Czechoslovak Mathematical Journal

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*Czechoslovak Mathematical Journal*, Vol. 40 (1990), No. 4, 648–658

Persistent URL: http://dml.cz/dmlcz/102418

Czechoslovak Mathematical Journal, 40 (115) 1990, Praha

# ON DIRECTED INTERPOLATION GROUPS

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(Received April 10, 1989)

In this paper there are investigated radical classes of directed interpolation groups. Next, two open problems on abelian directed groups with countable interpolation (which have been proposed by K. R. Goodearl [8]) are dealt with.

Radical classes of lattice ordered groups have been introduced in [12] and they were further investigated in the papers [3], [4], [13] – [16], [18]. The radical classes of intropolation groups can be defined analogously (cf. Section 1 below).

We denote by $\mathscr{G}$ and $\mathscr{I}$ the class of all lattice ordered groups and the class of all directed interpolation groups, respectively. Next let $R(\mathscr{G})$ and $R(\mathscr{I})$ be the collection of all radical classes of lattice ordered groups or the collection of all radical classes of directed interpolation groups, respectively. Both $R(\mathscr{G})$ and $R(\mathscr{I})$ are partially ordered by inclusion.

Sample results: $R(\mathscr{G})$ fails to be a subcollection of $R(\mathscr{I})$; in particular, $\mathscr{G} \in R(\mathscr{G})$, but $\mathscr{G}$ does not belong to $R(\mathscr{I})$. If $\mathscr{G}_1 = \{G_i\}_{i \in I}$ is any class of archimedean linearly ordered groups, then the radical class in $R(\mathscr{G})$ generated by $\mathscr{G}_1$ belongs to $R(\mathscr{I})$. This does not hold, in general, for non-archimedean linearly ordered groups. If $A \in R(\mathscr{I})$ and $\{B_i\}_{i \in I} \subseteq R(\mathscr{I})$, then $A \wedge (\bigvee_{i \in I} B_i) = \bigvee_{i \in I} (A \wedge B_i)$. There exists an injective mapping of the class of infinite cardinals into the class of all atoms of $R(\mathscr{I})$.

From the result of [11] it follows that each vaiety of lattice ordered groups belongs to $R(\mathscr{G})$. In particular, the class of all abelian lattice ordered groups belongs to $R(\mathscr{G})$. In the case of interpolation groups the situation is essentially different; it will be proved below that the class of all abelian interpolation groups does not belong to $R(\mathscr{I})$.

In the last section it is shown that the answers to the questions from [8] under considerations are "No".

## 1. PRELIMINARIES

The group operation in a lattice ordered group will be denoted additively; the commutativity of this operation is not assumed.

**1.1. Definition.** Let $\beta$ be a cardinal, $\beta \neq 0$. A partially ordered set $X$ is said to *satisfy the $\beta$-interpolation property* if, whenever $Y$ and $Z$ are nonempty subsets of $X$ with card $Y \leqq \beta$, card $Z \leqq \beta$ and $y \leqq z$ for each $y \in Y$ and each $z \in Z$, then there exists $x \in X$ such that $y \leqq x \leqq z$ for each $y \in Y$ and each $z \in Z$.

It is easy to verify that in the above definition the condition $y \leqq z$ can be replaced by $y < z$.

For $\beta = 2$ or $\beta = \aleph_0$; the $\beta$-interpolation property is denoted as Riesz interpolation property or countable interpolation property, respectively.

A partially ordered group satisfying the Riesz interpolation property is called a Riesz group (cf. [6], [7]) or an interpolation group (cf. [8]). Partially ordered groups with countable interpolation property have been investigated in [9]; cf. also [8], Chap. 16.

Let $\mathscr{G}$ be the class of all lattice ordered groups. For each $G \in \mathscr{G}$ we denote by $C_l(G)$ the set of all convex $l$-subgroups of $G$; this set is partially ordered by inclusion. Then $C_l(G)$ is a complete lattice (cf., e.g., [6]); the corresponding lattice operation will be denoted by $\bigwedge^l$ and $\bigvee^l$.

**1.2. Definition.** A nonempty subclass $A$ of $\mathscr{G}$ is said to be a *radical class of lattice ordered groups* if it satisfies the following conditions:

(i) $A$ is closed with respect to isomorphisms.

(ii) If $G_1 \in A$ and $G_2 \in C_l(G_1)$, then $G_2 \in A$.

(iii) If $G \in \mathscr{G}$ and $\{G_i\}_{i \in I} \subseteq C_l(G) \cap A$, then $\bigvee^l_{i \in I} G_i \in A$.

