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CYCLICALLY ORDERED GROUPS WITH UNIQUE ADDITION

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Linearly ordered groups with unique addition (for definitions, cf. below) were investigated by T. Ohkuma [8]. The case of lattice ordered groups having this property was dealt with by P. Conrad and M. Darnel [2], and by the author [4].

Cyclically ordered groups were studied in [1], [5], [6], [7], [9]–[13]. The notion of cyclically ordered group is a generalization of the notion of linearly ordered group.

An example of cyclically ordered group is the additive group K of all complex numbers z with $|z| = 1$, where the cyclic order is defined in a natural way.

In this note the notion of cyclically ordered group with unique addition will be introduced. Let $\mathscr{C}_u$ be the class of all cyclically ordered groups G such that (i) G fails to be linearly ordered, and (ii) G has a unique addition.

It will be proved that each element of $\mathscr{C}_u$ is isomorphic to a subgroup of K with the inherited cyclic order. Next it will be shown that there are exactly $2^{2^{\aleph_0}}$ nonisomorphic types of cyclically ordered groups belonging to $\mathscr{C}_u$.

1. PRELIMINARIES

A linearly ordered group $G_1 = (G; \leqq, +_1)$ is said *to have a unique addition*, if whenever $G_2 = (G; \leqq, +_2)$ is linearly ordered group such that the neutral element of the group $(G; +_1)$ is the same as the neutral element of the group $(G; +_2)$, then the operation $+_1$ coincides with the operation $+_2$.

For the notion of cyclically ordered group cf., e.g., Fuchs [3], Chap. IV. Section 6. In the present paper we shall apply the same terminology and denotations concerning cyclically ordered groups as in [7]; in particular, the group operation in a cyclically ordered group will be written additively and the relation of cyclic order will be denoted by the symbol $[x, y, z]$ (or, shortly, by $[\ ]$).

The cyclically ordered group K mentioned above can be described, up to isomorphism, as follows: K is the set of all reals x with $0 \leqq x < 1$, the operation $+$ is the addition mod 1, and for $a, b, c \in K$ we have $[a, b, c]$ if and only if

$$(1) \qquad a < b < c \quad \text{or} \quad b < c < a \quad \text{or} \quad c < a < b$$

is valid.

The relations between linearly ordered groups and cyclically ordered groups are well-known; cf., e.g., [6], Section 3.

For cyclically ordered groups we can apply a definition analogous to that applied above for linearly ordered groups, namely:

A cyclically ordered group $G_1 = (G; [\ ], +_1)$ will be said *to have a unique addition* if, whenever $G_2 = (G; [\ ], +_2)$ is a cyclically ordered group with $0_1 = 0_2$ (where 0_i is the neutral element of G_i $(i = 1, 2)$), then the operation $+_1$ coincides with the operation $+_2$.

2. ELEMENTS OF $\mathscr{C}_u$ AS SUBGROUPS OF K

Let us denote by $\mathcal{O}_u$ the class of all linearly ordered groups with unique addition. Next let $\mathscr{C}_u^0$ the class of all cyclically ordered groups with unique addition.

Let $G_1 = (G; [\ ], +)$ be a cyclically ordered group. In view of the Representation Theorem (which is due to Swierczkowski [10] (cf. also [7], Theorem 1.1)) there is a linearly ordered group L such that G_1 is isomorphic to a subgroup of the cyclically ordered group $K \otimes L$ (for denotations, cf. [7], Section 1). Hence without loss of generality we can suppose that G_1 is a subgroup of $K \otimes L$.

We denote by $G_1(K)$ and $G_1(L)$ the natural projection of G_1 into K or into L, respectively. Also, without loss of generality we can assume that $G_1(L) = L$.

2.1. Lemma. *Let $G_1(K) = \{0\}$. Then $G_1 \in \mathscr{C}_u^0$ if and only if $G_1 = L \in \mathcal{O}_u$.*

Proof. From $G_1(K) = \{0\}$ we obtain that $G_1(L) = G_1$, and thus in view of the above assumption, $G_1 = L$. Hence G_1 is linearly ordered. It is easy to verify that G_1 as cyclically ordered group has a unique addition if and only if G_1 as linearly ordered group has a unique addition.

Put $G_0 = \{g \in G: g(K) = 0\}$. Then in view of [6], Cor. 3.6, G_0 is the largest linearly ordered subgroup of G_1; moreover, G_0 is a normal subgroup of G_1 and it is c-convex in G_1 ([6], Section 4).

If $G_0 = \{0\}$, then for $g = (a, x) \in G$ we put $\varphi(g) = a$; it is easy to verify that φ is an isomorphism of G onto $G_1(K)$. Hence we have

2.2. Lemma. *Let $G_0 = \{0\}$. Then G_1 is isomorphic to a cyclically ordered subgroup of K.*

2.3. Lemma. *Let $G_1(K) \neq \{0\}$. Assume that G_1 belongs to $\mathscr{C}_u$. Then $G_0 = \{0\}$.*

Proof. By way of contradiction, suppose that $G_0 \neq \{0\}$. Since $G_1(K) \neq \{0\}$ there is $g \in G_1$ with $g = (a, y)$ such that $0 \neq a \in K$ and $y \in L$.

