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BOUNDED, ALMOST-PERIODIC, AND PERIODIC SOLUTIONS TO FULLY NONLINEAR TELEGRAPH EQUATIONS

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During the last years, many attempts have been made to describe the global behaviour of solutions to nonlinear partial differential equations of the hyperbolic type. We remark in passing that the mere existence of global solutions represents a difficult task solved in particular cases only. A characteristic feature of such problems is that they do not generally possess globally defined smooth solutions, no matter how smooth the data are.

It seems interesting, both mathematically and physically, to consider situations where some dissipative mechanism is present. In this case, the dissipation can prevail and ensure the global existence of smooth solutions provided that the data are small and smooth (see e.g. Kato [4], Matsumura [7], Milani [8], Shibata [11], and many others).

In the present paper, we deal with solutions $u = u(x, t)$, $x \in (0, L)$, $t \in R^1$ of the equation

(E) $$\mathscr{L}u + F(u_{xx}, u_x, u, u_{tt}, u_{xt}, u_t) = f(x, t)$$

where $\mathscr{L}u = u_{tt} + du_t - au_{xx}$, $a, d > 0$ together with the boundary conditions

(B) $$u(0, t) = u(L, t) = 0\,, \quad t \in R^1\,.$$

To put it more exactly, the global in time solutions will be studied (i.e. the solutions defined for all $t \in R^1$) on condition that, roughly speaking, the function f is small and smooth.

Several preliminary remarks are in order. To begin with, results of this type namely the existence of time periodic solutions to the above problem were obtained e.g. by Rabinowitz [10], Petzeltová-Štědrý [9], or Krejčí [5]. Their approach leans on the accelerated convergence method developed by Nash and Moser in order to cope with the "derivative loss", which prevents the use of classical inverse function theorems.

In his recent work [13], Štědrý succeeded in avoiding this rather complicated technique via decomposition of a solution in the form $u(x, t) = u_0(x) + \int_0^t u_t(x, s)\, ds$. In such a way, the existence and even uniqueness of small time periodic solutions can be stated.

There is also an interesting paper of Kato [3] where a new technique is suggested to treat the problems with the derivative loss.

A common feature of all quoted papers is that the authors prefer to work directly on the spaces of periodic functions, which resembles slightly the methods inherent to stationary problems. It seems interesting (and more natural) to look for global solutions of an evolutionary problem and then, as an added benefit of the method employed, to obtain the existence and uniqueness of bounded, almost-periodic, or periodic solutions.

One of the major stumbling blocks in this approach is represented by the beforementioned derivative loss resulting from the presence of the second derivative terms in the nonlinearity. In [12], Shibata and Tsutsumi investigated the local existence problem for fully nonlinear wave equation. To avoid the use of the Nash-Moser method, they transformed the equation to a system consisting of a quasilinear hyperbolic and nonlinear elliptic equation. This paper motivated my work and I am indebted to the authors for it.

Following [12], we set $u_t = v$ and differentiate the equation (E) with respect to the variable t. We arrive at the following system

$$(\mathrm{S}_1) \qquad \begin{aligned} &\mathscr{L}v + F_1'(D_x^2u, D^1v)\, v_{xx} + F_2'(D_x^2u, D^1v)\, v_x + \\ &+ F_3'(D_x^2u, D^1v)\, v + F_4'(D_x^2u, D^1v)\, v_{tt} + \\ &+ F_5'(D_x^2u, D^1v)\, v_{xt} + F_6'(D_x^2u, D^1v)\, v_t = f_t\,, \end{aligned}$$

$$(\mathrm{S}_2) \qquad -au_{xx} + F(D_x^2u, D^1v) = f - v_t - dv$$

for $x \in (0, L)$, $t \in R^1$ with

$$(\mathrm{B}) \qquad u(0, t) = v(0, t) = u(L, t) = v(L, t) = 0$$

for $t \in R^1$ (this step will be justified in Section 3).

The system (S_1), (S_2), (B) is much more simple to deal with since the equation (S_1) is, in fact, quasilinear in v. Thus, a classical iteration scheme may be used to get desirable results. Consequently, the basic strategy employed here leans on the solution of corresponding linear problems (see Section 4).

A simple application of compactness arguments based on embedding relations of Sobolev spaces yields then the existence of a unique (small) bounded solution provided f is small and smooth. Obviously, this fact can be directly exploited when the almost-periodic or periodic case is treated (Section 6, Section 7).

1. FUNCTION SPACES

To begin with, throughout the whole text the symbols c_i, $i = 1, 2, \ldots$ stand for strictly positive real constants, h_i, $i = 1, 2, \ldots$ are positive, continuous, nondecreasing functions defined on $[0, +\infty)$. By R^n we denote the standard n-dimensional Euclidean space.