Next, let $\mathscr{I}$ be the class of all directed interpolation groups. For $G \in \mathscr{I}$ let $C(G)$ be the set of all convex directed subgroups of $G$; we consider $C(G)$ as being partially ordered by inclusion. Then (cf. [17]) the set $C(G)$ is a complete lattice. The lattice operations in $C(G)$ will be denoted by $\wedge$ and $\vee$.

Now we can introduce the notion of radical class of directed interpolation groups in analogous way as in 1.2 (with the distinction that $C_l(G_1)$, $C_l(G)$ and $\bigvee^l_{i \in I} G_i$ are replaced by $C(G_1)$, $C(G)$ and $\bigvee_{i \in I} G_i$).

Let $R(\mathscr{G})$ and $R(\mathscr{I})$ be the collection of all radical classes of lattice ordered groups or the collection of all radical classes of directed interpolation groups, respectively. Both these collections are partially ordered by inclusion.

## 2. BASIC PROPERTIES OF $R(\mathscr{I})$

Let $A_{\min}$ be the class of all $G \in \mathscr{I}$ such that card $G = 1$. Then $A_{\min}$ is the least element of $R(\mathscr{I})$ and $\mathscr{I}$ is the greatest element of $R(\mathscr{I})$.

We need the following result.

**2.1. Proposition.** (Cf. [17].) *Let $G \in \mathscr{I}$, $G_1, G_2 \in C(G)$, $\{G_i\}_{i \in I} \subseteq C(G)$. Then $G_1 \wedge G_2 = G_1 \cap G_2$ and $G_1 \wedge (\bigvee_{i \in I} G_i) = \bigvee_{i \in I} (G_1 \wedge G_i)$.*

Let us also remark that the relation $\bigwedge_{i \in I} G_i = \bigcap_{i \in I} G_i$ need not be valid in general (cf. [17]).

For a nonempty subclass $\mathscr{I}_1$ of $\mathscr{I}$ we denote by

Sub $\mathscr{I}_1$ – the class of all $G \in \mathscr{I}$ having the property that there exists $G_1 \in \mathscr{I}_1$ such that $G$ is isomorphic to some element of $C(G_1)$;

Join $\mathscr{I}_1$ – the class of all $G \in \mathscr{I}$ having the property that there exist subgroups $G_i \in C(G)$ and $G_i^1 \in \mathscr{I}_1$ $(i \in I)$ such that $G = \bigvee_{i \in I} G_i$ and for each $i \in I$, $G_i$ is isomorphic to $G_i^1$.

**2.2. Lemma.** *Let $\emptyset \neq \mathscr{I}_1 \subseteq \mathscr{I}$. Denote $\mathscr{I}_1^- =$ Join Sub $\mathscr{I}_1$. Then*

(i) $\mathscr{I}_1^- \in R(\mathscr{I})$;

(ii) *for each $A \in R(\mathscr{I})$ with $\mathscr{I}_1 \subseteq A$ we have $\mathscr{I}_1^- \subseteq A$.*

Proof. It is obvious that $\mathscr{I}_1^-$ is closed with respect to isomorphisms.

Let $G_1 \in \mathscr{I}_1^-$ and $G_2 \in C(G_1)$. There exist $G_i \in C(G)$, $G_i \in \mathscr{I}_1$ and $H_i \in C(G_i^1)$ $(i \in I)$ such that $G_1 = \bigvee_{i \in I} G_i$ and for each $i \in I$, $G_i$ is isomorphic to $H_i$. Hence in view of 2.1 we have

$$G_2 = G_2 \wedge G_1 = G_2 \wedge (\bigvee_{i \in I} G_i) = \bigvee_{i \in I} (G_2 \wedge G_i)\,.$$

Then $G_2 \wedge G_i \in \operatorname{Sub} \mathscr{I}_1$ for each $i \in I$ and thus $G_2 \in \mathscr{I}_1^-$.

Next let $G \in \mathscr{I}$ and $\{G_i\}_{i \in I} \subseteq C(G) \cap I_1^-$. Put $\bigvee_{i \in I} G_i = H$. Then we have $H \in \operatorname{Join} \mathscr{I}_1^- = \operatorname{Join} \operatorname{Join} \operatorname{Sub} \mathscr{I}_1 = \operatorname{Join} \operatorname{Sub} \mathscr{I}_1 = \mathscr{I}_1^-$. Hence (i) holds. The assertion (ii) is an immediate consequence of the definition of $R(\mathscr{I})$.

**2.2.1. Remark.** In the second part of the above proof we have verified that Sub Join Sub $\mathscr{I}_1$ = Join Sub $\mathscr{I}_1$ for each nonempty subclass $\mathscr{I}_1$ of $\mathscr{I}$.

In view of 2.2 we say that $\mathscr{I}_1^-$ is a radical class of directed interpolation groups generated by $\mathscr{I}_1$; we also put $\mathscr{I}_1^- = T(\mathscr{I}_1)$. If $\mathscr{I}_1$ is a one-element set, $\mathscr{I}_1 = \{G\}$, then we denote $T(\mathscr{I}_1) = T(G)$.

