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TOLERANCE MODULAR VARIETIES OF SEMIGROUPS

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The aim of this paper is to describe all varieties of semigroups whose tolerance lattices are modular. The present result generalizes the issue from [1] to arbitrary semigroups. In addition we give a characterization of varieties of semigroups whose (principal) tolerances are congruences.

1. PRELIMINARIES

Recall that a tolerance on a semigroup S is a reflexive and symmetric subsemigroup of the direct product $S \times S$. By $\mathrm{Tol}(S)$ we denote the lattice of all tolerances on S with respect to set inclusion (see [2] and [3]). Denote by $\vee$ or $\wedge$ the join or meet in $\mathrm{Tol}(S)$, respectively. The meet evidently coincides with set intersection. For $M \subseteqq S \times S$, we denote by $T_S(M)$ (or simply $T(M)$) the least tolerance on S containing M. It is easy to show the following:

(1) $(x, y) \in T(M)$ if and only if $x = x_1x_2 \dots x_m$ and $y = y_1y_2 \dots y_m$, where either $(x_i, y_i) \in M$ or $(y_i, x_i) \in M$ or $x_i = y_i \in S$ for $i = 1, 2, \dots, m$.

(2) $A \vee B = T(A \cup B)$ for any $A, B \in \mathrm{Tol}(S)$.

By $\mathrm{Con}(S)$ we denote the lattice of all congruences on S. Clearly $\mathrm{Con}(S)$ is a subset of $\mathrm{Tol}(S)$, but it need not be a sublattice of $\mathrm{Tol}(S)$. Analogously for $M \subseteqq S \times S$, we denote by $C_S(M)$ (or simply $C(M)$) the least congruence on S containing M.

For any semigroup by $E(S)$ we denote the set of all idempotents of S. The notation S^1 stands for S if S has an identity, otherwise for S with an identity adjoined. By $\langle a \rangle_S$ (or simply $\langle a \rangle$) we denote the subsemigroup of S generated by $a \in S$.

Terminology and notation not defined here may be found in [4] and [5].

By $\mathscr{W}$ $(i_1 = i_2)$ we denote the variety of all semigroups satisfying the identity $i_1 = i_2$.

Lemma 1. *Let $\mathscr{V}$ be a variety of semigroups. Then every semigroup from $\mathscr{V}$ contains an idempotent if and only if $\mathscr{V} \subseteqq \mathscr{W}(x^n x^n = x^n)$ for a positive integer n.*

Proof. Suppose that every semigroup from $\mathscr{V}$ contains an idempotent. Let $a \in S \in \mathscr{V}$. Then $\langle a \rangle \in \mathscr{V}$ and so $a^m a^m = a^m$ for some positive integer m. The minimum of such m we denote by $m(a)$. If there exist a_i $(i = 1, 2, \ldots)$ such that $\langle a_i \rangle \in \mathscr{V}$ and $i \leqq m(a_i)$, then the direct product $\mathop{\mathsf{X}}\limits_{i=1}^{\infty} \langle a_i \rangle$ belongs to $\mathscr{V}$ and does not contain idempotents, which is a contradiction. Hence there exists a positive integer k such that $m(a) \leqq k$ for all a, where $\langle a \rangle \in \mathscr{V}$. Put $n = k!$. It is easy to show that $\mathscr{V} \subseteqq \mathscr{W}(x^n x^n = x^n)$.

By $\mathscr{Z}$ we denote the variety of all zero-semigroups, i.e. $\mathscr{Z} = \mathscr{W}(xy = uv)$. It is well known that $\mathscr{Z}$ is a minimal variety in the lattice of all semigroup varieties. Put $\mathscr{O} = \mathscr{W}(x = y)$.

Lemma 2. *Let $\mathscr{V}$ be a variety of semigroups. Then $\mathscr{V} \cap \mathscr{Z} = \mathscr{O}$ if and only if $\mathscr{V} \subseteqq \mathscr{W}(x^n x = x)$ for a positive integer n.*

Proof. Assume that $\mathscr{V} \cap \mathscr{Z} = \mathscr{O}$. Let $a \in S \in \mathscr{V}$. Then $\langle a \rangle \in \mathscr{V}$ and so $a = a^m a$ for some positive integer m. It follows from Lemma 1 that $\mathscr{V} \subseteqq \mathscr{W}(x^n x^n = x^n)$ for a positive integer n. If $a = a^m a$, then $a = (a^m)^n a = (a^n)^m a = a^n a$. Hence $\mathscr{V} \subseteqq \mathscr{W}(x^n x = x)$.