Let $w = (w_1, \ldots, w_m)$ be a (possible) vector function of x, t. The symbol $D^k w$ denotes the vector

$$D^k w = \left\{ \frac{\partial^{i+j} w_l}{\partial x^i \, \partial t^j} \,\middle|\, l = 1, \ldots, m, \; i, j \geqq 0, \; i + j \leqq k \right\}$$

and $D_y^k w$ (where $y = x$ or $y = t$) is

$$D_y^k w = \left\{ \frac{\partial^i w_l}{\partial y^i} \,\middle|\, l = 1, \ldots, m, \; 0 \leqq i \leqq k \right\}.$$

Here (and always) i, j, k, l stand for nonnegative integers.

Let us continue by listing some function spaces used in what follows. We start with the Lebesgue spaces $L_p = L_p(0, L)$, $p \in [1, +\infty]$ of integrable functions with associated norm $\| \, \|_p$ determined in a standard way. For $w = (w_1, \ldots, w_m)$, we set

$$|w| = \max \{ \|w_l\|_2 \mid l = 1, \ldots, m \} .$$

The symbols $H^k = H^k(0, L)$ denote the Sobolev spaces of those functions having derivatives up to the order k in L_2. Moreover, we determine

$$H_0^1 = \{ w \mid w \in H^1, \; w(0) = w(L) = 0 \} .$$

Next, we make use of vector functions ranging in a Banach space B. With I an interval of real numbers, we consider the space $W_p^k(I, B)$ of vector functions which derivatives up to the order k lie in the generalized Lebesgue space $L_p(I, B)$.

$C^k(I, B)$ is a Banach space containing all functions having derivatives up to the order k continuous and bounded on I.

The reader interested in precise definitions and basic properties of before-mentioned spaces can consult, for example, the monograph Vejvoda et al. [14].

Finally, the spaces of almost periodic functions are of interest. A continuous function $w: R^1 \to B$ is said to be *almost-periodic* if to every $\varepsilon > 0$ there corresponds a relatively dense set $\{\tau\}_\varepsilon \subset R^1$ such that

$$\sup \{ \|w(t + \tau) - w(t)\|_B \mid t \in R^1 \} \leqq \varepsilon$$

for all $\tau \in \{\tau\}_\varepsilon$. The space of all almost-periodic functions will be denoted $AP(R^1, B)$.

The above definition is taken over from [1] where the useful characterization can be found:

Lemma 1 (Bochner's criterion). *Let $w: R^1 \to B$ be a continuous function. w is then almost-periodic if, and only if, for an arbitrary real sequence $\{t_n\}_{n=1}^\infty$ there is a subsequence $\{t'_n\}_{n=1}^\infty$ such that the sequence of functions $w(\cdot + t'_n)$ converges uniformly in $t \in R^1$.*

For convenience, we are going to introduce the following sets. Identifying a function $w = w(x, t)$ with a vector function $w: t \to w(\cdot, t) \in L_2$ we set

$$X^k = \{ w \mid \text{each component of } D^k w \text{ belongs to } C(R^1, L_2) \} ,$$

$$X_0^k = \left\{ w \mid w \in X^k, \frac{\partial^i w}{\partial t^i} \in C(R^1, H_0^1) \text{ for } i = 0, \ldots, k-1 \right\},$$

$$X^k(\varepsilon) = \{ w \mid w \in X^k, |D^k w(t)| \leqq \varepsilon, t \in R^1 \}.$$

Similarly

$$Y^K = \{ w \mid \text{each component of } D^k w \text{ belongs to } L_\infty(R^1, L_2) \},$$

$$Y_0^k = \left\{ w \mid w \in Y^k, \frac{\partial^i w}{\partial t^i} \in L_\infty(R^1, H_0^1) \text{ for } i = 0, \ldots, k-1 \right\},$$

$$Y^k(\varepsilon) = \{ w \mid w \in Y^k, |D^k w(t)| \leqq \varepsilon \text{ for a.e. } t \in R^1 \}.$$

To conclude with, we are going to list auxiliary statements concerning the properties of substitution (Nemyckij) operators on the aforementioned spaces.

The first statement is easy to verify seeing that H^1, H^2 are Banach algebras and $H^1 \hookrightarrow C[0, L]$ according to the well known embedding relations.

Lemma 2. *Suppose $k = 1$ or $k = 2$, $v, w \in Y^k$.*

Then $vw \in Y^k$ and the following estimate holds

(1.1) $$|D^k v\, w(t)| \leqq c_1 |D^k v(t)|\, |D^k w(t)|$$

for a.e. $t \in R^1$.

Combining the preceding result together with the Taylor expansion formula we arrive at the following conclusion.