It will be proved below that there exists an injective mapping of the class of all cardinals into $R(\mathscr{I})$. In this sense, $R(\mathscr{I})$ is a "large" collection. Nevertheless, we shall apply for $R(\mathscr{I})$ the terminology of partially ordered sets, e.g., the notions of sup and inf. If $I$ is a nonempty class, $A_i \in R(\mathscr{I})$ for each $i \in I$, $A \in R(\mathscr{I})$, $A = \sup \{A_i\}_{i \in I}$, then we write also $A = \bigvee_{i \in I} A_i$. The symbol $\bigwedge_{i \in I} A_i$ stands for $\inf \{A_i\}_{i \in I}$.

**2.3. Lemma.** *Let $I$ be a nonempty class and for each $i \in I$ let $A_i \in R(\mathscr{I})$. Then we have*

(i) $\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i$;

(ii) $\bigvee_{i \in I} A_i = \operatorname{Join} \bigcup_{i \in I} A_i$.

Proof. Put $B = \bigcap_{i \in I} A_i$. Then $B = \emptyset$, since $A_{\min} \in A_i$ for each $i \in I$. If $D \in R(\mathscr{I})$ and $D = A_i$ for each $i \in I$, then $B \subseteq D$. It is obvious that $B \in R(\mathscr{I})$. Thus (i) holds.

Denote $E = \operatorname{Join} \bigcup_{i \in I} A_i$. We have $\operatorname{Sub} A_i = A_i$ for each $i \in I$, hence $E = \operatorname{Join} \operatorname{Sub} \bigcup_{i \in I} A_i$. Thus according to 2.2.1, $\operatorname{Sub} E = E$. Clearly $\operatorname{Join} E = E$.

Therefore from 2.2 we obtain $E \in R(\mathscr{I})$. If $F \in R(\mathscr{I})$ and $F \geqq A_i$ for each $i \in I$, then clearly $F \geqq E$. Hence (ii) is valid.

**2.3.1. Corollary.** *$R(\mathscr{I})$ is a complete lattice.*

**2.4. Proposition.** *Let $A \in R(\mathscr{I})$, $\{A_i\}_{i \in I} \subseteq R(\mathscr{I})$. Then*

$$A \wedge (\bigvee_{i \in I} A_i) = \bigvee_{i \in I} (A \wedge A_i) .$$

Proof. We have $A \wedge (\bigvee_{i \in I} A_i) \geqq \bigvee_{i \in I} (A \wedge A_i)$. Let $G \in A \wedge (\bigvee_{i \in I} A_i)$. Hence in view of 2.3, $G \in A$ and $G \in \bigvee_{i \in I} A_i = \text{Join} \bigcup_{i \in I} A_i$. Thus there exist $H_k (k \in K)$ in $C(G)$ such that $G = \bigvee_{k \in K} H_k$ and each $H_k$ belongs to some $A_{i(k)}$ $(i(k) \in I)$. Next, because of $G \in A$ we infer that $H_k \in A$ and hence $H_k \in A \wedge A_{i(k)}$ for each $k \in K$. Therefore $G \in \bigvee_{i \in I} (A \wedge A_i)$, completing the proof.

## 3. SOME FURTHER PROPERTIES OF $R(\mathscr{I})$

Since the basic properties of $R(\mathscr{I})$ are analogous to those of $R(\mathscr{G})$ the natural question arises what are the relations between $R(\mathscr{I})$ and $R(\mathscr{G})$. In particular we can ask whether $R(\mathscr{G})$ is a subclass of $R(\mathscr{I})$. The following consideration shows that the answer is "No".

**3.1. Example.** Let $Q$ be the additive group of all rational numbers with the natural linear order, $Y = X = Q$ and let $Z$ be any nonzero lattice ordered group. Put

$$G = (X \times Y) \circ Z ,$$

where $\times$ and $\circ$ have the usual meaning (the operation of the direct product and the operation of the lexicographic product). It is obvious that $G$ is an abelian directed group. We shall verify that $G$ belongs to $\mathscr{I}$.

Let $u_1, u_2, v_1, v_2 \in G$ such that $u_i \leqq v_j$ for $i, j = 1, 2$. We have to show that there exists $t \in G$ with $u_i \leqq t \leqq v_j$ for $i, j = 1, 2$. The case when either $u_1$ and $u_2$ are comparable or $v_1$ and $v_2$ are comparable is trivial. Thus it suffices to assume that $u_1$ is uncomparable with $u_2$ and that $v_1$ is uncomparable with $v_2$.

