUMS

Semilattices form the basis for a range of general algebraic construction techniques, known generically as "semilattice sums". The most elegant of these is the Płonka sum [RS2, Definition 236], introduced by Płonka [Pł1] under the name "sum of a direct system" as a generalization of A. H. Clifford's "strong semilattice of semigroups" [C], [Pe, I.8.7]. The description of Płonka sum in [RS2, 2.3] was based on meet semilattices, but for current purposes it is more natural to use join semilattices $(S, +)$, simultaneously considered as Ω-semilattices. As a partial order $(S, \leqq_+)$, the set S is the set of objects of a small category (S) having a unique morphism $x \to y$ if and only if $x \leqq_+ y$. Consider the category (Ω) whose objects are Ω-algebras, algebras (A, Ω) of a fixed type $\Omega \to \boldsymbol{N}$ [RS2, p. 5]. The morphisms of (Ω) are Ω-homomorphisms. Suppose given a covariant functor $F\colon (S) \to (\Omega)$. Then the *Płonka sum* of the algebras sF (for s in S) is the disjoint union $SF = \dot{\bigcup} \{sF \mid s \in S\}$ of the underlying sets, equipped with the Ω-algebra structure given by

$$\omega\colon s_1F \times \ldots \times s_nF \to s_1 \ldots s_n\omega F ;\tag{3.1}$$
$$(x_1, \ldots, x_n) \mapsto x_1(s_1 \to s_1 \ldots s_n\omega)\, F \ldots x_n(s_n \to s_1 \ldots s_n\omega)\, F\omega$$

for each n-ary ω in Ω and $s_1, \ldots, s_n$ in S. Note that there is an Ω-algebra homomorphism $\pi_F\colon (SF, \Omega) \to (S, \Omega)$, called the *projection*, having restrictions $\pi_F\colon (sF, \Omega) \to (\{s\}, \Omega)$. The subalgebra (sF, Ω) of the Płonka sum (SF, Ω) are called the *Płonka fibres*. Płonka sums give a way of constructing new Ω-algebras from given ones. As was proved by Płonka [Pł1] (see also [RS2, p. 34]), the identities satisfied by a Płonka sum over a non-trivial semilattice are precisely the regular identities satisfied by each of the fibres. On the other hand, the concept of Płonka sum provides the following structure theorem for algebras in the regularisation of an irregular variety.

Theorem 3.1 [Pł1, 2], [RS2, p. 35]. *Let $\mathscr{V}$ be a strongly irregular variety of Ω-algebras of fixed type. Then an Ω-algebra (A, Ω) is in the regularisation $\mathscr{V}^\sim$ of $\mathscr{V}$ if and only if it is a Płonka sum of $\mathscr{V}$-algebras over its Ω-semilattice replica (S, Ω).* ■

The conditions on a Płonka sum, although natural, are very strong, and do not obtain in general. It is not true that each Ω-algebra projecting onto an Ω-semilattice (S, Ω) is a Płonka sum over that semilattice. Here are some counterexamples.

Example 3.2. Let $\widetilde{\mathscr{G}_{3,2}}$ (see [R2]) be the variety of groupoids satisfying the identities

(3.2) $\quad x \,.\, (x \,.\, y) = x \,.\, y$

(3.3) $\quad x \,.\, (y \,.\, z) = x \,.\, (z \,.\, y)$

(3.4) $\quad (x \,.\, y) \,.\, z = (x \,.\, (y \,.\, z))^2$

(3.5) $\quad x \,.\, (y \,.\, (x \,.\, z)) = x \,.\, (y \,.\, z)$

(3.6) $\quad ((x \,.\, x) \,.\, (x \,.\, x)) \,.\, ((x \,.\, x) \,.\, (x \,.\, x)) = (x \,.\, x) \,.\, (x \,.\, x)\,.$

By results of [R2], the variety $\widetilde{\mathscr{G}_{3,2}}$ is the regularisation of an irregular variety $\mathscr{G}_{3,2}$ of groupoids defined by the identities (3.2)–(3.6) and

(3.7) $\quad x \,.\, y = x \,.\, z\,.$

The fibres G_s of the decomposition of a groupoid $(G, \cdot)$ in $\mathscr{G}_{3,2}$ over its semilattice replica $(S, \cdot)$ are in $\mathscr{G}_{3,2}$. However, as was shown by Płonka [Pł2], $(G, \cdot)$ is not necessarily a Płonka sum of the $(G_s, \cdot)$. For example, if $(G, \cdot) = (F(2), \cdot)$ is the free groupoid in $\widetilde{\mathscr{G}_{3,2}}$ on two generators x and y, then $(G, \cdot)$ may not be reconstructed as a Płonka sum of its fibres over its semilattice replica. (See [Pł2]). ■

Example 3.3. Let I and I^0 be the closed and open unit intervals $[0, 1]$ and $]0, 1[$ respectively. For each p in I^0, define a binary operation $\underline{p}$ on I as follows

(3.8) $\quad ab\underline{p} = a(1 - p) + bp.$

Then (I, I^0) is a barycentric algebra (as defined in [RS2]). The semilattice replica (S, I^0) of (I, I^0) has three elements, say $0, 1, a$, with $0, 1 <_+ a$. The corresponding fibres are $I_0 = \{0\}, I_1 = \{1\}$ and $I_a = I^0$. The algebra (I, I^0) cannot be reconstructed as a Płonka sum of I_0, I_1 and I_a. Indeed, if $0(0 \to a)\, F = x \in I^0$, then for any y in I^0 and p in I^0, $yp = 0y\underline{p} = 0(0 \to a)\, F\; y\underline{p} = yx\underline{p} = x(1 - p) + yp$, a contradiction. ■

Example 3.4. Replace the closed and open intervals I and I^0 from Example 3.3 by the closed and open unit intervals D and D^0 of dyadic numbers, i.e. $D = \{x \in I \mid x = k\, 2^{-n},\ k \in \boldsymbol{Z},\ n \in \boldsymbol{N}\}$, $D^0 = \{x \in I^0 \mid x = k\, 2^{-n},\ k \in \boldsymbol{Z},\ n \in \boldsymbol{N}\}$. It is known (see [Ku, V.1]) that in this case the operations defined by (3.8) may be reduced to the unique operation $\frac{1}{2}$. The groupoid $(D, \frac{1}{2})$ is a commutative binary mode (as defined in [RS], or "commutative idempotent medial groupoid" in the terminology of [JK2]). Using a similar argument to that in Example 3.3, one can show that the mode $(D, \frac{1}{2})$ (that is equivalent to (D, D^0)) decomposes over the 3-element semilattice replica $(S, +)$ of Example 3.3, and cannot be reconstructed as a Płonka sum over $(S, +)$. ■

In [RS2, Definition 623], a more general construction method called the *Lallement sum* was introduced, extending and adapting some semigroup-theoretical work of Lallement [L, 2.19]. In fact, as was shown by [RS2, Theorem 624], these Lallement sums have extremely broad applicability. The full generality of Lallement sums is not required here, but the concepts underlying them are needed.