Lemma 3. *Let $\Phi: U \subset R^m \to R^1$ be a function $\Phi \in C^4(\bar{U}, R^1)$ where U is an open ball centered in $0 \in R^m$. Consider functions $v = (v_1, \ldots, v_m)$, $w = (w_1, \ldots, w_m)$, both ranging in U, such that $v_i, w_i \in Y^k$, $i = 1, \ldots, m$ where $k = 1$ or $k = 2$.*

Then $\Phi \circ v$, $\Phi \circ w \in Y^k$ and

(1.2) $$|D^k(\Phi \circ v - \Phi \circ w)(t)| \leqq h_1(z(t))\, |D^k(v - w)(t)|$$

holds for a.e. $t \in R^1$. If, moreover, $\Phi'(0) = 0$, then

(1.3) $$|D^k(\Phi \circ v - \Phi \circ w)(t)| \leqq z(t)\, h_2(z(t))\, |D^k(v - w)(t)|$$

for a.e. $t \in R^1$ where we have denoted

$$z(t) = \max \{ |D^k v(t)|, |D^k w(t)| \}.$$

2. FORMULATION OF MAIN RESULTS

The main focus of this paper is on global existence and qualitative properties of solutions to the problem (E), (B).

Theorem 1 (bounded solutions). *Let F be a smooth function defined on some open neighbourhood of the point $0 \in R^6$. Denoting $D_i'F$, $i = 1, \ldots, 6$ the corresponding partial derivatives we assume that*

(F) $$F(0) = D_i'F(0) = 0, \quad i = 1, \ldots, 6.$$

Then there is $\varepsilon > 0$ such that for every $f \in Y^3(\varepsilon)$ there exists a function $u \in X_0^4$ satisfying (E), (B) *for all $t \in R^1$. Moreover, there is $\delta > 0$ such that the function u is the only solution of* (E), (B) *belonging to $X_0^4(\delta)$.*

Remark. In accordance with the well known embedding theorems, $u \in X_0^4$ implies that u is a classical bounded solution satisfying (E), (B) pointwise.

Corollary 1. (periodic solutions). *Under the assumptions of Theorem* 1, *suppose, in addition, that f is ω-periodic with respect to t.*

Then there is a unique (small) ω-periodic solution to the problem (E), (B).

Theorem 2. (almost-periodic solutions). *Let all assumptions made in Theorem* 1 *be satisfied. Suppose, moreover,*

$$f \in Y^3(\varepsilon) \cap AP(R^1, L_2) .$$

Then the solution u, which existence is guaranteed by Theorem 1, *is also almost-periodic, more specifically*

$$u \in X_0^4(\delta) \cap AP(R^1, H^2) .$$

As to the proofs of these statements, we refer to Section 6 concerning Theorem 1, and to Section 7 for Theorem 2.

3. EQUIVALENT FORMULATION OF THE PROBLEM

In this section, our aim is to clarify the correspondence between the problems (E), (B) and (S_1), (S_2), (B) outlined in the introduction.

Lemma 4. *Let a couple (u, v) solve* (S_1), (S_2), (B), *$u \in X_0^2$, $v \in X_0^3$, each component of $D_x^2 u$ lying in X^2, and*

$$(3.1) \qquad |D^2 D_x^2\, u(t)| < \delta\,, \quad |D^3\, v(t)| < \delta\,, \quad t \in R^1$$

for $\delta > 0$ small enough.

Then $v = u_t$.

Corollary 2. *Let (u, v) be a solution of* (S_1), (S_2), (B) *satisfying* (3.1) *for $\delta > 0$ sufficiently small.*

Then $u \in X_0^4$, u solving the problem (E), (B) *and vice versa, i.e. if $u \in X_0^4(\delta)$ is a solution of* (E), (B), *then* (of course) *the pair (u, u_t) satisfies* (S_1), (S_2), (B).

Proof of Lemma 4. Set $w = u_t$. Seeing that w, $D_x^2 w \in X^1 \subsetneq C([0, L] \times R^1)$ we are allowed to differentiate (S_2) with respect to t:

$$-a w_{xx} + F_1'(D_x^2 u, D^1 v)\, w_{xx} + \ldots + F_3'(D_x^2 u, D^1 v)\, w + $$
$$+ F_4'(D_x^2 u, D^1 v)\, v_{tt} + \ldots + F_6'(D_x^2 u, D^1 v)\, v_t = f_t - v_{tt} - d v_t\,.$$

Setting $z = w - v$ and comparing with (S_1) we get

$$\text{(a)} \qquad -az_{xx} + F'_1(D_x^2u, D^1v)\, z_{xx} + F'_2(D_x^2u, D^1v)\, z_x + F'_3(D_x^2u, D^1v)\, z = 0\,,$$

$$\text{(b)} \qquad z(0, t) = z(L, t) = 0\,.$$

Now, the coefficients F'_i, $i = 1, \ldots, 3$ are small together with their first derivatives in $C([0, L] \times R^1)$ in accordance with Lemma 3, (F), and (3.1). Consequently, the boundary-value problem (a), (b) possesses for each fixed $t \in R^1$ exactly one solution z, namely $z \equiv 0$. Q.E.D.

In view of Corollary 2, we are allowed to study the problem (S_1), (S_2), (B) instead of (E), (B). This fact will enable us to prove the results claimed in Section 2.