Let $u_1 = (x_1, y_1, z_1)$, $u_2 = (x_2, y_2, z_2)$, $v_1 = (x_3, y_3, z_3)$, $v_2 = (x_1, y_1, z_1)$. If $(x_1, y_1) = (x_2, y_2)$, then we take $t = (x_1, y_1, z_1 \vee z_2)$. In the case $(x_1, y_1) \neq (x_2, y_2)$ we put $t = (x_1 \vee x_2, y_1 \vee y_2, z_3 \wedge z_4)$. Then $t$ satisfies the desired condition. Thus $G \in \mathscr{I}$.

Let $G_1 = \{(x, y, z) \in G: y = 0\}$, $G_2 = \{(x, y, z) \in G: x = 0\}$. Thus $G_1$ and $G_2$ are convex directed subgroups of $G$, $G_1$ is isomorphic to $X \circ Z$ and $G_2$ is isomorphic to $Y \circ Z$ under a natural isomorphism. Both $G_1$ and $G_2$ are lattice ordered groups, $G_1$ is isomorphic to $G_2$ and in the lattice $C(G)$ the relation $G = G_1 \vee G_2$ is valid. It is obvious that $G \notin \mathscr{G}$.

**3.2. Corollary.** *Let $X$ be as in* 3.1 *and let $Z$ be a nonzero lattice ordered groups. Then $T(X \circ Z)$ fails to be a subclass of $\mathscr{G}$.*

Proof. Let $G$, $G_1$ and $G_2$ be as in 3.1. Both $G_1$ and $G_2$ are isomorphic to $X \circ Z$, hence they belong to $T(X \circ Z)$. In view of $G_1 \vee G_2 = G$ we obtain that $G$ is an element of $T(X \circ Z)$. Since $G \notin \mathscr{G}$, the relation $T(X \circ Z) \subseteq \mathscr{G}$ is not valid.

**3.3. Corollary.** $G \notin R(\mathscr{I})$.

Proof. Under the same denotations as in 3.2 we have $X \circ Z \in \mathscr{G}$, hence $T(X \circ Z) \subseteq \subseteq T(\mathscr{G})$. Next 3.2 yields that $\mathscr{G} \neq T(\mathscr{G})$. Therefore $\mathscr{G} \notin R(\mathscr{I})$.

Since $\mathscr{G} \in R(\mathscr{G})$, in view of 3.3 we can ask whether there exists an element $A$ in $R(\mathscr{G})$ with $A \neq A_{\min}$ such that $A$ belongs to $R(\mathscr{I})$. In the next section it will be shown that the answer to this question is positive.

## 4. ARCHIMEDEAN LINEARLY ORDERED CONVEX SUBGROUPS

For $\{0\} \neq G \in \mathscr{I}$ we denote by $A(G)$ the set of all elements $G_1 \in C(G)$ such that $G_1$ is linearly ordered and archimedean.

**4.1. Lemma.** *Let $G_1, G_2 \in A(G)$, $G_1 \neq G_2$. Then $G_1 \cap G_2 = \{0\}$.*

Proof. By way of contradiction, suppose that there exists $g \in G_1 \cap G_2$ with $g \neq 0$. Then without loss of generality we can assume that $g > 0$. Let $0 < g_1 \in G_1$. Because $G_1$ is archimedean there is a positive integer $n$ such that $g_1 < ng$. Now from $ng \in G_2$ we infer that $g_1 \in G_2$, which implies that $G_1 \subseteq G_2$. Analogously we obtain $G_2 \subseteq G_1$, whence $G_1 = G_2$, which is a contradiction.

Let $\{G_i\}_{i \in I}$ be a nonempty subset of $A(G)$ such that $G_i \neq \{0\}$ for each $i \in I$. We assume that for distinct elements $i(1)$ and $i(2)$ of $I$ the groups $G_{i(1)}$ and $G_{i(2)}$ must also be distinct. Put $H = \bigvee_{i \in I} G_i$.

**4.2. Lemma.** *Let $i(1)$ and $i(2)$ be distinct elements of $I$, $a \in G_{i(1)}$, $b \in G_{i(2)}$. Then $a + b = b + a$.*

Proof. It suffices to consider the case when $a > 0$ and $b > 0$. Since the mapping $\varphi: t \to -a + t + a$, (where $t$ runs over $G$) is an automorphism of the partially ordered group $G$, the subgroup $-a + G_{i(2)} + a$ is an element of $A(G)$. Put $b' = = -a + b + a$. Then $b + a = a + b'$ and $b' > 0$. There exist $a_1$ and $b_1$ in $G$ such that $b' = b_1 + a_1$ and $0 \leqq b_1 \leqq b$, $0 \leqq a_1 \leqq a$.