Let (T, Ω) be a sink in an Ω-algebra (A, Ω). A congruence θ on (A, Ω) is said *to preserve* the sink (T, Ω) if the restriction of the natural projection $A \to A^{\theta}$; $a \mapsto a^{\theta}$ to the subalgebra (T, Ω) injects. The algebra (A, Ω) is said to be an *envelope* of a subalgebra (T, Ω) if (T, Ω) is a sink of (A, Ω) such that equality is the only congruence on (A, Ω) preserving (T, Ω). For example, the closed unit interval (I, I^0) is an envelope of the open unit interval (I^0, I^0).

The version of Lallement sums to be used here is as follows (cf. [RS2, 6.2]).

Definition 3.5. Let S be an Ω-semilattice (S, Ω) and a join semilattice $(S, +, \leqq_+)$. Suppose given an envelope (E_s, Ω) of an Ω-algebra (A_s, Ω) for each element s in S. For each $s \leqq_+ t$ in $(S, \leqq_+)$, suppose given an Ω-homomorphism $\phi_{s,t}$: $(A_s, \Omega) \to$ $\to (E_t, \Omega)$ such that:

(a) $\phi_{s,s}$ is the injection of A_s into E_s;

(b) for each n-ary ω in Ω, and for $s_1, \ldots, s_n$ in S with $s_1 \ldots s_n\omega = s$,

$$(A_{s_1}\phi_{s_1,s}) \ldots (A_{s_n}\phi_{s_n,s})\, \omega \subseteqq A_s \, ;$$

(c) for each $s_1 \ldots s_n\omega = s \leqq_+ t$ in S and a_i in A_{s_i} (for $i = 1, \ldots, n$),

$$a_1\phi_{s_1,s} \ldots a_n\phi_{s_n,s}\omega\phi_{s,t} = a_1\phi_{s_1,t} \ldots a_n\phi_{s_n,t}\omega;$$

(d) for each s in S,

$$E_s = \{a\phi_{t,s} \mid t \leqq_+ s,\ a \in A_t\} \, .$$

Then the disjoint union $A = \dot{\bigcup} (A_s \mid s \in S)$ equipped with the operations

$$(3.9) \qquad \omega\colon A_{s_1} \times \ldots \times A_{s_n} \to A_s \, ; \quad (a_1, \ldots, a_n) \mapsto a_1\phi_{s_1,s} \ldots a_n\phi_{s_n,s}\omega \, ,$$

where $s = s_1 \ldots s_n\omega$, for each n-ary ω in Ω, is called the *coherent Lallement sum of the algebras* (A_s, Ω) *over the semillatice* (S, Ω) *by the mappings* $\phi_{s,t}$, or more briefly a *coherent Lallement sum.* ■

Note that the left hand side of the equality in (c) is defined, since condition (b) holds. The conditions (a)–(d) are best viewed as generalizations of the functoriality in Płonka sums, where the envelopes coincide with their sinks. A coherent Lallement sum has a *projection* π: $(A, \Omega) \to (S, \Omega)$, an Ω-homomorphism with restrictions $(A_s, \Omega) \to (\{s\}, \Omega)$. The subalgebras (A_s, Ω) of (A, Ω) are also called the *fibres* of the Lallement sum.

The significance of coherent Lallement sums resides in Theorem 624 of [RS2]. The special version of the theorem formulated here has a simpler proof.

Theorem 3.6. *Let* (A, Ω) *be an* Ω*-algebra having a homomorphism onto an*

Ω-semilattice (S, Ω) with corresponding fibres (A_s, Ω), $s \in S$. Then (A, Ω) is a coherent Lallement sum of (A_s, Ω) over (S, Ω).

Proof. The proof is based on the following observation (cf. Lemma 1.10 in [R1]). If (T, Ω) is a sink in an Ω-algebra (A, Ω), and θ a congruence of (A, Ω) preserving (T, Ω), then (A^θ, Ω) is an envelope of (T, Ω) iff θ is a maximal congruence of (A, Ω) preserving (T, Ω).

Now let $P_s = \bigcup \{A_t \mid t \leqq_+ s\}$. Obviously, P_s is a subalgebra of (A, Ω), and contains (A_s, Ω) as a sink. Let θ be the set of congruences on (P_s, Ω) preserving (A_s, Ω). The set θ is nonempty, since it contains the equality relation. Each chain of congruences θ_i $(i \in I)$ in θ is bounded above by the congruence $\bigcup \{\theta_i \mid i \in I\}$. By Zorn's Lemma, θ has a maximal element $\mu = \mu(s)$. By the observation at the beginning of the proof, $(E_s, \Omega) = (P_s^\mu, \Omega)$ is an envelope of (A_s, Ω). (Here and later, we identify the μ-class containing a_s in A_s with the element a_s.) For all $s \leqq_+ t$ in S, and a_s in A_s, define

$$\phi_{s,t}\colon A_s \to E_t\,; \quad a_s \mapsto a_s^{\mu(t)}\,.$$

Obviously, each mapping $\phi_{s,s}$ is an injection. For each n-ary ω in Ω, with $s_1 \ldots s_n\omega = s$ and a_i in A_{s_i} for $i = 1, \ldots, n$, one has $a_1\phi_{s_1,s} \ldots a_n\phi_{s_n,s}\omega = a_1^\mu \ldots a_n^\mu\omega = a_1 \ldots a_n\omega^\mu = a_1 \ldots a_n\omega \in A_s$. It follows that the conditions (a) and (b) of Definition 3.5 are satisfied. Now for each $s_1 \ldots s_n\omega = s \leqq_+ t$, one has $a_1\phi_{s_1,s} \ldots a_n\phi_{s_n,s}\omega\phi_{s,t} = a_1^{\mu(s)} \ldots a_n^{\mu(s)}\omega\phi_{s,t} = a_1 \ldots a_n\omega^{\mu(s)}\phi_{s,t} = a_1 \ldots a_n\omega\phi_{s,t} = a_1 \ldots a_n\omega^{\mu(t)} = a_1^{\mu(t)} \ldots a_n^{\mu(t)}\omega = a_1\phi_{s_1,t} \ldots a_n\phi_{s_n,t}\omega$, whence condition (c) of Definition 3.5 holds as well. The conditions (d) and (3.9) are obviously satisfied. In particular, (c) implies that the $\phi_{s,t}$ are homomorphisms. ∎

The next example of a coherent Lallement sum is in a sense opposite to the Płonka sum of (A_s, Ω) over (S, Ω), where the envelopes (E_s, Ω) of (A_s, Ω) coincide with (A_s, Ω).