4. THE LINEAR HYPERBOLIC EQUATION

Now, restrict our attention to solutions of the linearized equation related to (S_1). Consider the problem:

$$(L_1) \qquad \mathscr{L}v + a^1v_{xx} + a^2v_{xt} + a^3v_{tt} + b^1v_x + b^2v_t + b^3v = g\,,$$

$v = v(x, t)$, $x \in (0, L)$, $t \in [t_0, +\infty)$ together with the conditions

$$\text{(B)} \qquad v(0, t) = v(L, t) = 0\,, \quad t \in [t_0, +\infty)\,,$$

$$\text{(I)} \qquad v(x, t_0) = v^0(x)\,, \quad v_t(x, t_0) = v^1(x)\,, \quad x \in [0, L]\,.$$

As to the coefficients appearing, we require

$$(4.1) \qquad a^i \in Y^2(\alpha)\,, \quad i = 1, 2, 3\,.$$

$$(4.2) \qquad b^i \in Y^1(\alpha)\,, \quad i = 1, 2, 3\,.$$

Let us remark that there is a vast literature, hyperbolic problems of this type being concerned. Note in passing that much more general results were obtained concerning the regularity of coefficients and its relation to the existence question (see e.g. Arosio [2]) than these we are going to present here.

On the other hand, we were not able to find appropriate references fitting our specific situation. Nevertheless the methods employed are well known and we are going to point out principal ideas only, the details being available e.g. in the book of Lions-Magenes [6].

Lemma 5 (weak solutions). *Suppose that*

$$(4.3) \qquad v^0 \in H_0^1\,, \quad v^1 \in L_2\,, \quad g \in L_{2,\mathrm{loc}}((t_0, +\infty), L_2)\,.$$

Then there exists a weak solution v to the problem (L_1), (B), (I) uniquely determined in the class

$$v \in L_{1,\mathrm{loc}}((t_0, +\infty), H_0^1) \cap W^1_{1,\mathrm{loc}}((t_0, +\infty), L_2)$$

whenever $\alpha \in (0, \alpha_0)$ is sufficiently small.

Moreover, if

$$(4.4)\qquad g \in Y^0 ,$$

then $v \in C([t_0, +\infty), H_0^1) \cap C^1([t_0, +\infty), L_2)$ *and the energy estimate*

$$(4.5)\qquad e^{\beta t}|D^1 v(t)|^2 \leqq c_2[e^{\beta t_0}|D^1 v(t_0)|^2 + \int_{t_0}^t e^{\beta s}|g(s)|^2 \, ds]$$

holds for certain $\beta > 0$.

Proof. (a) Uniqueness: The uniqueness in the aforementioned class of functions can be shown using literally the same technique as in [2], [6].

(b) Existence: The existence may be stated with help of the Faedo-Galerkin method (we refer to [6] for details), the finite-dimensional approximation being based on the orthogonal system $\{\sin(n\pi/Lx)\}_{n=1}^{\infty}$.

Note that $D^1 a^i \in Y^1(\alpha) \circlearrowleft L_\infty([0, L] \times R^1)$ and, consequently, all standard energy estimates can be derived for the approximate problems.

Moreover, following the line of arguments from [2], the continuity of the couple (v, v_t) regarding the energy space $H_0^1 \times L_2$ can be shown.

The decay estimate (4.5) is a direct consequence of the presence of the damping term dv_t in the equation and is easy to obtain via multiplying the equation (in fact its finite-dimensional approximation) by $v_t + \gamma v$, $\gamma > 0$ sufficiently small. Q.E.D.

Lemma 6. (strong solutions). *Assume*

$$(4.6)\qquad v^0 \in H^2 \cap H_0^1 , \quad v^1 \in H_0^1 , \quad g \in Y^1 .$$

Then for all $\alpha \in (0, \alpha_0)$ *small enough, there exists a unique solution* v *of* (L_1), (B), (I),

$$(4.7)\qquad v \in \bigcap_{j=0}^{1} C^j([t_0, +\infty), H^{2-j} \cap H_0^1) \cap C^2([t_0, +\infty), L_2)$$

satisfying the estimate

$$(4.8)\qquad e^{\beta t}|D^2 v(t)|^2 \leqq c_3[e^{\beta t_0}|D^2 v(t_0)|^2 + \int_{t_0}^t e^{\beta s}|D^1 g(s)|^2 \, ds] .$$

Remark. $D^2 v(t_0)$ is determined with help of v^0, v^1, and the equation (L_1). Especially, we denote $v^2 = v_{tt}(\cdot, t_0)$.

Proof of Lemma 6. The regularity result (4.7) is achieved by a formal differentiation of (L_1) by t. Differentiating actually the Faedo-Galerkin approximation, this step is fully justified.