By $\mathscr{S}$ we denote the variety of all semilattices, i.e. $\mathscr{S} = \mathscr{W}(xy = yx) \cap \mathscr{W}(x^2 = x)$. It is well known that $\mathscr{S}$ is a minimal variety in the lattice of all semigroup varieties.

Lemma 3. *The following conditions for a variety $\mathscr{V}$ of semigroups are equivalent:*

1. $\mathscr{V} \cap \mathscr{Z} = \mathscr{O} = \mathscr{S} \cap \mathscr{V}$.
2. *Every semigroup from $\mathscr{V}$ is completely simple.*
3. *$\mathscr{V}$ is a subvariety of $\mathscr{W}(x^n x = x) \cap \mathscr{W}((xyx)^n = x^n)$ for a positive integer n.*

Proof. $1 \Rightarrow 2$. See Proposition 2 of [6].

$2 \Rightarrow 3$. Evidently $\mathscr{V} \cap \mathscr{Z} = \mathscr{O}$ and so, by Lemma 2, we have $\mathscr{V} \subseteqq \mathscr{W}(x^n x = x)$ for a positive integer n. Therefore $\mathscr{V} \subseteqq \mathscr{W}(x^n x^n = x^n)$. Let a, b be two elements of S from $\mathscr{V}$. According to Rees' theorem (Theorem 3.5 of [4]), aba and a belong to the same maxinral subgroup of S. Then $(aba)^n = a^n$. Hence $\mathscr{V} \subseteqq \mathscr{W}((xyx)^n = x^n)$.

$3 \Rightarrow 1$. It is clear

A semigroup S is said to be *(principal) tolerance trivial* if every (principal) tolerance on S is a congruence. See [7] and [8]. Recall that a tolerance A on S is principal if $A = T_S(\{a, b\})$ (or simply $T_S(a, b)$) for some pair of elements $a, b \in S$. Tolerance trivial semigroups in which a power of each element lies in a subgroup have been described in [9] and principal tolerance trivial commutative semigroups have been described in [10].

Let I and J be non-empty sets and let G be a group. Let $P: I \times J \to G$. Denote by $M(G, I, J, P) = G \times I \times J$ the Rees matrix semigroup with the following multiplication: $(g, i, j)(h, r, s) = (gp_{jr}h, i, s)$ where $g, h \in G$, $i, r \in I$, $j, s \in J$ and $p_{jr} = P(r, j)$.

Lemma 4. *A semigroup $M(G, I, J, P)$ is tolerance trivial if and only if* card $I \leqq 2$ *and* card $J \leqq 2$.

(See Lemma in [9].)

Lemma 5. *Every completely simple semigroup is principal tolerance trivial.*

Proof. Let S be a completely simple semigroup. According to Theorem 3.5 of [4], S is isomorphic to the Rees matrix semigroup $M(G, I, J, P)$. We can suppose that $S = M(G, I, J, P)$. Let $i \in I$ and $j \in J$. Put $G_{ij} = \{(g, i, j);\ g \in G\}$. It is known that G_{ij} is a subgroup of S.

Let Q be a principal tolerance on S. Then $Q = T_S(v, w)$, where $v \in G_{ab}$ and $w \in G_{cd}$ for some $a, c \in I$ and $b, d \in J$. We shall show that Q is transitive. Let $(x, y) \in Q$ and $(y, z) \in Q$. It follows from (1) that $x = x_1 x_2 \ldots x_m$, $y = y'_1 y'_2 \ldots y'_m$ and $y = y''_1 y''_2 \ldots y''_n$, $z = z_1 z_2 \ldots z_n$, where $x_i = y'_i \in S$ or $(x_i, y'_i) = (v, w)$ or $(x_i, y'_i) = (w, v)$ for all $i = 1, 2, \ldots, m$ and $y''_j = z_j \in S$ or $(y''_j, z_j) = (v, w)$ or $(y''_j, z_j) = (w, v)$ for all $j = 1, 2, \ldots, n$. Clearly $x \in G_{pq}$, $y \in G_{rs}$ and $z \in G_{tu}$ for some $p, r, t \in I$ and some $q, s, u \in J$. Then $y'_1 y'_m \in G_{rs}$ and $y''_1 y''_n \in G_{rs}$. If $r \notin \{a, c\}$, then $r = p = t$ and we put $I_0 = \{r\}$. If $r \in \{a, c\}$, then $p, t \in \{a, c\}$ and we put $I_0 = \{a, c\}$. Analogously we put $J_0 = \{s\}$ if $s \notin \{b, d\}$ and $J_0 = \{b, d\}$ if $s \in \{b, d\}$. Let $S_0 = M(G, I_0, J_0, P_0)$, where P_0 is the restriction of P to $I_0 \times J_0$. Therefore S_0 is a subsemigroup of S and $x, y, z \in S_0$. Put $Q_0 = Q \cap (S_0 \times S_0)$. It is easy to show that Q_0 is a tolerance on S_0. It follows from Lemma 4 that Q_0 is a congruence on S_0, because card $I_0 \leqq 2$ and card $J_0 \leqq 2$. Since $(x, y) \in Q_0$ and $(y, z) \in Q_0$, we have $(x, z) \in Q_0 \subseteqq Q$. Consequently Q is transitive.

