Ján Jakubík

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CONVERGENCES AND HIGHER DEGREES OF DISTRIBUTIVITY OF LATTICE ORDERED GROUPS AND OF BOOLEAN ALGEBRAS

JÁN JAKUBÍK, Košice

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All lattice ordered groups dealt with in the present note are assumed to be abelian. The partially ordered set of all convergences of a lattice ordered group G will be denoted by Conv G (cf. [2], [3]). Similarly, Conv B denotes the partially ordered set of all convergences of a Boolean algebra B (cf. [8]). In general, neither Conv G nor Conv B need be a lattice; Conv G is a lattice iff it possesses a greatest element and in such a case it is a complete lattice. An analogous result holds for Conv B.

In [7] it was shown that the existence of the greatest element in Conv G depends merely from the lattice properties of G and that the class of all lattice ordered groups having the largest convergence is a radical class (in the sense of [5]).

The following results were proved in [6] and [8]:

(A) If G is a completely distributive archimedean lattice ordered group, then Conv G is a complete lattice.

(B) If B is a completely distributive Boolean algebra, then Conv B is a complete lattice.

In the present note these results will be sharpened as follows:

(A_1) If G is an $(\aleph_0, 2)$-distributive lattice ordered group, then Conv G is a complete lattice.

(B_1) If B is an $(\aleph_0, 2)$-distributive Boolean algebra, then Conv B is a complete lattice.

The notion of bounded convergence in a lattice ordered group was introduced in [7]. Let $\mathrm{Conv}_b\, G$ be the set of all bounded convergences in G.

If $0 < e \in G$ and e is a singular element, then the interval $[0, e]$ of G is Boolean algebra. Put $[0, e] = B$. It will be shown that if e is, at the same time, a strong unit in G, then the partially ordered sets $\mathrm{Conv}_b\, G$ and Conv B are isomorphic. Next, Conv B is a complete lattice if and only if Conv G is a complete lattice.

Let G be a lattice ordered group. We recall briefly the basic notions concerning sequential convergences in G.

Let N be the set of all positive integers and let $G_n = G$ for each $n \in N$. We denote $\prod_{n\in N} G_n = G^N$. The elements of G^N (denoted, e.g., by (g_n)) are called *sequences in G*. If $g \in G$ and $g_n = g$ for each $n \in N$, then we denote $(g_n) = \text{const } g$.

Let α be a convex subsemigroup of the semigroup $(G^N)^+$ such that the following conditions are satisfied:

(I) If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.

(II) Let $(g_n) \in (G^N)^+$. If each subsequence of (g_n) has a subsequence belonging to α, then $(g_n) \in \alpha$.

(III) Let $g \in G$. Then const g belongs to α if and only if $g = 0$.

Under these assumptions α is said to be a *convergence in G*. The system of all convergences in G (partially ordered by inclusion) will be denoted by Conv G.

For $\alpha \in \text{Conv } G$, $(g_n) \in G^N$ and $g \in G$ we put $g_n \to_\alpha g$, if $(|g_n - g|) \in \alpha$.

A sequence $(g_n) \in (G^N)^+$ is said to be *regular* if there exists $\alpha \in \text{Conv } G$ such that $(g_n) \in \alpha$.

From the convexity of α in $(G^N)^+$ and from (II), (III) we obtain immediately:

1.1. Lemma. *Let (g_n) be a regular sequence in G and let (g_m) be a subsequence of (g_n). Then $\bigwedge g_m = 0$.*

Next, from the lemmas 3.2, 3.3 and 2.4 of [7] we obtain:

1.2. Lemma. *Let G be a lattice ordered group. The following conditions are equivalent:*

(i) Conv G *has no greatest element.*

(ii) *There are regular sequences (g_n), (h_n) in G and $0 < c \in G$ such that $g_n \vee h_n \geqq c$ for each $n \in N$.*

1.3. Lemma. *Let G be $(\aleph_0, 2)$-distributive. Then* Conv G *possesses the greatest element.*

Proof. By way of contradiction, assume that Conv G has no greatest element. Then in view of 1.2 there are sequences (g_n) and (h_n) in G such that the condition (ii) from 1.2 is satisfied. Put $g_{n0} = c \wedge g_n$ and $h_{n0} = c \wedge h_n$ for each $n \in N$. Then $c = g_{n0} \vee h_{n0}$ for each $n \in N$. Hence in view of $(\aleph_0, 2)$-distributivity of G we obtain

$$(1) \qquad 0 < c = (g_{10} \vee h_{10}) \wedge (g_{20} \vee h_{20}) \wedge \ldots .$$

Let I be the set of all mappings t_i of the set N into $\bigcup_{n\in N} \{g_{n0}, h_{n0}\}$ such that for each $n \in N$ we have $t_i(n) \in \{g_{n0}, h_{n0}\}$. Let us write t_{in} instead of $t_i(n)$. Let $i \in N$ be fixed. Then some of the following conditions is valid:

(a) the set $\{j \in N: t_{ij} = g_j\}$ is infinite;

(b) the set $\{j \in N: t_{ij} = h_j\}$ is infinite.