Repeating arguments of the proof of Lemma 5, the regularity of the function v_t is obtained. Note that, taking advantage of the choice of the basis, the approximate problems admit multiplication by the term v_{xx}. Thus, the hardest terms to cope with, namely

$$A = \int_0^L a_t^1(t)\, v_{xx}(t)\, v_{tt}(t) \, dx , \quad B = \int_0^L b_t^1(t)\, v_x(t)\, v_{tt}(t) \, dx ,$$

are estimated in the following way.

$$|A| \leqq \|a_t^1(t)\|'_\infty \, v_{xx}(t)\|_2 \, \|v_{tt}(t)\|_2 ,$$

$$|B| \leqq \|b_t^1(t)\|_2 \, \|v_x(t)\|_\infty \, \|v_{tt}(t)\|_2 \leqq$$
$$\leqq c_4 \|b_t^1(t)\|_2 \, \|v_{xx}(t)\|_2 \, \|v_{tt}(t)\|_2 \,,$$

the term $\|v_{xx}(t)\|_2$ being estimated from above with help of $\|v_{tt}(t)\|_2$, $\|v_{xt}(t)\|_2$ by means of the equation (L_1).

Consequently, all results claimed in Lemma 5 hold for v_t as well.

Finally, using (L_1) again, the regularity of v_{xx} is obtained. Q.E.D.

Lemma 7 (classical solutions). *Assume that* $b^1 = b^2 = b^3 = 0$ *and*

$$(4.9) \qquad v^0 \in H^3 \cap H_0^1 \,, \quad v^1 \in H^2 \cap H_0^1 \,, \quad g \in Y^2$$

together with the compatibility condition

$$(4.10) \qquad v^2 = v_{tt}(\cdot, t_0) \in H_0^1 \,.$$

Then for all $\alpha \in (0, \alpha_0)$ *small, the problem* (L_1), (B), (I) *possesses a unique (classical) solution* v,

$$(4.11) \qquad v \in \bigcap_{j=0}^{2} C^j([t_0, +\infty), H^{3-j} \cap H_0^1) \cap C^3([t_0, +\infty), L_2) \,,$$

v satisfying

$$(4.12) \qquad e^{\beta t} |D^3 v(t)|^2 \leqq c_5 [e^{\beta t_0} |D^3 v(t_0)|^2 + \int_{t_0}^t e^{\beta s} |D^2 g(s)|^2 \, ds] \,,$$

for certain $\beta > 0$.

Proof. Seeing that $b^1 = b^2 = b^3 = 0$, our basic strategy is to differentiate (L_1) and apply Lemma 6 to the function v_t.

To this end, we express

$$(a) \qquad v_{xx} = (a - a^1)^{-1} (v_{tt} + dv_t + a^3 v_{tt} + a^2 v_{xt} - g) \,.$$

Taking advantage of this equality and setting $w = v_t$ we deduce that w is a unique weak solution of the problem (L_2), (B), (I') with

$$(L_2) \qquad \mathcal{L}w + a^1 w_{xx} + a^2 w_{xt} + a^3 w_t + b^1 w_x + b^2 w_t + b^3 w =$$
$$= g_t + a_t^1 (a - a^1)^{-1} g \,,$$

$$(I') \qquad w(\cdot, t_0) = v^1 \,, \quad w_t(\cdot, t_0) = v^2$$

where b^i are determined by a^i and satisfy (4.2) in view of Lemma 2, 3.

Consequently, Lemma 6 yields the regularity of w. Using (a) we derive the regularity of v regarding the variable x. Q.E.D.

Having prepared all preliminary statements we are about to show the main result concerning the global existence of bounded solutions to the linearized problem.

Theorem 3 (bounded solutions). *Consider the problem*

$$(L) \qquad \mathcal{L}v + a^1 v_{xx} + a^2 v_{xt} + a^3 v_{tt} = g \quad \text{on} \quad (0, L) \times R^1$$

together with the condition (B) *where*

$$(4.13)\qquad a^i \in Y^2(\alpha), \quad \alpha \in (0, \alpha_0), \quad g \in Y^k, \quad i = 1, 2, 3, \quad k = 1 \text{ or } k = 2.$$

The problem (L), (B) *possesses a unique solution* v, $v \in X_0^{k+1}$ *such that*

$$(4.14)\qquad |D^{k+1} v(t)| \leqq c_6 \operatorname{ess\,sup} \{|D^k g(s)|, \ s \in R^1\}.$$

Proof. Take a function $\psi_n \in C^\infty(R^1)$,

$$\psi_n \begin{cases} = 0 & \text{on } (-\infty, -n] \\ \in [0, 1] & \text{on } [-n, -n+1] \\ = 1 & \text{on } [-n+1, +\infty). \end{cases}$$

According to Lemma 7, we are able to solve the initial boundary value problems

$$\text{(a)}\qquad \mathscr{L}v^n + a^1 v^n_{xx} + a^2 v^n_{xt} + a^3 v^n_{tt} = \psi_n g \quad \text{on} \quad (0, L) \times [-n, +\infty),$$

$$\text{(b)}\qquad v^n(0, t) = v^n(L, t) = 0, \quad t \in [-n, +\infty),$$

$$\text{(c)}\qquad v^n(\cdot, -n) = v^n_t(\cdot, -n) = 0.$$

We set $v^n = 0$ for $t \in (-\infty, -n]$ so that v^n may belong to the space X_0^{k+1}.