At first suppose that $-a + G_{i(2)} + a \neq G_{i(2)}$. Then in view of 4.1, $(-a + G_{i(2)} + a) \cap G_{i(2)} = \{0\}$. This yields that $b_1 = 0$ and hence $0 \neq b' = = a_1 \in G_{i(1)}$. Therefore $b' \in G_{i(1)}$ and hence, by applying 4.1 again, $G_{i(1)} = G_{i(2)}$, which is a contradiction. Thus $-a + G_{i(2)} + a = G_{i(2)}$ and so $b' \in G_{i(2)}$. Then $a_1 = 0$ which implies that $b' = b_1$ and hence $b' \leqq b$. By analogous reasoning we obtain that $b \leqq b'$, completing the proof.

**4.3. Lemma.** *Let $0 \neq h \in H$. Then there exists a finite nonempty subset $I_1$ of $I$ and elements $g_i \in G_i$ $(i \in I_1)$, $g_i \neq 0$ such that*

(i) $g = \Sigma g_i$ $(i \in I_1)$;

(ii) *if $i$ and $i(1)$ are distinct elements of $I_1$, then $g_i \notin G_{i(1)}$.*
*Next, $H$ is abelian.*

Proof. This follows from [17], Hilfsatz 4, and from 4.2.

**4.4. Lemma.** *Let $h$, $I_1$ and $g_i$ be as in 4.3. Then $h > 0$ if and only if $g_i > 0$ for each $i \in I_1$.*

Proof. The "if" part is obvious; let us investigate the "only if" part of the assertion. Let card $I_1 = n$. We proceed by induction on $n$.

For $n = 1$ the assertion obviously holds. Suppose that $n > 1$ and that the assertion is valid for $n - 1$. By way of contradiction, suppose that $g_{i(1)} < 0$ for some $i(1) \in I(1)$. Put $I_2 = I_1 \setminus \{i(1)\}$, $h' = \Sigma g_i$ $(i \in I_2)$. We have $0 < h = g_{i(1)} + h'$, hence $h' > 0$. Thus in view of the induction assumption the relation $g_i > 0$ must be valid for each $i \in I_2$.

According to

$$0 < -g_{i(1)} < \Sigma g_i \quad (i \in I_2)$$

there exist elements $h_i \in G$ $(i \in I_2)$ such that $0 \leqq h_i \leqq g_i$ is valid for each $i \in I_2$ and

$$-g_{i(1)} = \Sigma h_i \quad (i \in I_2)\,.$$

Let $i \in I_2$. Then from the convexity of $G_i$ we obtain that $h_i \in G_i$; moreover, $0 \leqq h_i \leqq -g_{i(1)}$, hence $h_i \in G_{i(1)}$. Thus according to 4.1, $g_i = 0$ for each $i \in I$, implying that $-g_{i(1)} = 0$, which is a contradiction.

By an obvious modification of the proof of 4.4 we obtain:

**4.5. Lemma.** *Let $h \in H$. Let $I_1$ be a finite subset of $I$ and let $g_i \in G_i$ for each $i \in I_1$ such that $h = \Sigma g_i$ $(i \in I_1)$. The following conditions are equivalent:*

(i) $h \geqq 0$;

(ii) $g_i \geqq 0$ *for each* $i \in I$.

*The corresponding dual assertion is also valid.*

**4.6. Corollary.** *Let $h, I_1$ and $g_i$ be as in 4.5. If $h = 0$, then $g_i = 0$ for each $i \in I_1$.*

**4.7. Corollary.** *Let $h \in H$, $h \neq 0$. Then the set $I_1$ and the elements $g_i$ $(i \in I_1)$ (the existence of which was proved in 4.3) are uniquely determined.*

From 4.3, 4.5 and 4.7 we infer

**4.8. Lemma.** *$H$ is a weak direct product of the system $\{G_i\}_{i \in I}$.*

*Now let $\mathscr{G}_1 = \{G_j\}$ $(j \in J)$ be a class of archimedean linearly ordered groups which is closed with respect to isomorphisms. Then* Sub $\mathscr{G}_1 = \mathscr{G}_1 \cup A_{\min}$. *Thus 4.8 yields:*

**4.9. Proposition.** *Let $\mathscr{G}_1 = \{G_j\}$ $(j \in J)$ be a nonempty class of archimedean linearly ordered groups. Let $G \in \mathscr{I}$. Then the following conditions are equivalent:*

(i) $G \in$ Join Sub $\mathscr{G}_1$.

(ii) *Either* $G = \{0\}$ *or* $G$ *is a weak direct product of some elements belonging to* $\mathscr{G}_1$.

From 4.9, 2.2 and [12], 2.1 and 2.2 it follows:

**4.10. Theorem.** *Let* $\mathscr{G}_1$ *be a nonempty class of archimedean linearly ordered groups. Then* Join Sub $\mathscr{G}_1 \in R(\mathscr{I}) \cap R(\mathscr{G})$.

