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HAKE'S PROPERTY OF A MULTIDIMENSIONAL GENERALIZED RIEMANN INTEGRAL

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Throughout, $m \geqq 1$ is a fixed integer. The set of all real numbers is denoted by $\boldsymbol{R}$, and the m-fold cartesian product of $\boldsymbol{R}$ is denoted by $\boldsymbol{R}^m$. For $x = (\xi_1, \ldots, \xi_m)$ in $\boldsymbol{R}^m$ we let $|x| = \max\{|\xi_1|, \ldots, |\xi_m|\}$, and in $\boldsymbol{R}^m$ we use exclusively the metric induced by the norm $|x|$. The distance from a point $x \in \boldsymbol{R}^m$ to a set $E \subset \boldsymbol{R}^m$ is denoted by $\operatorname{dist}(x, E)$. If $E \subset \boldsymbol{R}^m$, then E^0, ∂E, $d(E)$, and $|E|_k$, $k = 0, \ldots, m$, denote, respectively, the interior, boundary, diameter, and the k-dimensional Hausdorff measure of E (as is customary, a "measure" means an "outer measure"). We define the Hausdorff measure as in [1; Section 2.10.2, p. 171], so that $|E|_k$ is the counting measure of E if $k = 0$, and the k-dimensional Lebesgue measure of E if $k \geqq 1$ and $E \subset \boldsymbol{R}^k$. Instead of $|E|_m$ we write $|E|$.

A *k-plane*, $k = 0, \ldots, m - 1$, is a k-dimensional linear submanifold H of $\boldsymbol{R}^m$ which is parallel to k distinct coordinate axes. When the dimension k of H is not specified, we talk about a *plane* H, whose dimension is denoted by $\dim H$. A *regulator* is a pair $(\varepsilon, \mathscr{H})$ where $0 < \varepsilon < 1/2$ and $\mathscr{H}$ is a finite family of planes.

An *interval* is a set $A = \prod_{i=1}^m [a_i, b_i]$ where $a_i, b_i \in \boldsymbol{R}$ and $a_i < b_i$, $i = 1, \ldots, m$. Intervals are *nonoverlapping* if their interiors are disjoint. Given a regulator $(\varepsilon, \mathscr{H})$, we call an interval A an $(\varepsilon, \mathscr{H})$*-interval* whenever

$$\varepsilon < \sup\left\{\frac{|A \cap H|_{\dim H}}{[d(A)]^{\dim H}} : H \in \mathscr{H} \cup \{\boldsymbol{R}^m\},\ A \cap H \neq \emptyset\right\}.$$

A *figure* is a finite union of intervals. A *division* of a figure F is a finite collection $\{A_1, \ldots, A_n\}$ of nonoverlapping intervals whose union is F. A *partition* of an interval A is a set $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ such that $\{A_1, \ldots, A_p\}$ is a division of A and $x_i \in A_i$, $i = 1, \ldots, p$. When all $A_1, \ldots, A_p$ are $(\varepsilon, \mathscr{H})$-intervals for a regulator $(\varepsilon, \mathscr{H})$, then P is called an $(\varepsilon, \mathscr{H})$*-partition*. If δ is a positive function on A and $d(A_i) < \delta(x_i)$, $i = 1, \ldots, p$, we say that P is δ*-fine*.

Let f be a function on an interval A and let $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a partition of A. We set

$$\sigma(f, P) = \sum_{i=1}^{p} f(x_i)\,|A_i|$$

and say that f is *integrable* on A if there is a real number I satisfying the following condition: given a regulator $(\varepsilon, \mathscr{H})$, we can find a positive function δ on A so that $|\sigma(f, P) - I| < \varepsilon$ for each δ-fine $(\varepsilon, \mathscr{H})$-partition P of A. The number I, which is uniquely determined by f (cf. [6; Corollary 2.5]), is called the *integral* of f over A, denoted by $\int_A f$.

It was established in [6] that over intervals, the integral is a well behaved extension of the Lebesgue integral which yields a very general divergence theorem (see [6; Theorem 5.4]). Our goal is to show that under suitable conditions, the integrability on an interval A follows from the integrability on intervals contained in the interior of A. In dimension one, such a result was established by H. Hake for the Perron integral (cf. [2] and also [8; Chapter VIII, Lemma (3.1), p. 247]), whose Riemann-like definition was given by Kurzweil (see [6]) and independently by Henstock (see [3]). A direct proof for the one-dimensional generalized Riemann integral can be found, e.g., in [7; Lemma 7, (v)].

If C and D are nonempty compact subsets of $\boldsymbol{R}^m$, the positive real number

$$\varrho(C, D) = \max\{\sup_{x\in C} \operatorname{dist}(x, D), \sup_{y\in D} \operatorname{dist}(y, C)\}$$

is called the *Hausdorff distance* between C and D. We say that a sequence $\{F_n\}$ of figures *converges* to an interval A if $F_n \subset A^0$, $n = 1, 2, \ldots$, $\lim \varrho(F_n, A) = 0$, and $\sup |\partial F_n|_{m-1} < +\infty$.