Example 3.7. Let (A, Ω) be an Ω-algebra with semilattice replica (S, Ω) and corresponding fibres (A_s, Ω). Additionally, for each s in S, suppose that the only congruence on (P_s, Ω) preserving (A_s, Ω) is the equality relation. Then for each s in S, the envelope (E_s, Ω) of (A_s, Ω) is isomorphic to (P_s, Ω). Obviously, (A, Ω) cannot be reconstructed as a Płonka sum of (A_s, Ω) over (S, Ω). ∎

In the proof of Theorems 4.1 and 5.5 to be given in Section 4 and 5, a certain additional condition on coherent Lallement sums is considered. This condition is:

$$\text{(3.10)} \qquad \forall r, r' \leqq_+ s \leqq_+ t\,, \quad \forall a \in A_r\,, \quad \forall b \in A_{r'}\,,$$
$$a\phi_{r,s} = b\phi_{r',s} \Rightarrow a\phi_{r,t} = b\phi_{r',t}\,.$$

The significance of condition (3.10) resides in the following

Proposition 3.8. *If a coherent Lallement sum satisfies condition* (3.10), *then it is a subalgebra of a Płonka sum of its envelopes.*

Proof. A functor $F\colon (S) \to (\Omega)$ will be defined. For s in S, the object sF is the envelope (E_s, Ω). For $s \leqq_+ t$, a morphism $(s \to t) F: E_s \to E_t$ is needed. Now by (d) of Definition 3.5, each element x of E_s is of the form $a\phi_{r,s}$ for a in A_r with $r \leqq_+ s$. Define $x(s \to t) F$ to be $a\phi_{r,t}$. This is certainly an element of E_t. The definition is good, since if $x = a\phi_{r,s} = b\phi_{r',s}$ for b in $A_{r'}$, with $r' \leqq_+ s$, then $a\phi_{r,t} = b\phi_{r',t}$ by condition (3.10). To see that $(s \to t) F$ is an Ω-homomorphism, consider $x_1 = a_1\phi_{r_1,s}, \ldots, x_n = a_n\phi_{r_n,s}$ in E_s, where $r_1, \ldots, r_n \leqq_+ s$. Then for an n-ary ω in Ω,

$$x_1(s \to t) F \ldots x_n(s \to t) F\omega = a_1\phi_{r_1,s} \ldots a_n\phi_{r_n,s}\omega =$$
$$= a_1\phi_{r_1,r_1\ldots r_n\omega} \ldots a_n\phi_{r_n,r_1\ldots r_n\omega}\omega\phi_{r_1\ldots r_n\omega,t} =$$
$$= a_1\phi_{r_1,r_1\ldots r_n\omega} \ldots a_n\phi_{r_n,r_1\ldots r_n\omega}\omega\phi_{r_1\ldots r_n\omega,s}(s \to t) F =$$
$$= a_1\phi_{r_1,s} \ldots a_n\phi_{r_n,s}\omega(s \to t) F = x_1 \ldots x_n\omega(s \to t) F\,,$$

the second and fourth equalities holding by (c) of Definition 3.5, while the third holds by the definition of $(s \to t) F$. Thus $(s \to t) F$ is indeed an Ω-homomorphism. Also $x(s \to s) F = a\phi_{r,s} = x$, so $(s \to s) F$ is the identity on sF. Now suppose $s \leqq_+ t \leqq_+ u$ in S. Then $x(s \to t) F\, (t \to u) F = a\phi_{r,t}(t \to u) F = a\phi_{r,u} = x(s \to u)F$, completing the verification that $F: (S) \to (\Omega)$ is a functor. It remains to check that the Lallement sum (A, Ω) is a subalgebra of the Płonka sum by the functor F. Consider elements $a_1 = a_1\phi_{r_1,r_1}, \ldots, a_n = a_n\phi_{r_n,r_n}$ of A. Then for an n-ary ω in Ω, $a_1 \ldots a_n\omega$ in the Płonka sum is calculated as

$$a_1(r_1 \to r_1 \ldots r_n\omega) F \ldots a_n(r_n \to r_1 \ldots r_n\omega) F\omega =$$
$$= a_1\phi_{r_1,r_1\ldots r_n\omega} \ldots a_n\phi_{r_n,r_1\ldots r_n\omega}\omega\,,$$

which by (3.9) is just $a_1 \ldots a_n\omega$ in the Lallement sum. ■

4. EMBEDDING OF ALGEBRAS IN $\mathscr{V} \circ \mathscr{S}\ell$ INTO PŁONKA SUMS

Let $\mathscr{V}$ be an irregular variety of Ω-algebras of a fixed type. Let $\mathscr{S}\ell$ denote the variety of Ω-semilattices. By the considerations at the end of Section 2, it follows that each Ω-algebra (A, Ω) in $\mathscr{V} \circ \mathscr{S}\ell$ has a natural decomposition over its semilattice replica, in which the fibres are members of the variety $\mathscr{V}$. By Theorem 3.6, (A, Ω) may be reconstructed as a coherent Lallement sum of the fibres. This section characterizes those Ω-algebras in $\mathscr{V} \circ \mathscr{S}\ell$ that are subalgebras of Płonka sums of $\mathscr{V}$-algebras. The class of subalgebras of Płonka sums of V-algebras is a subclass of the regularisation $\mathscr{V}^\sim$ of $\mathscr{V}$. If $\mathscr{V}$ is a strongly irregular variety, then the following three classes coincide: (a) the regularisation $\mathscr{V}^\sim$ of $\mathscr{V}$; (b) the class of Płonka sums of V-algebras; and (c) the class of subalgebras of Płonka sums of $\mathscr{V}$-algebras. This is not true for an arbitrary variety of Ω-algebras. As was proved by Saliĭ (see [S1] and [S2]), if $\mathscr{V}$ is an irregular variety of semigroups, then $\mathscr{V}^\sim$ is the class (c) of subalgebras of Płonka sums of $\mathscr{V}$-algebras, which does not necessarily coincide with the class (b) of Płonka sums of $\mathscr{V}$-algebras. In [GKW], it was shown that in general the regularisation $\mathscr{V}^\sim$ of an irregular variety $\mathscr{V}$ does not necessarily consist of subalgebras of

Płonka sums of $\mathscr{V}$-algebras. An example was given of an algebra with one binary and one unary operation in the regularisation $\mathscr{V}^{\sim}$ of the variety $\mathscr{V}$ defined by the unique identity $xy = x^2$, such that the algebra cannot be reconstructed as a subalgebra of a Płonka sum of $\mathscr{V}$-algebras.

In what follows we assume that $\mathscr{V}$ is defined by a set of regular identities and the identity $x \circ y = x \circ z$.

Theorem 4.1. *Let (A, Ω) be an Ω-algebra in the class $\mathscr{V}^{\sim}$. Let (S, Ω) be its Ω-semilattice replica, with corresponding fibres (A_s, Ω) $(s \in S)$ in the variety $\mathscr{V}$. Then (A, Ω) is a subalgebra of a Płonka sum of $\mathscr{V}$-algebras if and only if (A, Ω) is a coherent Lallement sum of (A_s, Ω) over (S, Ω) by mappings $\phi_{p,r}$ that satisfy the additional conditions*:

(i) $a\phi_{r,s} = b\phi_{r',s} \Rightarrow a\phi_{r,t} = b\phi_{r',t}$;

(ii) $(a \circ b)\, \phi_{r+r',s} = (a \circ a)\, \phi_{r,s}$

for a in A_r, b in $A_{r'}$, r, r', s, t in S with $r, r' \leqq_+ s \leqq_+ t$.