A variety $\mathscr{V}$ of semigroups is (principal) tolerance trivial if every semigroup S from $\mathscr{V}$ has this property.

Theorem 1. *The following conditions for a variety $\mathscr{V}$ of semigroups are equivalent:*

1. *$\mathscr{V}$ is principal tolerance trivial.*
2. *$\mathscr{V} = \mathscr{Z}$ or $\mathscr{V}$ is a subvariety of $\mathscr{W}(x^n x = x) \cap \mathscr{W}((xyx)^n = x^n)$ for a positive integer n.*
3. *$\mathscr{V} = \mathscr{Z}$ or every semigroup from $\mathscr{V}$ is completely simple.*

Proof. $1 \Rightarrow 2$. Let V be a principal tolerance trivial variety of semigroups. It follows from Theorem of [10] that every chain C (i.e. a semilattice C in which we have $ef \in \{e, f\}$ for all $e, f \in C$) from $\mathscr{V}$ satisfies card $C \leqq 2$. This implies that $\mathscr{V} \cap \mathscr{S} = \mathscr{O}$ and so according to Theorem of [10] we have:

(3) Every commutative semigroup from $\mathscr{V}$ is either a zero-semigroup or a group.

If $\mathscr{V} \cap \mathscr{Z} = \emptyset$, then Lemma 3 implies that $\mathscr{V} \subseteqq \mathscr{W}(x^n x = x) \cap \mathscr{W}((xyx)^n = x^n)$ for a positive integer n.

Now, we can suppose that $\mathscr{Z} \subseteqq \mathscr{V}$. We shall show that

(4) Every commutative group from $\mathscr{V}$ is trivial.

Assume by way of contradiction that a non-trivial group G belongs to $\mathscr{V}$. Choose $Z \in \mathscr{Z}$ such that $\operatorname{card} Z = 2$. Then $Z \times G$ is a commutative semigroup from $\mathscr{V}$ which contradicts (3).

Let S be an arbitrary semigroup from $\mathscr{V}$. Let $a \in S$. Clearly $\langle a \rangle$ is commutative and belongs to $\mathscr{V}$. It follows from (3) and (4) that $a^2 = a^3$.

We have

(5) $$\mathscr{V} \subseteqq \mathscr{W}(x^2 = x^3)\,.$$

Further, we shall prove that $\operatorname{card} E(S) = 1$. By way of contradiction suppose that S contains at least two idempotents (say e and f). We have $Z = \{0, z\}$, where $uv = 0$ for all $u, v \in Z$. Evidently $S \times Z \in \mathscr{V}$ and so $S \times Z$ is principal tolerance trivial. Put $Q = T_{S \times Z}((e, z), (f, 0))$. Then $((e, 0), (f, 0)) = ((e, z), (f, 0))^2 \in Q$ and so $((e, z), (e, 0)) \in Q$, because Q is transitive. It follows from (1) that $(e, z) = u_1 u_2 \dots u_m$ and $(e, 0) = v_1 v_2 \dots v_m$, where either $u_i = v_i \in S \times Z$ or $(u_i, v_i) = ((e, z), (f, 0))$ or $(u_i, v_i) = ((f, 0), (e, z))$ for $i = 1, 2, \dots, m$. Clearly $m = 1$ and so $e = f$ which is a contradiction. By (5) we have