We define a mapping $\psi: G \to G$ as follows. We choose a fixed $z \in G_0$, $z \neq 0$ and for each $g_1 \in G$ we put

$$\psi(g_1) = g_1 + z \quad \text{if} \quad g_1 \in g + G_0\,, \quad \text{and}$$
$$\psi(g_1) = g_1 \quad \text{otherwise}\,.$$

Then ψ is an automorphism of the cyclically ordered set $(G; [\,])$ such that $\psi(0) = 0$. Now we define a binary operation $+_1$ on G by putting

$$g_1 +_1 g_2 = \psi(\psi^{-1}(g_1) + \psi^{-1}(g_2))$$

for each $g_1, g_2 \in G$. The operation $+_1$ does not coincide with $+$ (since $g + g \neq g +_1 g$) and $(G; [\,], +_1)$ is a cyclically ordered group whose neutral element is 0; in this way we arrived at a contradiction.

From 2.2 and 2.3 we obtain as a corollary

2.4. Theorem. *Let G_1 be a cyclically ordered group belonging to $\mathscr{C}_u$. Then G_1 is isomorphic to a cyclically ordered subgroup of K.*

3. THE POWER OF THE CLASS $\mathscr{C}_u$

Let us denote by R the additive group of all reals with the natural linear order.

3.1. Lemma. *Let $(G'; \leqq, +) \in \mathcal{O}_u$. Assume that $(G'; \leqq, +)$ is an l-subgroup of R such that $1 \in G'$. Put $G = \{x \in G' : 0 \leqq x < 1\}$ and let $+_1$ be the binary operation on G defined as to be the addition* mod 1. *For $x, y, z \in G$ put $[x, y, z]$ if the relation* (1) *is valid Then*

(i) $G_1 = (G; [\,], +_1)$ *is a cyclically ordered group;*

(ii) *if $G \neq \{0\}$; then $G_1 \in \mathscr{C}_u$.*

Proof. (i) can be verified by a routine calculation; it will be omitted. Assume that $G \neq \{0\}$. Then G_1 fails to be a linearly ordered group. It remains to show that G_1 has a unique addition.

Let $+_2$ be a binary operation on G such that $G_2 = (G; [\,], +_2)$ is a cyclically ordered group such that its neutral element is 0. Assume that $+$ is the original group operation on R (i.e., the addition of reals); hence $+$ is also the group operation of G'. Let us define a new binary operation $+^2$ on G' in the following manner:

Let $x, y \in G'$. There are uniquely determined integers x_0, y_0 and uniquely determined elements x^1, y^1 of G such that $x = x^0 + x^1$ and $y = y^0 + y^1$. We put

(2) $$x +^2 y = (x^0 + y^0) + z^1 ,$$

where $z^1 = x^1 +_2 y^1$ if $x^1 +_2 y^1 < 1$, and $z^1 = (x^1 +_2 y^1) - 1$ otherwise. Then the set G' with the natural linear order and with the operation $+^2$ is a linearly ordered group with the neutral element 0. Since G' has a unique addition, the operation $+^2$ coincides with the operation $+$ on G'. Thus from (2) we infer that the operation $+_2$ on G must coincide with the operation $+_1$. Therefore G_1 has a unique addition.

3.2. Lemma. *Let $(G'', \leqq, +)$ be an l-subgroup of R with $1 \in G''$. Let $G^* = \{x \in G'' : 0 \leqq x < 1\}$ and let us define the cyclically ordered group $G_3 = (G^*, [\,], +_2)$ analogously as we did for G_1 in* 3.1 *with the distinction that we now have G^* instead of G'. Suppose that the cyclically ordered groups G_1 and G_3*

are isomorphic. Then the linearly ordered groups $(G'; \leqq, +)$ *and* $(G^*; \leqq, +)$ *are isomorphic as well.*

Proof. Let φ be an isomorphism of G_1 onto G_3. For $x \in G'$ let x^0 and x^1 be as in the proof of 3.2. Put

$$\psi(x) = x^0 + \varphi(x^1) .$$

Then ψ is an isomorphism of $(G'; \leqq, +)$ onto $(G^*; \leqq, +)$.

The following theorem is the main result of [8].

3.3. Theorem. (Ohkuma) *There exists a subset* $\{G^i: i \in I\}$ *of* $\mathcal{O}_u$ *such that*

(i) card $I = 2^{2^{\aleph_0}}$;

(ii) *if* $i(1)$ *and* $i(2)$ *are distinct elements of* I, *then* $G^{i(1)}$ *fails to be isomorphic to* $G^{i(2)}$;

(iii) *for each* $i \in I$, G^i *is an l-subgroup of* R *and contains all rational numbers.*

(The assertions (i) and (ii) are expressed in Theorem 3 of of [8] (cf. also [2]); (iii) follows from the constructions established in Section 2 and Section 3 of [8].)

If $(G'; \leqq, +) = G^i$ for some $i \in I$, then let $\chi(G^i) = G_1$, where G_1 is as in 3.1. In view of 3.2, if $i(1)$ and $i(2)$ are distinct elements of I, then $\chi(G^{i(1)})$ is not isomorphic to $\chi(G^{i(2)})$.

Now from 2.4, 3.1 and 3.3 we obtain:

3.4. Theorem. *There are exactly* $2^{2^{\aleph_0}}$ *nonisomorphic types of cyclically ordered groups belonging to* $\mathcal{C}_u$.

Next, from 3.3, 3.4 and 2.1 we infer that in 3.4 the class $\mathcal{C}_u$ can be replaced by $\mathcal{C}_u^0$.

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