Proof. $(\Leftarrow)$ Note that the condition (i) is just (3.10). By Proposition 3.8, (A, Ω) is a subalgebra of the Płonka sum of the envelopes (E_s, Ω) of (A_s, Ω) over (S, Ω). Now for $x = a\phi_{r,s}$ and $y = b\phi_{r',s}$ in E_s, by Lemma 4.2 of [R2] and the condition (ii), one has that $x \circ y = a\phi_{r,s} \circ b\phi_{r',s} = (a \circ b)\, \phi_{r+r',s} = (a \circ a)\, \phi_{r,s} = a\phi_{r,s} \circ$ $\circ\, a\phi_{r,s} = x \circ x$. It follows that all (E_s, Ω) belong to the variety $\mathscr{V}$.

$(\Rightarrow)$ Let (A, Ω) be a subalgebra of a Płonka sum (TF, Ω) of algebras tF (for t in T) in $\mathscr{V}$ by a functor $F\colon (T) \to (\Omega)$. Obviously, (S, Ω) is a subsemilattice of (T, Ω). For each s in S, define

$$E_s = \{a(r \to s)\, F \mid r, s \in S,\ r \leqq_+ s,\ a \in A_r\}\,.$$

Let $p_1, \ldots, p_n, p, s$ be in S with $p = p_1 + \ldots + p_n \leqq_+ s$. For a_i in A_{p_i}, $i = 1, \ldots, n$, and n-ary ω in Ω, $a_1(p_1 \to s)\, F \ldots a_n(p_n \to s)\, F\omega = a_1(p_1 \to p)\, F \ldots a_n(p_n \to p)\, F\, \omega$ $(p \to s)\, F = a_1 \ldots a_n\, \omega\, (p \to s)\, F$ is in E_s, whence E_s is a subalgebra of (sF, Ω). Obviously, A_s is a subalgebra of (E_s, Ω). Moreover A_s is a sink in (E_s, Ω). Indeed, for a in A_s, $a_1(p_1 \to s)\, F \ldots a_{i-1}(p_{i-1} \to s)\, F\, a\, a_{i+1}(p_{i+1} \to s)\, F \ldots a_n(p_n \to s)\, F\omega =$ $= a_1 \ldots a_{i-1} a\, a_{i+1} \ldots a_n \omega$ is in A, and hence in A_s.

Now let $\mu(s)$ be the relation on $P_s = \bigcup \{A_r \mid r \leqq_+ s\}$ defined by

$$a\, \mu(s)\, b \quad \text{if} \quad a(p \to s)\, F = b(q \to s)\, F$$

for $p, q \leqq_+ s$, a in A_p, b in A_q. It is easy to check that $\mu(s)$ is a congruence on (P_s, Ω) preserving (A_s, Ω). Suppose $\lambda \geqq \mu(s)$ is a congruence on (P_s, Ω) preserving (A_s, Ω), and $a\, \lambda\, b$ for a in A_p, b in A_q and $p, q \leqq_+ s$. Then since $a\, \lambda\, a(p \to s)\, F$ and $b\, \lambda\, b(q \to s)\, F$, it follows that $a(p \to s)\, F = b(q \to s)\, F$, implying that $\lambda \leqq \mu(s)$. Therefore $\lambda = \mu(s)$, and $\mu(s)$ is maximal preserving (A_s, Ω). It follows that (E_s, Ω) is an envelope of (A_s, Ω).

For r and s in S with $r \leqq_+ s$, define $\phi_{r,s}\colon A_r \to E_s$ by $a\phi_{r,s} = a(r \to s)\, F$. It is easy to see that the mappings $\phi_{r,s}$ satisfy all the conditions (a)–(d) of Definition 3.5. Now for a in A_r, b in $A_{r'}$, r, r', s, t in S with $r, r' \leqq_+ s \leqq_+ t$, one has that $a\, \phi_{r,s} =$

$= b\ \phi_{r',s}$ implies that

$$a\ \phi_{r,t} = a(r \to t)\ F = a(r \to s)\ F(s \to t)\ F = b(r' \to s)\ F(s \to t)\ F =$$
$$= b'(r' \to t)\ F = b\ \phi_{r',t} \quad \text{and} \quad (a \circ b)\ \phi_{r+r',s} =$$
$$= (a \circ b)\ (r \to s)\ F = a(r \to s)\ F \circ b(r \to s)\ F =$$
$$= a(r \to s)\ F \circ a(r \to s)\ F = (a \circ a)\ (r \to s)\ F =$$
$$= (a \circ a)\ \phi_{r,s}\ .$$

The second equality follows by Lemma 4.2 in [R2]. Consequently, (A, Ω) is a coherent Lallement sum of (A_s, Ω) over (S, Ω) by mappings $\phi_{r,s}$ satisfying (i) and (ii). ■

Remark 4.2. In general, an Ω-algebra (A, Ω) in the variety $\widetilde{V}$ may be reconstructed as a coherent Lallement sum of $\mathscr{V}$-algebras (A_s, Ω) over its Ω-semilattice replica (S, Ω) by means of many systems of homomorphisms $\phi_{s,t}$. If, for each system, any of the conditions (i) and (ii) of Theorem 4.1 is not satisfied, then (A, Ω) cannot be embedded into a Płonka sum of $\mathscr{V}$-algebras. Note that the condition (ii) is not satisfied in the example given in [GKW].

Remark 4.3. Consider once more the variety $\mathscr{G}_{3,2}^{\sim}$ of groupoids defined in Example 3.2. The structure of groupoids in $\mathscr{G}_{3,2}^{\sim}$, and more generally, in the variety $\mathscr{G}^{\sim}$ defined by the identities (3.2)–(3.5), i.e. the regularisation of the variety $\mathscr{G}$ of groupoids defined by the unique identity $xy = xz$, was described in [R2] by means of certain special coherent Lallement sums. We give here an example of a groupoid in $\mathscr{G}_{3,2}^{\sim}$ that is not a subalgebra of a Płonka sum of $\mathscr{G}_{3,2}$-groupoids. The free $\mathscr{G}_{3,2}^{\sim}$-groupoid $(F(2), \cdot)$ on two generators x and y consists of elements $x, x^2, x^4, y, y^2, y^4, xy, (xy)^2, yx, (yx)^2$, where $x^2 = x\,.\,x$ and $x^4 = x^2\,.\,x^2$. $(F(2), \cdot)$ decomposes over its semilattice replica into the classes $\{x, x^2, x^4\}$, $\{y, y^2, y^4\}$, and $\{xy, yx, (xy)^2, (yx)^2\}$. It is easy to see that the set $A = \{x, x^2, x^4, xy, (xy)^2\}$ is a subalgebra of $(F(2), \cdot)$.