In view of (4.8), we get

$$|D^{k+1} v^n(t)| \leqq c_8 \operatorname{ess\,sup} \{|D^k g(s)| \mid s \in R^1\}.$$

The right-hand side of the above relation being bounded, there is an accumulation point $v \in Y_0^{k+1}$ of the sequence $\{v^n\}_{n=1}^\infty$ with respect to the weak-star topology induced on $L_\infty(R^1, L_2)$.

Dealing with a linear equation we infer v solves (L), (B) on R^1.

The regularity results achieved in Lemma 7 imply, in fact, $v \in X_0^{k+1}$.

The uniqueness of a bounded solution is an easy consequence of (4.12). Q.E.D.

5. THE NONLINEAR ELLIPTIC PROBLEM

We are going to discuss the "elliptic" equation corresponding to (S_2).

Lemma 8. *Let* $f \in Y^3(\varepsilon)$, $v \in X^2(\delta)$, $\varepsilon, \delta > 0$ *being sufficiently small.*

Then we are able to find a uniquely determined (small) function $u \in X_0^1$, $D_x^2 u \in (X^1)^3$, $u = \mathscr{G}(v, f)$, u *being a solution of the equation* (S_2), (B) *on* R^1 *for* f, v *fixed.*

Furthermore, we have the estimates

$$(5.1)\qquad |D^k D_x^2(u^1 - u^2)(t)| \leqq c_9 |D^{k+1}(v^1 - v^2)(t)|,$$

$$(5.2)\qquad |D^k D_x^2 u^0(t)| \leqq c_{10} |D^k f(t)|$$

where $k = 0, 1$, $t \in R^1$, $u^i = \mathscr{G}(v^i, f)$, $i = 0, 1, 2$, $v^0 = 0$.

In case $v \in X^3(\delta)$, *we obtain* $u \in X_0^2$, $D_x^2 u \in (X^2)^3$, *and* (5.1), (5.2) *hold even for* $k = 2$.

Proof. For fixed v, f consider an operator S determined for all u,

$$u \in \mathscr{B}(\gamma) = \{u \in X_0^1, \ |D^1 D_x^2 u(t)| \leqq \gamma\}, \quad \gamma > 0$$

as a unique solution of the linear problem

(a) $$-a(Su)_{xx} + F(D_x^2 u, D^1 v) = f - v_t - dv \quad \text{on} \quad (0, L) \times R^1,$$

(b) $$Su(0, t) = Su(L, t) = 0, \quad t \in R^1.$$

For $k = 0, 1$ we deduce

(c) $$|D^k D_x^2(Su^1 - Su^2)(t)| \leqq c_{11}|D^k(F(D_x^2 u^1, D^1 v) - F(D_x^2 u^2, D^1 v))(t)| \leqq$$
(according to (1.3), (F))
$$\leqq c_{11} \max\{\delta, \gamma\} \, h_2(\max\{\delta, \gamma\}) \, D^k D_x^2(u^1 - u^2)(t)|.$$

Next, we set $u^2 = 0$ in (c) to obtain

(d) $$|D^k D_x^2 S \, u(t)| \leqq |D^k D_x^2 \, S(0)(t)| + c_{11} \max\{\delta, \gamma\} \times$$
$$\times \, h_2(\max\{\delta, \gamma\}' \, |D^k D_x^2 \, u(t)| \leqq$$
$$\leqq c_{12}[\varepsilon + \delta + \max\{\delta, \gamma\} \, h_2(\max\{\delta, \gamma\}) \, |D^k D_x^2 \, u(t)|].$$

Choosing $\varepsilon, \delta > 0$ appropriately small we can see that

$$S: \mathscr{B}(\gamma) \to \mathscr{B}(\gamma)$$

is a contractive mapping for certain (small) $\gamma > 0$. Consequently, there is a uniquely determined fixed point of S in $\mathscr{B}(\gamma)$ – a solution of (S_2), (B).

In order to establish (5.1), we estimate

$$|D^k D_x^2(u^1 - u^2)(t)| \leqq c_{13}|D^k(F(D_x^2 u^1, D^1 v^1) - F(D_x^2 u^2, D^1 v^2))(t)| +$$
$$+ |D^{k+1}(v^1 - v^2)(t)|.$$

With Lemma 3 in mind, we obtain

$$|D^k(F(D_x^2 u^1, D^1 v^1) - F(D_x^2 u^2, D^1 v^2)(t)| \leqq$$
$$\leqq c_{14} \max\{\gamma, \delta\} \, h_2(\max\{\delta, \gamma\}) \, [|D^k D_x^2(u^1 - u^2)(t)| +$$
$$+ |D^{k+1}(v^1 - v^2)(t)|],$$

which yields immediately (5.1). Q.E.D.