(6) $$\mathscr{V} \subseteqq \mathscr{W}(x^2 = y^2)\,.$$

Finally, we shall show that S is a zero-semigroup. It follows from (5) and (6) that there exists an element h of S such that $h = u^2 = uh = hu$ for all $u \in S$. Assume by way of contradiction that there exist $a, b \in S$ such that $ab \neq h$. Then $a \neq h$ and $a \notin S^1 aS \cup SaS^1$. Indeed, if $a = cad$, where $c, d \in S^1$ and $cd \in S$, then $a = c^2 ad^2 = h$, a contradiction. Hence we have $a \neq ab$. Put $Q = T_S(a, ab)$. Then $(ab, h) = (a, ab) \,.\, (b, b) \in Q$. Since Q is a transitive tolerance on S, we have $(a, h) \in Q$ and so, by (1), we get $a = c_1 c_2 \dots c_m$ and $h = d_1 d_2 \dots d_m$, where either $c_i = d_i \in S$ or $(c_i, d_i) = (a, ab)$ or $(c_i, d_i) = (ab, a)$ for $i = 1, 2, \dots, m$. Since $a \neq h$, we have $a \in S^1 aS \cup SaS^1$, which is a contradiction. Therefore $ab = h$. Hence S is a zero-semigroup. Consequently $\mathscr{V} = \mathscr{Z}$.

$2 \Rightarrow 1$. It is easy to show that the variety $\mathscr{Z}$ is principal tolerance trivial (see Theorem of [10]). If $\mathscr{V}$ is a subvariety of $\mathscr{W}(x^n x = x) \cap \mathscr{W}((xyx)^n = x^n)$ for a positive integer n, then it follows from Lemma 3 and Lemma 5 that $\mathscr{V}$ is principal tolerance trivial.

$2 \Leftrightarrow 3$. See Lemma 3.

3. TOLERANCE TRIVIAL VARIETIES

By $\mathscr{L}$ ($\mathscr{R}$, respectively) we denote the variety of all left (right, respectively) semigroups, i.e. $\mathscr{L} = \mathscr{W}(xy = x)$ and $\mathscr{R} = \mathscr{W}(xy = y)$. It is well known that $\mathscr{L}$ and $\mathscr{R}$ are minimal varieties in the lattice of all semigroup varieties.

Following [11], [12] we shall say that a semigroup S has transferable tolerances (congruences) if for any elements a, b, c of S there exists an element d of S such that $T_S(a, b) = T_S(c, d)$ ($C_S(a, b) = C_S(c, d)$, respectively). A variety $\mathscr{V}$ of semigroups has transferable tolerances (congruences) if each semigroup S from $\mathscr{V}$ has this property.

Recall that a semigroup S is said to be *tolerance* (*congruence*) *modular* if the lattice $\operatorname{Tol}(S)$ ($\operatorname{Con}(S)$, respectively) is modular. A variety $\mathscr{V}$ of semigroups is called *tolerance* (*congruence*) *modular* if each semigroup S from $\mathscr{V}$ has this property. See [13].

Theorem 2. *The following conditions for a variety $\mathscr{V}$ of semigroups are equivalent:*

1. *$\mathscr{V}$ is tolerance trivial.*
2. *For any semigroup S from $\mathscr{V}$* $\operatorname{Con}(S)$ *is a sublattice of* $\operatorname{Tol}(S)$.
3. $\mathscr{V} \cap \mathscr{Z} = \mathscr{V} \cap \mathscr{S} = \mathscr{V} \cap \mathscr{L} = \mathscr{V} \cap \mathscr{R} = \mathscr{O}$.
4. *Every semigroup from $\mathscr{V}$ is a group.*
5. *$\mathscr{V}$ is a subvariety of $\mathscr{W}(x^n y = y) \cap \mathscr{W}(yx^n = y)$ for a positive integer n.*
6. *$\mathscr{V}$ has transferable tolerances.*
7. *$\mathscr{V}$ has transferable congruences.*
8. *$\mathscr{V}$ is congruence permutable.*
9. *$\mathscr{V}$ is congruence modular.*

Proof. $1 \Rightarrow 2$. Clearly.