Let $(B, \cdot)$ be a groupoid with multiplication defined by the following table

	a	a^2	b	b^2
a	a^2	a^2	b	b
a^2	a^2	a^2	b^2	b^2
b	b^2	b^2	b^2	b^2
b^2	b^2	b^2	b^2	b^2

It is easy to check that the mapping $h: A \to B$ defined by $xh = a$, $x^2h = x^4h = a^2$, $(xy)\,h = b$, $(xy)^2\,h = b^2$ is a homomorphism. Hence the groupoid $(B, \cdot)$ is in the variety $\mathscr{G}_{3,2}^{\sim}$. The semilattice replica of $(B, \cdot)$ consists of two elements 0 and 1 with $0 <_+ 1$, the corresponding fibres being $B_0 = \{a, a^2\}$ and $B_1 = \{b, b^2\}$. Let $\phi_{0,1}$: $B_0 \to E_1$ be any Lallement homomorphism. The elements $a\ \phi_{0,1}$ and $a^2\ \phi_{0,1}$ are not in B_1. Indeed, if $a^2\ \phi_{0,1}$ were in B_1, then

$$b = a\ b = a\ \phi_{0,1}\ b = a\ \phi_{0,1}\ a\ \phi_{0,1} = a^2\ \phi_{0,1} = (a^2a^2)\ \phi_{0,1} =$$
$$= a^2\ \phi_{0,1}\ a^2\ \phi_{0,1} = b^2\ ,$$

a contradiction. If $a\,\phi_{0,1}$ were in B_1, then $a^2\,\phi_{0,1}$ would also be in B_1. Consequently, neither $a\,\phi_{0,1}$ nor $a^2\,\phi_{0,1}$ is in B_1. Now $a^2\,\phi_{0,1} \notin B_1$ and $(ab)\,\phi_{1,1} = ab = b \in B_1$. It follows that the condition (ii) of Theorem 4.1 is not satisfied, and $(B, \cdot)$ cannot be embedded into a Płonka sum of $\mathscr{G}_{3,2}$-groupoids. ∎

Remark 4.4. More generally, if for each decomposition of an Ω-algebra (A, Ω) into a coherent Lallement sum of $\mathscr{V}$-algebras (A_s, Ω) over the semilattice replica (S, Ω), one has $(a \circ a)\,\phi_{r,s} \notin A_s$ for some r, s in S with $r <_+ s$ and a in A_r, then $(a \circ a)\,\phi_{r,s} \neq a \circ b = (a \circ b)\,\phi_{s,s}$ for any b in A_s. It follows that (A, Ω) cannot be a subalgebra of Płonka sum of $\mathscr{V}$-algebras. In particular, the Ω-algebras considered in Example 3.7, as well as those of Examples 3.3 and 3.4, are not subalgebras of Płonka sums of $\mathscr{V}$-algebras. ∎

Remark 4.5. A Lallement sum is said to be *strict* if $E_s = A_s$ for each s in S. (See [RS2]. p. 136.) If (A, Ω) is a coherent strict Lallement sum of (A_s, Ω) over (S, Ω) by $\phi_{r,s}$, then the condition (ii) of Theorem 4.1 is always satisfied, since in this case

$$(a \circ b)\,\phi_{r+r',s} = a\phi_{r,s} \circ b\,\phi_{r',s} = a\,\phi_{r,s} \circ a\,\phi_{r,s} = (a \circ a)\,\phi_{r,s}\,.$$

The condition (i) guarantees that (A, Ω) is a Płonka sum of (A_s, Ω). ∎

Remark 4.6. Let (A, Ω) satisfy the assumptions of Theorem 4.1, and let (A, Ω) be idempotent. Then the condition (ii) means that for $r' = s$, $a \circ b = (a \circ b)\,\phi_{s,s} = (a \circ a)\,\phi_{r,s} = a\,\phi_{r,s}$. If this holds, the envelope E_s coincides with the algebra A_s. It follows that if (A, Ω) is a subalgebra of a Płonka sum of $\mathscr{V}$-algebras (A_s, Ω), then (A, Ω) is itself a Płonka sum of the (A_s, Ω). ∎

5. EMBEDDING COHERENT LALLEMENT SUMS OF CANCELLATIVE MODES INTO PŁONKA SUMS

In this section we consider Ω-modes (A, Ω) of a fixed type belonging to the Mal'cev product $\mathscr{C} \circ \mathscr{S}\ell$ where $\mathscr{C}$ is the class of all cancellative Ω-modes. Recall that an Ω-algebra (A, Ω) is said to be *cancellative* [RS, p. 149] if for each n-ary ω in Ω, $a_1, \ldots, a_n$, b, c in A and each $i = 1, \ldots, n$,

$$a_1 \ldots a_{i-1}\, b\, a_{i+1} \ldots a_n\, \omega = a_1 \ldots a_{i-1}\, c\, a_{i+1} \ldots a_n\, \omega \tag{5.1}$$
$$\text{implies that} \quad b = c\,.$$

Note that if $\mathscr{V}$ is a variety contained in $\mathscr{C} \circ \mathscr{S}\ell$ and $\mathscr{C}' = \mathscr{C} \cap \mathscr{V}$, then $\mathscr{C}'$ is a quasi-variety, but in general not necessarily a variety.

In this section, it will be shown that each Ω-mode in the class $\mathscr{C} \circ \mathscr{S}\ell$ is a subalgebra of a Płonka sum of $\mathscr{C}$-modes. In particular, if $\mathscr{V}$ is a variety contained in $\mathscr{C} \circ \mathscr{S}\ell$, and the class $\mathscr{V}'$ of $\mathscr{V}$-algebras having no nontrivial semilattice homomorphic image is contained in $\mathscr{C}'$, then each $\mathscr{V}$-algebra is a subalgebra of a Płonka sum of $\mathscr{C}'$-algebras. In [RS3] we have proved that each barycentric algebra is a subalgebra of a Płonka sum of convex sets. Convex sets are precisely the cancellative barycentric algebras. In Section 6, we use results of this section to prove some structure theorems of similar type for commutative binary modes.

Let (A, Ω) be a mode in the class $\mathscr{C} \circ \mathscr{S}\ell$. Suppose given a homomorphism of (A, Ω) onto an Ω-semilattice (S, Ω) with corresponding fibres (A_s, Ω) in $\mathscr{C}$ for s in S. Set $P_s = \bigcup \{A_t \mid t \leqq_+ s\}$. Suppose given a *subtype* σ, a mapping $\sigma\colon \Omega \to N$ with $\omega\sigma \leqq \omega\tau$ for each ω in Ω. Define a relation $\mu = \mu(s, \sigma)$ on P_s by:

(5.2) $$b\mu c \Leftrightarrow \forall \omega \in \Omega\,, \quad \forall a_1, \ldots, a_{\omega\sigma-1}, a_{\omega\sigma+1}, \ldots, a_{\omega\tau} \in A_s\,,$$
$$a_1 \ldots a_{\omega\sigma-1}\, b\, a_{\omega\sigma+1} \ldots a_{\omega\tau}\omega = a_1 \ldots a_{\omega\sigma-1}\, c\, a_{\omega\sigma+1} \ldots a_{\omega\tau}\omega\,.$$

Lemma 5.1. *For each subtype σ, the relation $\mu(s, \sigma)$ is the largest congruence on (P_s, Ω) preserving (A_s, Ω).*

Proof. It is obvious that $\mu(s, \sigma)$ is an equivalence relation.