Let f be a function on a set $E \subset \boldsymbol{R}^m$ such that f is integrable on each interval $A \subset E$. If $F \subset E$ is a figure, we set

$$\int_F f = \sum_{D\in\mathscr{D}} \int_D f$$

where $\mathscr{D}$ is a division of F. By the additivity of the integral (see [6; Proposition 3.6]), the number $\int_F f$ does not depend on the choice of $\mathscr{D}$. Using this observation, we can conveniently formulate our result.

Proposition. *Let f be a function on an interval A which is integrable on each interval $B \subset A^0$. If a finite $\lim \int_{F_n} f$ exists for every sequence $\{F_n\}$ of figures converging to A, then all these limits have the same value I, f is integrable on A, and $\int_A f = I$.*

Proof. If $\{B_n\}$ and $\{C_n\}$ are sequences of figures converging to A, then so is the sequence $\{B_1, C_1, B_2, C_2, \ldots\}$. By our assumption

$$\lim \int_{B_n} f = \lim \int_{C_n} f = I,$$

and it remains to show that f is integrable on A and $\int_A f = I$. According to [6; Corollary 4.2], we may assume that $f(x) = 0$ for each $x \in \partial A$.

Choose a regulator $(\varepsilon, \mathscr{H})$ with $\partial A \subset \bigcup \mathscr{H}$, and construct a locally finite countable family $\{K_1, K_2, \ldots\}$ of nonoverlapping intervals with $\bigcup_{n=1}^{\infty} K_n = A^0$. If $x \in A^0$ then the collection $\mathscr{K}_x = \{K_n : x \in K_n\}$ is finite and $x \in (\bigcup \mathscr{K}_x)^0$. For $n = 1, 2, \ldots$, let $\mathscr{H}_n$ be a finite collection of planes which is closed with respect to nonempty intersections and assume that $\mathscr{H} \subset \mathscr{H}_n$ and $\partial K_n \subset \bigcup \mathscr{H}_n$. By Henstock's lemma (see [6; Lemma 4.3]), there is a $\delta_n: K_n \to (0, 1)$ such that

$$\sum_{i=1}^{p} \left| f(x_i) |A_i| - \int_{A_i} f \right| < \frac{\varepsilon}{2^{n+1}}$$

for each δ_n-fine $(\varepsilon, \mathscr{H}_n)$-partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ of K_n, $n = 1, 2, \ldots$. Making the δ_n's smaller, we may assume that the following conditions are satisfied:

(1) $\delta_n(x) < \operatorname{dist}(x, H)$ for each $H \in \mathscr{H}_n$ and each $x \in K_n - H$;

(2) $\delta_n(x) < \operatorname{dist}(x, A - \bigcup \mathscr{K}_x)$ for each $x \in A^0$;

(3) $\delta_n(x) = \delta_{n'}(x)$ whenever $x \in K_n \cap K_{n'}$.

Let $h = \sum_{H \in \mathscr{H}} |A \cap H|_{\dim H}$, and find an $\eta \in (0, 1)$ so that $\left| \int_F f - I \right| < \varepsilon/2$ for each figure $F \subset A^0$ with $\varrho(A, F) < \eta$ and $|\partial F|_{m-1} < 2mh/\varepsilon$. In view of condition (3), we can define a $\delta: A \to (0, 1)$ by setting

$$\delta(x) = \begin{cases} \delta_n(x) & \text{if} \quad x \in K_n, \quad n = 1, 2, \ldots, \\ \eta & \text{if} \quad x \in \partial A. \end{cases}$$

Let P be a δ-fine $(\varepsilon, \mathscr{H})$-partition of A, and let

$$Q = \{(B, x) \in P : x \in \partial A\} \cup \bigcup_{x \in A^0} \{(B \cap K_n, x) : K_n \in \mathscr{K}_x,\ (B \cap K_n)^0 \neq \emptyset\}.$$

Then Q is a δ-fine partition of A and $\sigma(f, P) = \sigma(f, Q)$. Using [6; Lemma 3.5 and Corollary 2.5], it is easy to verify that the collection $Q_n = \{(C, x) \in Q : C \subset K_n\}$ is a subset of a δ_n-fine $(\varepsilon, \mathscr{H}_n)$-partition of K_n, $n = 1, 2, \ldots$; in particular,

$$\sum_{(C,x) \in Q_n} \left| f(x) |C| - \int_C f \right| < \frac{\varepsilon}{2^{n+1}}$$

by the choice of δ_n. The interval A is the union of nonoverlapping figures

$$F = \bigcup_{n=1}^{\infty} \left(\bigcup \{C : (C, x) \in Q_n\} \right) \quad \text{and} \quad F' = \bigcup \{B : (B, x) \in P,\ x \in \partial A\}.$$