According to 2.1, in both the cases (a) and (b) we have

$$t_{i1} \wedge t_{i2} \wedge t_{i3} \wedge \ldots = 0 ,$$

hence

$$\bigvee_{t_i \in I} (t_{i1} \wedge t_{i2} \wedge t_{i3} \wedge \ldots) = 0 . \tag{2}$$

The relation (1) and (2) show that G is not $(\aleph_0, 2)$-distributive, which is contradiction.

From 1.3 and from [4] we infer that (A_1) holds.

Let us remark that if G is $(\aleph_0, 2)$-distributive, then it need not be archimedean (e.g., it suffices to take a non-archimedean linearly ordered group).

2. THE CASE OF BOOLEAN ALGEBRAS

Let B be a Boolean algebra. For each $n \in N$ let $B_n = B$. The direct product (in lattice-theoretic sense) of lattices B_n $(n \in N)$ will be denoted by B^N. The elements of B^N are denoted, e.g., as (b_n) and they will be called *sequences in* B.

The notion of sequential convergence in B was introduced in [8] (Definition 1.1). Let Conv B be the system of all sequential convergences in B; this system is partially ordered by inclusion.

For $\alpha \in \operatorname{Conv} B$ we denote by α_0 the set of all $(x_n) \in \alpha$ such that $x_n \to_\alpha 0$. Let $\operatorname{Conv}_0 B$ be the set of all α_0, where α runs over the system Conv B. The set $\operatorname{Conv}_0 B$ is partially ordered by inclusion. In [8] it was shown that the mapping $\alpha \to \alpha_0$ $(\alpha \in \operatorname{Conv} B)$ is an isomorphism of Conv B onto $\operatorname{Conv}_0 B$. The elements of $\operatorname{Conv}_0 B$ are called *0-convergences* in B.

From 1.5 in [8] it follows that for a subset β of B^N the following conditions are equivalent:

(i) $\beta \in \operatorname{Conv}_0 B$.

(ii) β is an ideal of the lattice B^N such that the condition (I), (II) and (III) are satisfied (where α and G are replaced by β or B, respectively).

Since Conv B and $\operatorname{Conv}_0 B$ are isomorphic, by proving (B_1) it suffices to prove the corresponding assertion for $\operatorname{Conv}_0 B$.

A sequence (x_n) in B will be called *regular in B* if there is $\beta \in \operatorname{Conv}_0 B$ such that $(x_n) \in \beta$.

The assertion of Lemma 1.1 remains valid if G is replaced by B (let us denote this modified assertion as 2.1). Similarly, we can formulate the assertion 2.2 which is analogous to 1.2.

2.2. Lemma. *Let B be a Boolean algebra. The following conditions are equivalent:*

(i) $\operatorname{Conv}_0 B$ *has no greatest element.*

(ii) *There are regular sequences* (g_n) *and* (h_n) *in B and* $0 < c \in B$ *such that* $g_n \vee h_n \geqq c$ *for each* $n \in N$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. The implication (i) $\Rightarrow$ (ii) is contained in the proof of 3.4 in [8].

Next, by replacing 1.1 and 1.2 in the proof of 1.3 by 2.1 and 2.2 respectively we obtain that the following assertion analogous to 1.3 holds:

2.3. Lemma. *Let B be $(\aleph_0, 2)$-distributive. Then* Conv B *possesses the greatest element.*

The above lemma and Theorem 3.6 of [8] yield that (B_1) is valid.

The equation whether the $(\aleph_0, 2)$-distributivity of B is necessary for Conv B to be complete remains open. The corresponding question for lattice ordered groups remains open as well.

3. SINGULAR STRONG UNIT

Again, let G be an abelian lattice ordered group, $G \neq \{0\}$. We recall the following definitions (cf. [1]):

An element $0 < x \in G$ is called *singular* if, whenever $y \in G$, $0 < y < x$, then $(x - y) \wedge y = 0$.

Let $0 < e \in G$. The element e is said to be a *weak unit* in G, if whenever $0 < y \in G$, then $e \wedge y > 0$. Next, e is called a *strong unit in G* if for each $y \in G$ there is $n \in N$ such that $y < ne$. Every strong unit in G is a weak unit in G.