Now for $j = 1, \ldots, m$, let $t_j \leqq_+ s$ and $u_j \leqq_+ s$. Let $b_j \in A_{t_j}$ and $c_j \in A_{u_j}$. Suppose $b_j\, \mu\, c_j$, i.e. for each n-ary ω in Ω, $a_1, \ldots, a_n$ in A_s, $a_1 \ldots a_{\omega\sigma-1}\, b_j\, a_{\omega\sigma+1} \ldots a_n\, \omega = a_1 \ldots a_{\omega s-1}\, c_j\, a_{\omega\sigma+1} \ldots a_n\omega$. Then by the idempotent and entropic laws,

$$\begin{aligned} a_1 \ldots a_{\omega\sigma-1}(b_1 \ldots b_{\omega'\tau}\omega')\, a_{\omega\sigma+1} \ldots a_n\omega &= (a_1 \ldots a_1\omega') \ldots \\ \ldots (a_{\omega\sigma-1} \ldots a_{\omega\sigma-1}\omega')\,(b_1 \ldots b_{\omega'\tau}\omega')\,&(a_{\omega\sigma+1} \ldots a_{\omega\sigma+1}\omega') \ldots \\ \ldots (a_n \ldots a_n\omega')\, \omega &= (a_1 \ldots a_{\omega\sigma-1}\, b_1\, a_{\omega\sigma+1} \ldots a_n\omega) \ldots \\ \ldots (a_1 \ldots a_{\omega\sigma-1} b_{\omega'\tau} a_{\omega\sigma+1} \ldots a_n\omega)\, \omega' &= \\ = (a_1 \ldots a_{\omega\sigma-1}\, c_1\, a_{\omega\sigma+1} \ldots a_n\omega) \ldots \\ \ldots (a_1 \ldots a_{\omega\sigma-1} c_{\omega'\tau} a_{\omega\sigma+1} \ldots a_n\omega)\, \omega' &= \\ = a_1 \ldots a_{\omega\sigma-1}(c_1 \ldots c_{\omega'\tau}\omega')\, a_{\omega\sigma+1} \ldots a_n\omega\,. \end{aligned}$$

It follows that μ is a congruence on (P_s, Ω). By the cancellativity of (A_s, Ω), it is immediate that μ preserves (A_s, Ω).

Let λ be another congruence on (P_s, Ω) preserving (A_s, Ω). Let $b\, \lambda\, c$ for b in A_t, c in A_u, where t, $u \leqq_+ s$. Then for $a_1, \ldots, a_n$ in A_s and n-ary ω in Ω, $a_1 \ldots a_{\omega\sigma-1}\, b\, a_{\omega\sigma+1} \ldots a_n\omega\, \lambda\, a_1 \ldots a_{\omega\sigma-1}\, c\, a_{\omega\sigma+1} \ldots a_n\omega$. But since both these elements are in A_s, and λ preserves (A_s, Ω), it follows that $a_1 \ldots a_{\omega\sigma-1}\, b\, a_{\omega\sigma+1} \ldots \ldots a_n\omega = a_1 \ldots a_{\omega\sigma-1}\, c\, a_{\omega\sigma+1} \ldots a_n\omega$. Consequently $b\, \mu\, c$, and μ is the largest congruence on (P_s, Ω) preserving (A_s, Ω). ■

By Lemma 5.1, it follows that $\mu = \mu(s)$ does not depend on the choice of the subtype σ in Definition (5.2).

Proposition 5.2. *The mode (A, Ω) is a coherent Lallement sum of (A_s, Ω) over (S, Ω). For each s in S, $(E_s, \Omega) = (P_s, \Omega)^{\mu(s)}$ is an envelope of (A_s, Ω). The homomorphisms $\phi_{r,s}$ are given by*

$$\phi_{r,s}\colon A_r \to E_s\,, \quad a \mapsto a^{\mu(s)}\,.$$

Proof. The proof is analogous to the proof of Theorem 3.6. The only difference is that for each s in S, one takes as a maximal congruence on (P_s, Ω) the congruence defined by (5.2). ■

Lemma 5.3. *For each s in S, the envelope (E_s, Ω) of (A_s, Ω) is cancellative.*

Proof. Let $a_1, \ldots, a_n$ be in A_{s_j}, $j = 1, \ldots, n$, and let $s_j, t, u \leqq_+ s$, $b \in A_t$ and $c \in A_u$. For n-ary ω in Ω, let $(a_1 \ldots a_{j-1} b a_{j+1} \ldots a_n\omega)^\mu = (a_1 \ldots a_{j-1} c a_{j+1} \ldots a_n\omega)^\mu$, i.e. for each m-ary ω' in Ω, $d_1, \ldots, d_m$ in A_s, and $1 \leqq i \leqq m$,

$$d_1 \ldots d_{i-1}(a_1 \ldots a_{j-1}\, b\, a_{j+1} \ldots a_n\omega)\, d_{i+1} \ldots d_m\omega' =$$
$$= d_1 \ldots d_{i-1}(a_1 \ldots a_{j-1}\, c\, a_{j+1} \ldots a_n\omega)\, d_{i+1} \ldots d_m\omega' .$$

By the idempotent and entropic laws,

$$d_1 \ldots d_{i-1}(a_1 \ldots a_{j-1}\, x\, a_{j+1} \ldots a_n\omega)\, d_{i+1} \ldots d_m\omega' =$$
$$= (d_1 \ldots d_{i-1} a_1 d_{i+1} \ldots d_m\omega') \ldots (d_1 \ldots d_{i-1} a_{j-1} d_{i+1} \ldots d_m\omega')$$
$$(d_1 \ldots d_{i-1}\, x\, d_{i+1} \ldots d_m\omega')(d_1 \ldots d_{i-1} a_{j+1} d_{i+1} \ldots d_m\omega') \ldots$$
$$\ldots (d_1 \ldots d_{i-1} a_n d_{i+1} \ldots d_m\omega')\, \omega ,$$

where $x = b$ or $x = c$. Moreover for $k = 1, \ldots, n$ and $a_j = x$, each element $d_1 \ldots d_{i-1}\, a_k\, d_{i+1} \ldots d_m\omega'$ belongs to A_s. Since (A_s, Ω) is cancellative, it follows that $d_1 \ldots d_{i-1}\, b\, d_{i+1} \ldots d_m\omega' = d_1 \ldots d_{i-1}\, c\, d_{i+1} \ldots d_m\omega$, whence $b^\mu = c^\mu$. Consequently, (E_s, Ω) is cancellative. ■

Lemma 5.4. *Let s, t, u, v be in S and $u, v \leqq_+ t \leqq_+ s$. Let e be in A_u and f in A_v. If $e^{\mu(t)} = f^{\mu(t)}$, then $e^{\mu(s)} = f^{\mu(s)}$.*