Let us remark that the above theorem need not hold for linearly ordered groups which fail to be archimedean (cf. Example 3.1).

**4.11. Theorem.** *Let* $G_1$ *be a nonzero archimedean linearly ordered group. Put* $\mathscr{G}_1 = \{G_1\}$, $A =$ Join Sub $\mathscr{G}_1$. *Then* $A$ *is an atom of the lattice* $R(\mathscr{I})$.

Proof. In view of 2.2 we have $A \in R(\mathscr{I})$. Since $G_1 \in A$, the relation $A \neq A_{\min}$ is valid. Let $A_{\min} \neq B \in R(\mathscr{I})$, $B \leqq A$. Thus there is $G \in B$ with $G \neq \{0\}$. Then $G \in A =$ Join Sub $\{G_1\}$. Hence there are $G_j$ $(j \in J)$ in $C(G)$ such that $G = \bigvee_{j \in J} G_j$ and each $G_j$ is isomorphic to $G_i$. Therefore $G_i \in B$ and so $B = A$.

If $G_1$ and $G_2$ are nonzero archimedean linearly ordered groups and if $G_1$ is not isomorphic to $G_2$, then $G_2 \notin$ Join Sub $\{G_1\}$. Since there are infinitely many nonzero archimedean linearly ordered groups which are mutually nonisomorphic, we obtain

**4.12. Corollary.** *The lattice* $R(\mathscr{I})$ *has infinitely many atoms.*

This result will be sharpened in the next section.

## 5. A FURTHER TYPE OF ATOMS IN $R(\mathscr{I})$

We denote by $A_1$ the class of all nonzero archimedean linearly ordered groups $G_1$ such that no element of $G_1$ covers 0. Next let $\mathscr{K}$ be the class of all groups $K$ with a trivial partial order (i.e., for $k \in K \in \mathscr{K}$ we have $k \geqq 0$ iff $k = 0$). For $G_1 \in A_1$ and $K_1 \in \mathscr{K}$ we put $H(G_1, K_1) = G_1 \circ K_1$. Then $H(G_1, K_1) \in \mathscr{I}$. Let us denote by $\mathscr{H}$ the class of all $H(G_1, K_1)$ which can be constructed in this way (where $G_1$ runs over $A_1$ and $K_1$ runs over $\mathscr{K}$).

If $H(G_1, K_1)$ and $H(G_2, K_2)$ belong to $\mathscr{H}$ and are isomorphic, then $G_1$ is isomorphic to $G_2$ and $K_1$ is isomorphic to $K_2$.

If $H \in \mathscr{H}$, then $C(H) = \{H, \{0\}\}$. From this fact we obtain (by the same argument as in the proof of 4.11)

**5.1. Lemma.** *Let* $H \in \mathscr{H}$, $A = T(H)$. *Then* $A$ *is an atom of the lattice* $R(\mathscr{I})$.

Let $H = (G_1, K_1) \in \mathscr{H}$. Put $\beta = \max\{\text{card } G_1, \text{card } K_1\}$. Then we have

**5.2. Lemma.** *Let* $0 < h \in H$. *Then* card $[0, h] = \beta$.

This can be generalized as follows:

**5.3. Lemma.** Let $H$ be as in 5.2 and let $\{0\} \neq H' \in T(H)$, $0 < h' \in H'$. Then card $[0, h] = \beta$.

Proof. There exist $G_j \in C(H')$ $(j \in J)$ such that $H' = \bigvee_{j \in J} G_j$ and each $G_j$ is isomorphic to $H$. Next there exists a finite subset $J_1$ of $J$ and elements $0 < g_j \in G_j$ for each $j \in J_1$ such that $h' = \Sigma g_j$ $(j \in J_1)$. According to 5.2 we have card $[0, g_j] = \beta$ for each $j \in J_1$. Hence card $[0, h] \geqq \beta$. Next, it is clear that the cardinal $\beta$ is infinite. For each $x \in [0, h']$ there exist elements $x_j \in [0, g_j]$ $(j \in J_1)$ such that $x = \Sigma x_j$ $(j \in J_1)$. Hence there exists an injective mapping of the set $[0, h']$ into $\Pi[0, g_j]$ $(j \in J_1)$. Since $J_1$ is finite we have card $[0, h'] \leqq$ card $\Pi[0, g_j]$ $(j \in J_1) = \beta$.

**5.4. Lemma.** *Let $H$ be as in 5.2. Let $H_1 = H(G_1, K_1) \in \mathscr{H}$, card $K_1 > \beta$. Then $H_1$ does not belong to $T(H)$.*

Proof. This is a consequence of 5.2 and 5.3.