$2 \Rightarrow 3$. Let $S \in \mathscr{Z} \cup \mathscr{L} \cup \mathscr{R}$. It is easy to show that $\operatorname{Tol}(S)$ is the set of all reflexive and symmetric relations on S and $\operatorname{Con}(S)$ is the set of all equivalences on S. Clearly $\operatorname{Con}(S)$ is no sublattice of $\operatorname{Tol}(S)$ whenever $\operatorname{card} S \geqq 3$. Thus we have $\mathscr{V} \cap \mathscr{Z} = \mathscr{V} \cap \mathscr{L} = \mathscr{V} \cap \mathscr{R} = \mathscr{O}$.

In [14] it is proved that $\operatorname{Con}(S)$ is a sublattice of $\operatorname{Tol}(S)$ for a commutative separative semigroup S if and only if S is either a group or a group with zero. This implies that $\operatorname{card} S \leqq 2$ for every S from $\mathscr{V} \cap \mathscr{S}$ and so $\mathscr{V} \cap \mathscr{S} = \mathscr{O}$.

$3 \Rightarrow 4$. It follows from Lemma 3 that every semigroup S from $\mathscr{V}$ is completely simple and so, by Theorem 3.5 of [4], S is isomorphic to the Rees matrix semigroup $M(G, I, J, P)$. If $\operatorname{card} I \geqq 2$, then $\mathscr{L} \cap \mathscr{V} \neq \mathscr{O}$, which is a contradiction. Therefore $\operatorname{card} I = 1$. Analogously we have $\operatorname{card} J = 1$ and so S is a group.

$4 \Rightarrow 5$. Apply Lemma 3.

$5 \Rightarrow 6$. It follows from Theorem 1 of [11], where $p(x, y, z) = xy^{n-1}z$ and $q(x_1, x_2, x_3, x_4, x_5) = x_1 x_5^{n-1} x_4$.

$6 \Rightarrow 7$. Trivial.

$7 \Rightarrow 8$. Let $S \in \mathscr{Z} \cup \mathscr{S} \cup \mathscr{L} \cup \mathscr{R}$ and $\operatorname{card} S = 3$. It is easy to show that S has no transferable congruences and S does not belong to $\mathscr{V}$. The rest of the proof follows from $3 \Rightarrow 4$.

$8 \Rightarrow 9$. According to Theorem 1 of [8], $\mathscr{V}$ is tolerance trivial. The rest of the proof follows from $1 \Rightarrow 4$.

$9 \Rightarrow 1$. If $S \in \mathscr{Z} \cup \mathscr{L} \cup \mathscr{R}$, then Con (S) is the set of all equivalences on S and so $\mathscr{Z}$, $\mathscr{L}$ and $\mathscr{R}$ are not congruence modular. According to Theorem 2 of [15], S is not congruence modular. The rest of the proof follows from $3 \Rightarrow 4$, because every tolerance on a group is transitive.

Corollary 1. *A variety $\mathscr{V}$ of commutative semigroups is principal tolerance trivial if and only if $\mathscr{V} = \mathscr{Z}$ or $\mathscr{V}$ is tolerance trivial.*

The proof follows from Theorem 1, Theorem 2 and $\mathscr{W}(x^n y = y) \cap \mathscr{W}(yx^n = y) \cap \cap \mathscr{W}(xy = yx) = \mathscr{W}(x^n x = x) \cap \mathscr{W}((xyx)^n = x^n) \cap \mathscr{W}(xy = yx)$.

4. TOLERANCE MODULAR VARIETIES

Theorem 3. *A variety $\mathscr{V}$ of semigroups is tolerance modular if and only if $\mathscr{V}$ is a subvariety of*

$$\mathscr{W}((xy)^{n+1} = xy) \cap \mathscr{W}((xyx)^n = x^n)$$

for a positive integer n.

Before the proof we formulate three lemmas.

Lemma 6. *A variety $\mathscr{V}$ of commutative semigroups is tolerance modular if and only if $\mathscr{V}$ is a subvariety of $\mathscr{W}(xyz^n = xy)$ for a positive integer n.*

See Theorem 1 in [1].

Lemma 7. *Let $P = \{p, q, r, 0\}$ be a four-element semigroup with the multiplication table*

	p	q	r	0
p	0	r	0	0
q	0	0	0	0
r	0	0	0	0
0	0	0	0	0

Then the lattice Tol $(P \times P)$ *is not modular.*

Proof. Put $A = T_{P \times P}((p, p), (p, 0))$, $B = T_{P \times P}((q, q), (0, q))$ and $C = T_{P \times P}((0, r), (r, 0)) \vee A$. We have $((0, r), (r, 0)) = ((p, p), (p, 0)) . ((0, q), (q, q)) \in (A \vee B) \wedge C$. We shall show that $((0, r), (r, 0)) \notin A \vee (B \wedge C)$.