Proof. Let $a_1, \ldots, a_n$ be in A_t, $b_1, \ldots, b_m$ in A_s. Let Ω be n-ary and ω' m-ary in Ω. Let

$$a_1 \ldots a_{i-1}\, e\, a_{i+1} \ldots a_n\omega = a_1 \ldots a_{i-1}\, f\, a_{i+1} \ldots a_n\omega .$$

Then by the idempotent and entropic laws,

$$b_1 \ldots b_{j-1}(a_1 \ldots a_{i-1}\, x\, a_{i+1} \ldots a_n\omega)\, b_{j+1} \ldots b_m\omega' =$$
$$= (b_1 \ldots b_{j-1}\, a_1\, b_{j+1} \ldots b_m\omega') \ldots (b_1 \ldots b_{j-1} a_{i-1} b_{j+1} \ldots b_m\omega')$$
$$(b_1 \ldots b_{j-1}\, x\, b_{j+1} \ldots b_m\omega')(b_1 \ldots b_{j-1} a_{i+1} b_{j+1} \ldots b_m\omega') \ldots$$
$$\ldots (b_1 \ldots b_{j-1}\, a_n\, b_{j+1} \ldots b_m\omega')\, \omega ,$$

where $x = e$ or $x = f$. Moreover for $k = 1, \ldots, n$ and $a_i = x$, each element $b_1 \ldots \ldots b_{j-1}\, a_k\, b_{j+1} \ldots b_m\omega'$ belongs to A_s. Since (A_s, Ω) is cancellative, it follows that

$$b_1 \ldots b_{j-1}\, e\, b_{j+1} \ldots b_m\omega' = b_1 \ldots b_{j-1}\, f\, b_{j+1} \ldots b_m\omega' ,$$

whence $e^{\mu(s)} = f^{\mu(s)}$. ■

Now the main theorem of this section can be proved easily.

Theorem 5.5. *Let (A, Ω) be an Ω-mode in the class $\mathscr{C} \circ \mathscr{S}\ell$, with a homomorphism onto an Ω-semilattice (S, Ω) with corresponding fibres (A_s, Ω) in the class $\mathscr{C}$. Then*

(A, Ω) *is a subalgebra of a Płonka sum of* $\mathscr{C}$*-modes over the same* Ω*-semilattice* (S, Ω).

Proof. By Proposition 5.2, (A, Ω) is a coherent Lallement sum of (A_s, Ω) with envelopes $(E_s, \Omega) = (P_s, \Omega)^\mu$ over $(S, \cdot)$ by mappings $\phi_{r,s}\colon A_r \to E_s$, $a \mapsto a^\mu$. By Lemma 5.3, these envelopes are cancellative. By Lemma 5.4, the condition (3.10) is satisfied. Now Theorem 5.5 follows by Proposition 3.8. ■

6. STRUCTURE THEOREMS FOR COMMUTATIVE BINARY MODES

In this section we use the previous results to describe the structure of commutative binary modes (called commutative idempotent medial groupoids in [JK1], [JK2]), algebras $(A, \cdot)$ with a commutative, idempotent and entropic multiplication. Let $\mathscr{C}\ell m$ be the variety of all such modes. Each mode $(A, \cdot)$ in the variety $\mathscr{C}\ell m$ has a decomposition into algebraically open submodes $(A_s, \cdot)$ over the semilattice $(S, \cdot)$ of principal walls, its semilattice replica. By Theorem 3.6, (A, Ω) is a coherent Lallement sum of (A_s, Ω) over $(S, \cdot)$. However, the full classification of algebraically open commutative binary modes is not known. (For some partial results, see [JK2].) By Proposition 2.5, the semilattice of principal walls of $(A, \cdot)$ is isomorphic to the semilattice of principal sinks, and the algebraically open fibres do not contain proper non-empty sinks. By Proposition 1.4 in [JK2], each commutative binary mode not containing a proper non-empty sink is cancellative. This leads to the following structure theorem for commutative binary modes, an easy consequence of Theorem 5.5.

Theorem 6.1. *Each commutative binary mode is a subalgebra of a Płonka sum of cancellative commutative binary modes.* ■

Note that the class of cancellative binary modes is a quasivariety, but not a variety.

For modes in non-trivial subvarieties of the variety $\mathscr{C}\ell m$, one can give a more precise structure theorem. Before formulating it, we recall some facts about the lattice of subvarieties of $\mathscr{C}\ell m$. The lattice of varieties of commutative binary modes was described in [JK3], and it was shown there that it decomposes into two disjoint parts: an "irregular" one consisting of all irregular varieties, and a "regular" one consisting of the variety $\mathscr{C}\ell m$ of all commutative binary modes and the regularisations of all the irregular varieties. The regular part is a principal filter in the lattice of subvarieties of $\mathscr{C}\ell m$ generated by the variety of semilattices. If $(A, \cdot)$ is in an irregular variety, then $(A, \cdot)$ has no epimorphism onto the two-element semilattice. By Proposition 2.5 it follows that $(A, \cdot)$ is algebraically open. Moreover, $(A, \cdot)$ has the structure of a commutative quasigroup mode (as defined in [RS2, p. 93]). (Cf. [D], [R1], [RS1].) The structure of commutative quasigroup modes is well known (see for instance [JK1], [JK2], [RS2, 4.3]). They are equivalent to unital $Z[\frac{1}{2}]$-modules. By results of Ježek, Kepka [JK3], it is known that each irregular variety $\mathscr{V}$ of commutative binary modes is strongly irregular. This, together with Theorem 3.1, gives the following structure theorem for commutative binary modes in non-trivial subvarieties of $\mathscr{C}\mathscr{B}m$.

Theorem 6.2. *Each commutative binary mode in a non-trivial subvariety $\mathscr{V}$ of $\mathscr{C}\mathscr{B}m$ is a Płonka sum of commutative quasigroup modes in the largest irregular variety contained in $\mathscr{V}$.* ∎

Note that if $(A, \cdot)$ is in the variety $\mathscr{C}\mathscr{B}m$, but not in any non-trivial subvariety of $\mathscr{C}\mathscr{B}m$, then the fibres of the decomposition over the semilattice replica of $(A, \cdot)$ are not necessarily quasigroups. (Cf. [JK1].) However, as we will show in the final structure theorem, such groupoids may be embedded into a Płonka sum of commutative qusigroup modes. The proof is based on the following Theorem 6.3 due to Ježek and Kepka [JK2, Thm 5.3.1, p. 54]. At first recall that a non-empty subset A of a groupoid $(B, \cdot)$ is *closed* if $a \in A$, $b \in B$ and $ab \in A$ or $ba \in A$ imply that $b \in A$. Also A is *dense* in $(B, \cdot)$ if $(B, \cdot)$ is the only closed subgroupoid of $(B, \cdot)$ containing A.