Let $G_1$ be a fixed element of $A_1$. For each infinite cardinal $\beta_1$ there exists $K(\beta_1) \in \mathscr{K}$ with card $K(\beta_1) = \beta_1$. Put $H(\beta_1) = H(G_1, K(\beta_1))$.

**5.5. Proposition.** *The mapping $\beta_1 \to T(H(G_1, K(\beta_1)))$ of the class of all infinite cardinals into the class $R(\mathscr{I})$ is injective.*

Proof. This follows immediately from 5.4.

Next, 5.1 and 5.5 yields

**5.6. Corollary.** *There exists an injective mapping of the class of all infinite cardinals into the class of all atoms of $R(\mathscr{I})$.*

For $A \in R(\mathscr{I})$ we denote by $\mathscr{A}(A)$ the collection of all $B \in R(\mathscr{I})$ such that $B$ covers $A$ in $R(\mathscr{I})$ (i.e., $A < B$ and there is no $C$ in $R(\mathscr{I})$ with $A < C < B$).

In view of 5.6, the collection $\mathscr{A}(A_{\min})$ is "large". Next, from 5.6 and 2.4 we obtain

**5.7. Proposition.** *Let $n$ be a positive integer and let $A_1, A_2, \ldots, A_n$ be atoms of $R(\mathscr{I})$. Then there exists an injective mapping of the class of all infinite cardinals into $\mathscr{A}(A_1 \vee A_2 \vee \ldots \vee A_n)$.*

For each infinite cardinal $\beta$ we denote by $A_\beta$ the collection of all $G \in \mathscr{I}$ such that card $[0, h] = \beta$ whenever $0 < h \in G$.

**5.8. Lemma.** Let $\beta$ be an infinite cardinal. Then $A_\beta \in R(\mathscr{I})$.

Proof. If $G \in A_\beta$ and $H \in C(G)$, then clearly $H \in A_\beta$. Let $G_1 \in \mathscr{I}$, $\{G_i\}_{i \in I} \subseteq C(G) \cap A_\beta$ and $\bigvee_{i \in I} G_i = G_1$. Then by the same method as in the proof of 5.3 we can verify that $G_1$ belongs to $A_\beta$.

**5.9. Proposition.** *Let $\beta$ be an infinite cardinal. Then there exists an injective mapping of the class of all infinite cardinals into $\mathscr{A}(A_\beta)$.*

Proof. It suffices to verify that there exists an injective mapping of the class of all cardinals greather that $\beta$ into $\mathscr{A}(A_\beta)$. For each cardinal $\beta_1 > \beta$ let $T(H(G_1, K(\beta_1)))$ be as above. Put

$$f(\beta_1) = T(H(G_1, K(\beta_1))) \vee A_\beta \,.$$

From 5.1, 5.3 and 2.4 it follows that $f$ has the desired properties.

## 6. ABELIAN INTERPOLATION GROUPS

We denote by $A$ and $A_i$ the class of all abelian lattice ordered groups or the class of all abelian interpolation groups, respectively.

From the result of Holland [11] it follows that the relation $A \in R(\mathcal{G})$ is valid. In this section we shall investigate the question whether $A_i$ belongs to $R(\mathcal{I})$.

**6.1. Example.** As usual, we denote by $Q$ the additive group of all rational numbers with the natural linear order. Put $X = Y = Z = Q$. Let $G$ be the set of all triples $(x, y, z)$ with $x \in X$, $y \in Y$ and $z \in Z$. We define the operation $+$ in $G$ as follows. For $(x, y, z)$ and $(x_1, y_1, z_1)$ in $G$ we set

$$(x, y, z) + (x_1, y_1, z_1) = (x + x_1,\ y + y_1,\ z + z_1 + x_1 y)\,.$$

Then $(G; +)$ is a nonabelian group with the neutral element $0 = (0, 0, 0)$. Next we put $(x, y, z) \geqq 0$ if some of the following conditions is valid:

(i) $x > 0$ and $y \geqq 0$;

(ii) $x \geqq 0$ and $y > 0$;

(iii) $x = y = 0$ and $z \geqq 0$.

Then $G$ turns out to be a non-abelian interpolation group.

Denote

$$G_1 = \{(x, y, z) \in G : x = 0\}\,, \quad G_2 = \{(x, y, z) \in G : y = 0\}\,.$$

Both $G_1$ and $G_2$ are directed convex subgroups of $G$; next, both $G_1$ and $G_2$ are abelian. We obviously have

$$(*) \qquad G_1 \vee G_2 = G\,.$$

**6.2. Lemma.** Join Sub $A_i \neq A_i$.

Proof. Let $G_1$, $G_2$ and $G$ be as in 6.1. Then $G_1$ and $G_2$ belong to $A_i$. Hence in view of $(*)$ the relation $G \in \text{Join}\, A_i$ is valid. Clearly Sub $A_i = A_i$ thus $G \in \text{Join Sub}\, A_i$. Since $G$ is nonabelian, it does not belong to $A_i$.