6. THE PROOF OF THEOREM 1

First of all, we claim that, in view of Corollary 2, we may confine ourselves to the system (S_1), (S_2), (B).

Existence. We try to solve the problem taking advantage of a standard iterative technique.

Starting with $v^1 = 0$ we determine a sequence $\{v^n\}_{n=1}^{\infty}$ of approximate solutions using

$$(\mathrm{S}_1^n) \qquad \mathscr{L}v^{n+1} + F_1'(D_x^2u^n, D^1v^n)\, v_{xx}^{n+1} + F_4'(\dots)\, v_{tt}^{n+1} + F_5'(\dots)\, v_{xt}^{n+1} = = -F_2'(\dots)\, v_x^n - F_3'(\dots)\, v^n - F_6'(\dots)\, v_t^n + f_t \quad \text{on} \quad (0, L) \times R^1 \,,$$

$$(\mathrm{B}) \qquad v^{n+1}(0, t) = v^{n+1}(L, t) = 0 \,, \quad t \in R^1$$

where the function $u^n = \mathscr{G}(v^n, f)$ is a unique solution of

$$(\mathrm{S}_2^n) \qquad -au_{xx}^n + F(D_x^2u^n, D^1v^n) = f - v_t^n - dv^n \quad \text{on} \quad (0, L) \times R^1 \,,$$

$$(\mathrm{B}) \qquad u^n(0, t) = u^n(L, t) = 0 \,, \quad t \in R^1 \,.$$

Our present goal rests on the choice of $\varepsilon > 0$ appropriately small so that the relation

$$(6.1) \qquad f \in Y^3(\varepsilon)$$

may ensure solvability of (S_1^n), (S_2^n) with

$$(6.2) \qquad v^n \in X_0^3(\delta) \,, \quad n = 1, 2, \dots$$

for sufficiently small $\delta > 0$.

Assume that we have already determined the functions v^k, $k = 1, \dots, n$ satisfying (6.2).

By means of Lemma 8, the function u^n can be found, u^n being determined by (S_2^n), (B).

Inserting u^n, v^n into (S_1^n) we are able to get v^{n+1} provided we can verify (4.13), namely

$$F_i'(D_x^2u^n, D^1v^n) \in Y^2(\alpha) \,, \quad i = 1, 4, 5$$

for $\alpha > 0$ small enough.

In view of Lemma 3, we get

$$(6.3) \qquad |D^2F_i'(D_x^2u^n, D^1v^n)(t)| \leqq$$

(according to (1.2), (F))

$$\leqq h_1(\max\{|D^2D_x^2\, u^n(t)|, \delta\}) \max\{|D^2D_x^2\, u^n(t)|, \delta\} \leqq$$

(in accordance with (5.1), (5.2))

$$\leqq h_3(\max\{\varepsilon, \delta\}) \max\{\varepsilon, \delta\} \,.$$

Thus for $\varepsilon, \delta > 0$ small, (4.13) is satisfied and we can apply Theorem 3 to obtain the function $v^{n+1} \in X_0^3$.

As a final step, we are about to show (6.2) for v^{n+1}. With (1.1) in mind, we deduce

$$|D^3\, v^{n+1}(t)| \leqq c_6 \operatorname*{ess\,sup}_{s \in R^1} \{ \max_{i=2,3,6} |D^2F_i'(D_x^2u^n, D^1v^n)(s)|\, \delta + \varepsilon \} \leqq$$

(according to (6.3))

$$\leqq c_{15}(h_3(\max\{\varepsilon, \delta\}) \max\{\varepsilon, \delta\}\, \delta + \varepsilon) \,.$$

Consequently, (6.2) holds for v^{n+1} whenever $\varepsilon, \delta > 0$ are chosen appropriately small.

We conclude that there are sequences $\{u^n\}_{n=1}^{\infty}$, $\{v^n\}_{n=1}^{\infty}$ determined by (S_1^n), (S_2^n),

(B) such that

$$(6.4) \qquad v^n \in X_0^3(\delta), \quad u^n \in X_0^1, \quad D_x^2 u^n \in (X^2(\delta))^3 .$$

As a consequence, passing to subsequences as the case may be, we get the existence of a pair (u, v),

$$(6.5) \qquad D^2 D_x^2 u^n \to D^2 D_x^2 u ,$$

$$(6.6) \qquad D^3 v^n \to D^3 v$$

componentwise with respect to the weak-star topology on the space $L_\infty(R^1, L_2)$. Moreover, we have

$$D_x^2 u^n, D^1 v^n \in X^2(\delta) \subsetneq H^2_{\mathrm{loc}}((0, L) \times R^1) .$$

Seeing that the embedding $H^2(K) \circlearrowleft C(K)$, $K \subset R^2$ compact, is completely continuous, we derive

$$(6.7) \qquad D_x^2 u^n \to D_x^2 u , \quad D^1 v^n \to D^1 v$$

in the strong topology of $C_{\mathrm{loc}}([0, L] \times R^1)$.