Theorem 6.3. *Let $(A, \cdot)$ be entropic and cancellative. Then there is a homomorphism $f: (A, \cdot) \to (Q, \cdot)$ embedding A into an entropic quasigroup $(Q, \cdot, /, \backslash)$. The quasigroup Q has the following properties:*

(i) *A is dense in $(Q, \cdot)$;*

(ii) *if $g: (A, \cdot) \to (B, \cdot)$ is a homomorphism into an entropic quasigroup $(B, \cdot, /, \backslash)$, then there is a unique homomorphism $h: (Q, \cdot) \to (B, \cdot)$ such that $fh = g$.*

(iii) *$(A, \cdot)$ and $(Q, \cdot)$ satisfy the same identities.* ∎

Theorem 6.4. *Each commutative binary mode is a subalgebra of a Płonka sum of commutative quasigroup modes.*

Proof. Let $(A, \cdot)$ be a commutative binary mode, with fibres $(A_s, \cdot)$ over its semilattice replica $(S, \cdot)$. By Theorems 5.5 and 6.1, $(A, \cdot)$ is a subalgebra of a Płonka sum of cancellative commutative binary modes $(E_s, \cdot)$ over $(S, \cdot)$ by the functor $F: (S) \to (\{\cdot\})$ defined in Proposition 3.8. Each $(E_s, \cdot)$ is an envelope of $(A_s, \cdot)$. Now a new functor $G: (S) \to (\{\cdot\})$ will be defined. By Theorem 6.3, for each s in S there is an embedding $f_s: (E_s, \cdot) \to (Q_s, \cdot)$ into a uniquely determined commutative quasigroup mode $(Q_s, \cdot, /, \backslash)$ satisfying the conditions (i)–(iii). For each s in S, the object sG is the quasigroup $(Q_s, \cdot)$. Now if, for $s \leqq_+ t$, $(s \to t)F: (E_s, \cdot) \to$ $\to (E_t, \cdot) \leqq (Q_t, \cdot)$, then by the condition (ii), there is a unique homomorpism $h_{s,t}: (Q_s, \cdot) \to (Q_t, \cdot)$ such that $f_s h_{s,t} = (s \to t)F$. Define $(s \to t)G: (Q_s, \cdot) \to (Q_t, \cdot)$ to be $h_{s,t}$. It is clear that $(s \to s)G$ is the identity on $(Q_s, \cdot)$. Now let $s \leqq_+ t \leqq_+ u$. Then by Theorem 6.3, since $f_s h_{s,t} h_{t,u} = (s \to u)F = f_s h_{s,u}$, it follows that $h_{s,t} h_{t,u} =$ $= h_{s,u}$, completing the verification that $G: (S) \to (\{\cdot\})$ is a functor. It remains to check that the Lallement sum $(A, \cdot)$ is a subalgebra of the Płonka sum by the functor G. Consider elements $a_1 = a_1\phi_{s_1,s_1}, \ldots, a_n = a_n\phi_{s_n,s_n}$ of A. Then for an n-ary ω

in Ω, $a_1 \dots a_n\omega$ in the Płonka sum is calculated as

$$a_1(s_1 \to s_1 \dots s_n\omega)\, G \dots a_n(s_n \to s_1 \dots s_n\omega)\, G\omega =$$
$$= a_1\phi_{s_1,s_1\dots s_n\omega} \dots a_n\phi_{s_n,s_1\dots s_n\omega}\omega$$

which by (3.9) is just $a_1 \dots a_n\omega$ in the Lallement sum.

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References

[C] *A. H. Clifford:* Semigroups admitting relative inverses, Ann. Math. *42* (1941), 1037—1049.

[Co] *P. M. Cohn:* Universal Algebra, Reidel, Dordrecht, 1981.

[D] *J. Dudek:* Varieties of idempotent commutative groupoids, Fund. Math. *120* (1983), 193—204.

[DG] *J. Dudek, E. Graczyńska:* The lattice of varieties of algebras. Bull. Acad. Polon. Sci. Sér. Sci. Math. *29* (1981), 337—340.

[GKW] *E. Graczyńska, D. Kelly, P. Winkler:* On the regular part of varieties of algebras, Algebra Universalis *23* (1986), 77—84.

[JK1] *J. Ježek, T. Kepka:* Ideal free CIM-groupoids and open convex sets, Springer Lecture Notes in Mathematics 1004, Universal Algebra and Lattice Theory, 1983, 166—175.

[JK2] *J. Ježek, T. Kepka:* Medial Groupoids, Rozpravy ČSAV, Rada mat. a přir. ved. *93/2* (1983), Academia, Praha.

[JK3] *J. Ježek, T. Kepka:* The lattice of varieties of commutative idempotent abelian distributive groupoids, Algebra Universalis *5* (*1975*), 225—237.

[Ku] *M. Kuczma:* An introduction to the Theory of Functional Equations and Inequalities, P.W.N, Warszawa, 1985.

[L] *G. Lallement:* Demi-groupes réguliers, Ann. Math. Pura Appl. *77* (1967), 47—129.

[Ma1] *A. I. Mal'cev:* Algebraic Systems, Springer-Verlag, Berlin 1973.

[Ma2] *A. I. Mal'cev:* Multiplication of classes of algebraic systems (Russian), Sibirsk. Math. Zh. *8* (1967), 346—365.

[Me] *I. I. Mel'nik:* Normal closures of perfect varieties of universal algebras (Russian), Ordered sets and lattices, Izdat. Saratov. Univ., Saratov (1971), 56—65, MR *52* # 3011.

[Mi] *B. Melnik:* Theory of filters, Commun. Math. Phys. *15* (1969), 1—46.

[W] *W. D. Neumann:* On the quasivariety of convex subsets of affine spaces, Arch. Math. *21* (1970), 11—16.

[Pe] *M. Petrich:* Lectures in Semigroups, Akademie-Verlag, Berlin, 1977.

[Pł1] *J. Płonka:* On a method of construction of abstract algebras, Fund. Math. *61* (1967), 183—189.

[Pł2] *J. Płonka:* On equational classes of abstract algebras defined by regular equations, Fund. Math. *64* (1969), 241—247.

[Po] *F. Poyatos:* Generalizacion de un teorema de J. A. Green a algebras universales, Rev. Math. Hisp.-Amer. (4)40 (1980), 193—205.

[R1] *A. Romanowska:* Constructing and reconstructing of algebras, Demonstratio Math. *18* (1985), 209—230.

[R2] *A. Romanowska:* On regular and regularised varieties, Algebra Universalis *23* (1986), 215—241.

[RS1] *A. Romanowska, J. D. H. Smith:* From affine to projective geometry via convexity, Springer Lecture Notes in Mathematics 1149, Universal Algebra and Lattice Theory, 1985, 255—269.

[RS2] *A. Romanowska, J. D. H. Smith:* Modal Theory-an Algebraic Approach to Order, Geometry and Convexity, Heldermann Verlag, Berlin, 1985.

[RS3] *A. Romanowska, J. D. H. Smith:* On the structure of barycentric algebras, Houston J. Math., to appear.

[S1] *W. W. Salii:* Equationally normal varieties of semigroups (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. *84* (1969), 61—68, MR 41 # 8555.

[S2] *W. W. Salii:* A theorem on homomorphisms of strong semilattices of semigroups (Russian), Theory of semigroups and its applications, V. V. Vagner (ed.), Izd. Saratov. Univ. *2* (1970), 69—74, MR 53 # 10959.

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