**6.3. Corollary.** *$A_i$ fails to be a radical class of directed interpolation groups.*

Proof. This is a consequence of 2.2 and 6.3.

## 7. DIRECTED GROUPS WITH COUNTABLE INTERPOLATION

All partially ordered groups considered in the present section are assumed to be abelian.

For the following three definitions cf. [8].

**7.1. Definition.** Let $G$ be a partially ordered group and let $n$ be a positive integer. We say that $G$ is *n-perforated* if there exists an element $x \in G$ such that $nx \geqq 0$ but $x \ngeqq 0$; otherwise, $G$ is $n$-unperforated. If $G$ fails to be $n$-perforated for each positive integer $n$, then $G$ is said to be *unperforated.*

**7.2. Definition.** Let $G$ be a directed unperforated interpolation group. Then $G$ is said to be a *dimension group.*

**7.3. Definition.** A partially ordered group $G$ is said to be *monotone σ-complete* provided that every ascending sequence $x_1 \leqq x_2 \leqq \ldots$ in $G$ which is bounded above in $G$ has a supremum in $G$.

In [8], p. 320 the following open problems were proposed:

(A) Is every directed group with countable interpolation unperforated?

(B) Is every directed group with countable interpolation isomorphic to a quotient group of a monotone $\sigma$-complete dimension group?

Let $\alpha$ be an ordinal. We recall some notions concerning $\eta_\alpha$-sets (cf., e.g., [10] or [19]).

Let $X$ be a linearly ordered set and let $P \neq \emptyset$, $Q \neq \emptyset$ be subsets of $X$. The sets $P$ and $Q$ are said to be *neighbours in* $X$ if $p_1 < q_1$ for each $p_1 \in P$ and each $q_1 \in Q$, and there does not exist any $x \in X$ such that $p < x < q$ for each $(p, q) \in P \times Q$. $X$ is said to be an *$\eta_\alpha$-set* if it satisfies the following conditions:

(i) If $Y \subseteq X$ and card $Y < \aleph_\alpha$, then $X$ is neither cofinal nor coinitial with $Y$;

(ii) If $Y_1$ and $Y_2$ are subsets of $X$, card $Y_i < \aleph_\alpha$ for $i = 1, 2$, then $Y_1$ and $Y_2$ fail to be neighbours in $X$.

Linearly ordered groups or fields having the property that the underlying sets are $\eta_\alpha$-sets were investigated in [1], [2], [5], [19].

**7.4. Proposition.** (Cf. [2].) *For each infinite ordinal there exists a linearly ordered group $G(\alpha)$ which is an $\eta_\alpha$-set.*

Let $H$ be a group with a trivial partial order and let $G(\alpha)$ be as in 7.4. We put $G = \{(x, y): x \in G(\alpha) \text{ and } y \in H\}$. The operation $+$ in $G$ is defined compentwise. For $(x_1, y_1)$ and $(x_2, y_2)$ in $G$ we put $(x_1, y_1) < (x_2, y_2)$ iff $x_1 < x_2$. Then $G$ turns out to be a directed group. Next, since $G(\alpha)$ is an $\eta_\alpha$-set, it obviously satisfies the $\beta$-interpolation property for each cardinal $\beta$ with $\beta < \aleph_\alpha$. Hence for each such $\beta$, $G$ satisfies the $\beta$-interpolation property as well.

Now let $H = \{0, 1, 2, \ldots, n - 1\}$, the operation $+$ on $H$ being defined as addition mod $n$. Put $z = (0, 1) \in G$. Then the relation $z \geqq (0, 0)$ fails to be valid in $G$, but $nz = (0, 0)$. Thus $G$ is $n$-perforated. Hence we have

**7.5. Proposition.** *Let $\alpha$ be an ordinal and let $n$ be a positive integer. Then there exists a directed $n$-perforated group $G$ satisfying the $\beta$-interpolation property for each cardinal $\beta$ with $\beta < \aleph_\alpha$.*

As a corollary we obtain that the answer to the question (A) above is "No".

**7.6. Proposition.** (Cf. [8], Proposition 3.1.) *If $K_1$ is an ideal in a dimension group $K_2$, then the quotient group $K_2/K_1$ is a dimenion group as well.*

**7.7. Proposition.** *Let $\alpha$ be an ordinal and let $\beta$ be a cardinal with $\beta < \aleph_\alpha$. There* *exists a directed group G satisfying the β-interpolation property such that G is not a quotient group of any dimension group.*

Proof. Let $G$ be as above. Since $G$ fails to be a dimension group, it suffices to apply 7.6.

This proposition yields that the answer to the question (B) above is negative.

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