Similarly, we get $D^2 v^n \in X^1(\delta) \subsetneq H^1_{\mathrm{loc}}((0, L) \times R^1)$, and consequently

$$(6.8) \qquad D^2 v^n \to D^2 v$$

componentwise in the strong topology on $L_{p,\mathrm{loc}}((0, L) \times R^1)$, $p < +\infty$.

As a result of (6.7), (6.8), we deduce that $u \in Y_0^1$, $D_x^2 u \in (Y^2(\delta))^3$, $v \in Y_0^3(\delta)$ is a solution of (S_1), (S_2), (B).

According to the regularity result for linear problems achieved in Theorem 3, we get, in fact

$$(6.9) \qquad u \in X_0^1 , \quad D_x^2 u \in (X^2(\delta))^3 , \quad v \in X_0^3(\delta) ,$$

Lemma 8 being taken into account.

Uniqueness. Suppose we have two pairs (u^1, v^1), (u^2, v^2) satisfying (S_1), (S_2), (B), and (6.9).

Setting $w = v^1 - v^2$ we obtain from (S_1)

$$(6.10) \qquad \begin{aligned} &\mathscr{L}w + F_1'(D_x^2 u^1, D^1 v^1)\, w_{xx} + \ldots + F_5'(D_x^2 u^1, D^1 v^1)\, w_{xt} = \\ &= (F_1'(D_x^2 u^2, D^1 v^2) - F_1'(D_x^2 u^1, D^1 v^1))\, v_{xx}^2 + \ldots + \\ &\quad (F_5'(D_x^2 u^2, D^1 v^2) - F_5'(D_x^2 u^1, D^1 v^1))\, v_{xt}^2 + \\ &\quad F_2'(D_x^2 u^2, D^1 v^2)\, v_x^2 - F_2'(D_x^2 u^1, D^1 v^1)\, v_x^1 + \ldots + \\ &\quad F_6'(D_x^2 u^2, D^1 v^2)\, v_t^2 - F_6'(D_x^2 u^1, D^1 v^1)\, v_t^1 . \end{aligned}$$

Evoking the relation (4.14) for $k = 1$ we obtain

$$|D^2 w(t)| \leqq$$

(according to Lemma 2, 3)

$$\leqq \sup_{s \in R^1} \{\delta\, h_4(\delta) [|D^1 D_x^2(u^1 - u^2)(s)| + |D^2(v^1 - v^2)(s)|]\} \leqq$$

(using (5.1))

$$\leqq \delta h_5(\delta) \sup \{|D^2\, w(s)| \mid s \in R^1\} .$$

If $\delta > 0$ is small enough, we deduce

$$|D^2\, w(t)| \leqq \tfrac{1}{2} \sup \{|D^2\, w(s)| \mid s \in R^1\} \tag{6.11}$$

which yields $w = 0$, and (by (5.1)) $u^1 = u^2$.

Theorem 1 has been proved.

7. THE PROOF OF THEOREM 2

Suppose

$$f \in Y^3(\varepsilon) \cap AP(R^1, L_2) . \tag{7.1}$$

Assume that

$$u \notin AP(R^1, H^2) .$$

In view of Lemma 1 there is a sequence $\{t_n\}_{n=1}^{\infty} \subset R^1$ such that

$$f(\cdot + t_n) \to \hat{f} \quad \text{uniformly in} \quad C(R^1, L_2) \tag{7.2}$$

and two subsequences $\{t'_n\}_{n=1}^{\infty}$, $\{t''_n\}_{n=1}^{\infty}$ satisfying

$$\|u(s_n + t'_n) - u(s_n + t''_n)\|_{H^2} \geqq K > 0 \tag{7.3}$$

for a certain sequence $\{s_n\}_{n=1}^{\infty}$.

Repeating arguments from the proof of existence in Section 6 we can extract subsequences such that

$$u' = \lim_{n\to\infty} u(\cdot + s_n + t'_n) , \quad v' = \lim_{n\to\infty} (v \cdot + s_n + t''_n) ,$$

$$u'' = \lim_{n\to\infty} u(\cdot + s_n + t'_n) , \quad v'' = \lim_{n\to\infty} (v \cdot + s_n + t''_n)$$

the pairs $(u', v'), (u'', v'')$ solving (S_1), (S_2), (B) with the right-hand side $f = f'$, $f = f''$ respectively where

$$f' = \lim_{n\to\infty} f(\cdot + s_n + t'_n) , \quad f'' = \lim_{n\to\infty} (\cdot + s_n + t''_n)$$

uniformly in $C(R^1, L_2)$.

But according to (7.2), we have necessarily $f' = f''$. On the other hand, combining (6.7) together with (7.3) we deduce

$$u' \neq u'' .$$

Thus, we have obtained a contradiction with uniqueness claimed in Theorem 1.

The proof of Theorem 2 is complete.

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