Now $\varrho(A, F) < \eta$ and

$$|\partial F|_{m-1} \leqq |\partial F'|_{m-1} \leqq \sum \{|\partial B|_{m-1} : (B, x) \in P, \quad x \in \partial A\} \leqq \sum_{b \in \mathscr{B}} |\partial B|_{m-1},$$

where $\mathscr{B} = \{B : (B, x) \in P,\ B \cap (\bigcup \mathscr{H}) \neq \emptyset\}$. Since each $B \in \mathscr{B}$ is an $(\varepsilon, \mathscr{H})$-interval, there is a k_B-plane $H_B \in \mathscr{H}$ such that $B \cap H \neq \emptyset$ and $|B \cap H_B|_{k_B} / [d(B)]^{k_B} > \varepsilon$ (cf. [6; Section 2.1]). As $d(B) < 1$ and $k_B \leqq m - 1$, we see that

$$|\partial B|_{m-1} \leqq 2m[d(B)]^{m-1} \leqq 2m[d(B)]^{k_B} < \frac{2m}{\varepsilon} |B \cap H_B|_{k_B}$$

and consequently,

$$\sum_{B \in \mathscr{B}} |\partial B|_{m-1} < \frac{2m}{\varepsilon} \sum_{B \in \mathscr{B}} |B \cap H_B|_{k_B} \leqq \frac{2m}{\varepsilon} \sum_{B \in \mathscr{B}} \sum_{H \in \mathscr{H}} |B \cap H|_{\dim H} =$$

$$= \frac{2m}{\varepsilon} \sum_{H \in \mathscr{H}} \sum_{B \in \mathscr{B}} |B \cap H|_{\dim H} \leqq \frac{2m}{\varepsilon} h.$$

It follows that

$$\left|\sigma(f, P) - I\right| \leqq \left|\sum_{n=1}^{\infty} \sum_{(C,x)\in Q_n} f(x)\left|C\right| - \int_F f\right| + \left|\int_F f - I\right| <$$
$$< \varepsilon/2 + \sum_{n=1}^{\infty} \sum_{(C,x)\in Q_n} \left|f(x)\left|C\right| - \int_C f\right| < \varepsilon\,,$$

and the proof is completed.

Remark. The reader may compare the Proposition with [4; Theorem (6.3)].

If f is integrable on an interval A and $\{A_n\}$ is a sequence of *intervals* converging to A, it follows from [6; Proposition 4.10] that $\lim \int_{A_n} f = \int_A f$. However, in the Proposition a sequence of figures *cannot* be replaced by a sequence of intervals. Indeed, for $(x, y) \in \boldsymbol{R}^2$ set

$$f(x, y) = \begin{cases} |y|/xy & \text{if} \quad x \neq 0 \quad \text{and} \quad 0 < |y| < 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and let $A = [0, 1] \times [-2, 2]$. Then f is integrable on each interval $B \subset A^0$ and $\lim \int_{A_n} f = 0$ for every sequence $\{A_n\}$ of intervals converging to A. Yet, f is not integrable on A because it is not integrable on $[0, 1]^2$.

On the other hand, the next example shows that if f is integrable on an interval A and $\{F_n\}$ is a sequence of *figures* converging to A, a finite $\lim \int_{F_n} f$ may not exist.

Example. Let $m = 2$, $A = [0, 1]^2$, and for $k = 1, 2, \ldots$, let

$$A_{k_+} = [3\,.\,2^{-k-1}, 2^{-k+1}] \times [2^{-k-1}, 2^{-k}]\,,$$
$$A_{k_-} = [2^{-k-1}, 2^{-k}] \times [3\,.\,2^{-k-1}, 2^{-k+1}]\,.$$

By [6; Example 6.2], the function f defined by

$$f(x) = \begin{cases} \pm 2^{2(k+1)}/k & \text{if} \quad x \in A_{k_\pm}\,, \\ 0 & \text{if} \quad x \in A - \bigcup_{k=1}^{\infty} A_{k_\pm}\,, \end{cases}$$

is integrable on A. If $F_n = [2^{-2n-1}, 1]^2 - \bigcup_{k=n}^{\infty} (A_{k_-})^0$, then $\varrho(A, F_n) \leqq 2^{-n}$ and $|\partial F_n|_1 \leqq 6$, $n = 1, 2, \ldots$. Thus $\{F_n\}$ is a sequence of figures converging to A, and $\lim \int_{F_n} = +\infty$ since

$$\int_{F_n} f = \sum_{k=n}^{2n} \int_{A_{k+}} f = \sum_{k=n}^{2n} (1/n) \geqq \log n\,, \quad n = 1, 2, \ldots\,.$$

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