It is easy to verify that an element $0 < x \in G$ is singular if and only if the interval $[0, x]$ of G is a Boolean algebra.

A subset α_1 of $(G^N)^+$ will be called *regular* if there exists $\alpha \in \text{Conv}\, G$ such that $\alpha_1 \subseteqq \alpha$. Analogously we define the regularity of a subset of B^N, where B is a Boolean algebra.

3.1. Lemma. *Let $0 < e \in G$ such that* (i) *e is a weak unit in G, and* (ii) *e is singular. Denote $B = [0, e]$ and let $\alpha_1 \subseteqq B^N$. Then the following conditions are equivalent:*

(a) *α_1 is regular with respect to G.*

(b) *α_1 is regular with respect to B.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from 1.2 and 2.2.

Let $\text{Conv}_b\, G$ be the set of all $\alpha \in \text{Conv}\, G$ having the property that whenever $(x_n) \in \alpha$, then (x_n) is bounded in G. The set $\text{Conv}_b\, G$ is partially ordered by inclusion.

3.2. Proposition. (Cf. [7], Theorem 4.8.) *The following conditions are equivalent:*

(i) Conv G *has a greatest element.*

(ii) $\text{Conv}_b\, G$ *has a greatest element.*

Let e and B be as in 3.1 and let $\alpha_1 \in \text{Conv}_0\, B$. We denote by $T(\alpha_1)$ the least element of Conv G which is larger or equal to α_1; such an element does exist in view of 3.1. Then we have

3.3. Lemma. *Let $(x_0) \in (G^N)^+$. Under the above assumptions and denotations, the following conditions are equivalent:*

(i) $(x_n) \in T(\alpha_1)$.

(ii) *There are* $m \in N$ *and* $(z_n) \in \alpha_1$ *such that* $x_n \leqq mz_n$ *for each* $n \in N$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. Let (i) ve valid. By similar reasoning as in the proof of Lemma 2.5 in [7] we obtain that there are $m_1 \in N$ and $(y_n^1), (y_n^2), \ldots$ $\ldots, (y_n^k) \in \alpha_1$ such that

$$x_n \leqq m_1(y_n^1 + y_n^2 + \ldots + y_n^k) \quad \text{for each} \quad n \in N .$$

Thus in view of Lemma 2.4 in [7] there is $m \in N$ such that

$$x_n \leqq m(y_n^1 \vee y_n^2 \vee \ldots \vee y_n^k) \quad \text{for each} \quad n \in N .$$

Since $(y_n^1 \vee y_n^2 \vee \ldots \vee y_n^k) \in \alpha_1$, it suffices to put $z_n = y_n^1 \vee y_n^2 \vee \ldots \vee y_n^k$.

Throughout this section, the above denotations will be applied.

3.4. Corollary. *Let* $\alpha_1 \in \mathrm{Conv}_0 B$. *Then* $T(\alpha_1) \in \mathrm{Conv}_b G$.

Proof. Let $(x_n) \in T(\alpha_1)$ and let m be as in 3.3 (ii). Then $x_n \leqq me$ for each $n \in N$, hence (x_n) is bounded in G.

3.5. Lemma. *Let* $x, y \in [0, e]$, $m \in N$, $x \leqq my$. *Then* $x \leqq y$.

Proof. By way of contradiction, assume that $x \nleqq y$. Then (since $[0, e]$ is a Boolean algebra) there is $x_1 \in [0, e]$ such that $0 < x_1 \leqq x$ and $x_1 \wedge y = 0$. Hence $x_1 \wedge$ $\wedge\ my = 0$, which is a contradiction.

3.6. Lemma. *Let* $\alpha_1, \beta_1 \in \mathrm{Conv}_0 B$. *Then we have*

$$\alpha_1 \leqq \beta_1 \Leftrightarrow T(\alpha_1) \leqq T(\beta_1) .$$

Proof. The implication $\alpha_1 \leqq \beta_1 \Rightarrow T(\alpha_1) \leqq T(\beta_1)$ is obvious. Hence it suffices to verify that if $\alpha_1 \nleqq \beta_1$, then $T(\alpha_1) \nleqq T(\beta_1)$.

Assume that $\alpha_1 \nleqq \beta_1$. Hence there exists $(t_n) \in \alpha_1 \setminus \beta_1$. Clearly $(t_n) \in T(\alpha_1)$. We shall show that (t_n) does not belong to $T(\beta_1)$. By way of contradiction, suppose that $(t_n) \in T(\beta_1)$. Thus in view of 3.3 there are $m \in N$ and $(z_n) \in \beta_1$ such that

$$t_n \leqq mz_n \quad \text{for each} \quad n \in N .$$

By applying 3.5 we obtain that

$$t_n \leqq z_n \quad \text{for each} \quad n \in N .$$

Hence $(t_n) \in \beta_1$, which is a contradiction.

From 3.4 and 3.5 we obtain:

3.7. Theorem. *Let* $0 < e \in G$ *be a singular weak unit in* F. *Then the mapping* $\alpha_1 \to T(\alpha_1)$ (*where* α_1 *runs over* $\mathrm{Conv}_0 [0, e]$) *is an isomorphism of the partially ordered set* $\mathrm{Conv}_0 [0, e]$ *into the partially ordered set* $\mathrm{Conv}_b G$.

3.8. Lemma. *Let* $0 < e \in G$ *be a singular strong unit in* G. *Let* $\alpha \in \mathrm{Conv}_b G$. *Put* $\alpha_1 = \alpha \cap B^N$, *where* $B = [0, e]$. *Then* $\alpha_1 \in \mathrm{Conv}_0 B$ *and* $T(\alpha_1) = \alpha$.

Proof. The verification of the relation $\alpha_1 \in \mathrm{Conv}_0 B$ is easy. Since $\alpha_1 \subseteq \alpha$, we have $T(\alpha_1) \subseteq T(\alpha) = \alpha$. Let $(x_n) \in \alpha$. Because of $\alpha \in \mathrm{Conv}_b (G)$, there is $0 < g \in G$

such that $x_n \leqq g$ for each $n \in N$. Next, e is a strong unit in G and thus there is $m \in N$ such that $g \leqq me$. Therefore

$$x_n \leqq me \quad \text{for each} \quad n \in N .$$

Let n be fixed. There are $x_{n1}, x_{n2}, \ldots, x_{nm}$ in G such that

(1) $0 \leqq x_{nj} \leqq e$ for $j = 1, 2, \ldots, m$,

(2) $x_n = x_{n1} + x_{n2} + \ldots + x_{nm}$.

Thus according to Lemma 2.4, [7] there is $m_1 \in N$ such that

(3) $x_n \leqq m_1(x_n \vee x_{n2} \vee \ldots \vee x_{nm})$.

In view of (1) and (2) we have $(x_{n1}), (x_{n2}), \ldots, (x_{nk}) \in \alpha_1$, whence $(z_n) \in \alpha_1$, where $z_n = x_{n1} \vee x_{n2} \vee \ldots \vee x_{nm}$ for each $n \in N$. Thus (3) yields that (x_n) belongs to $T(\alpha_1)$, completing the proof.

The following theorem is a consequence of 3.7 and 3.8.

Theorem 3.9. *Let $0 < e \in G$ be a singular strong unit in G. Then the mapping $\alpha_1 \to T(\alpha_1)$ (where α_1 runs over* $\mathrm{Conv}_0\,[0, e]$*) is an isomorphism of the partially ordered set* $\mathrm{Conv}_0\,[0, e]$ *onto the partially ordered set* $\mathrm{Conv}_0\, G$.

Next, 3.9 and 3.2 yield:

Corollary 3.10. *Let $0 < e \in G$ be a singular strong unit in G. Then* $\mathrm{Conv}_0\,[0, e]$ *is a complete lattice iff* Conv G *is a complete lattice.*

References

[1] *P. Conrad:* Lattice ordered groups, Tulane University 1970.

[2] *M. Harminc:* Sequential convergences on abelian lattice-ordered groups. Convergence structures 1984. Mathematical Research, Band 24, Akdemie Verlag, Berlin; 1985, 153—158.

[3] *M. Harminc:* The cardinality of the system of all sequential convergences on an abelian lattice ordered group. Czech. Math. J. *37*, 1987, 533—546.

[4] *M. Harminc:* Sequential convergences on lattice ordered groups. Czech. Math. J. *39*, 1989, 232—238.

[5] *J. Jakubík:* Radical mappings and radical classes of lattice ordered groups. Symposia Mathematica, Vol. 21, Academic Press, London and New York, 1977, 451—477.

[6] *J. Jakubík:* Convergences and complete distributivity of lattice ordered groups. Math. Slovaca *38*, 1988, 269—272.

[7] *J. Jakubík:* Lattice ordered groups having a largest convergence. Czech. Math. J. *39*, 1989, 717—729.

[8] *J. Jakubík:* Sequential convergences in Boolean algebras. Czech. Math. J. *38*, 1988, 520—530.

Author's address: 040 01 Košice, Grešákova 6, Czechoslovakia (Matematický ústav SAV, dislokované pracovisko v Košiciach).