On the contrary, suppose that $((0, r), (r, 0)) \in A \vee (B \wedge C)$.

Case 1. $((0, r), (r, 0)) \in A$. It follows from (1) that $((0, r), (r, 0)) = ((u_1, p), (p, v_1)) . ((u_2, q), (q, v_2))$, where either $(u_1, p) = (p, v_1)$ or $((u_1, p), (p, v_1)) = ((p, p), (p, 0))$ and $(u_2, q) = (q, v_2)$. Then $u_1 = p$ and $u_2 = q$ and so $0 = u_1 u_2 = pq = r$, which is a contradiction.

Case 2. $((0, r), (r, 0)) \in B$. According to (1), we have $((0, r), (r, 0)) = ((u_1, p), (p, v_1)) . ((u_2, q), (q, v_2))$, where $(u_1, p) = (p, v_1)$ and either $(u_2, q) = (q, v_2)$ or

$((u_2, q), (q, v_2)) = ((0, q), (q, q))$. Therefore $v_1 = p$ and $v_2 = q$ and so $0 = v_1 v_2 = pq = r$, a contradiction.

Case 3. $((0, r), (r, 0)) \notin A \cup B$. Then, by (1) and (2), the only possibility is $((0, r), (r, 0)) = ((p, p), (p, 0)) \cdot ((0, q), (q, q))$ and $((0, q), (q, q)) \in C$, which is impossible.

Therefore the lattice $\mathrm{Tol}(P \times P)$ is not modular.

Lemma 8. *Let $Q = \{p, q, r, s\}$ be a four-element semigroup with the multiplication table*

	p	q	r	s
p	s	r	s	s
q	q	q	q	q
r	r	r	r	r
s	s	s	s	s

Then the lattice $\mathrm{Tol}(Q \times Q)$ *is not modular.*

Proof. Put $A = T_{Q \times Q}((q, p), (s, p))$, $B = T_{Q \times Q}((p, s), (p, q))$ and $C = T_{Q \times Q}((r, s), (s, q)) \vee A$. We have $((r, s), (s, q)) = ((p, s), (p, q)) \cdot ((q, p), (s, p)) \in (A \vee B) \wedge C$. We shall show that $((r, s), (s, q)) \notin A \vee (B \wedge C)$.

On the contrary, suppose that $((r, s), (s, q)) \in A \vee (B \wedge C)$.

Case 1. $((r, s), (s, q)) \in A$. According to (1) we have $((r, s), (s, q)) = \prod_{i=1}^{m} ((a_i, b_i), (c_i, d_i))$, where either $(a_i, b_i) = (c_i, d_i)$ or $((a_i, b_i), (c_i, d_i)) = ((q, p), (s, p))$ or $((a_i, b_i), (c_i, d_i)) = ((s, p), (q, p))$ for $i = 1, 2, \ldots, m$. It is easy to show that $b_1 \in \{p, s\}$ and $d_1 = q$, which is a contradiction.

Case 2. $((r, s), (s, q)) \in B$. It follows from (1) that $((r, s), (s, q)) = \prod_{i=1}^{m} ((a_i, b_i), (c_i, d_i))$, where $(a_i, b_i) = (c_i, d_i)$ or $((a_i, b_i), (c_i, d_i)) = ((p, s), (p, q))$ or $((a_i, b_i), (c_i, d_i)) = ((p, q), (p, s))$ for $i = 1, 2, \ldots, m$. It is easy to show that $((a_1, b_1), (c_1, d_1)) = ((p, s), (p, q))$ and so $m \geqq 2$. Then $\prod_{i=2}^{m} a_i = q$ and so $a_2 = q$. Hence we have $c_2 = q$ and so $s = pq \prod_{i=3}^{m} c_i = r$, a contradiction.

Case 3. $((r, s), (s, q)) \notin A \cup B$. By (1) and (2) we have $((r, s), (s, q)) = \prod_{i=1}^{m} ((a_i, b_i), (c_i, d_i))$, where $((a_i, b_i), (c_i, d_i)) \in A \cup (B \cap C)$. Then $a_1 \in \{p, r\}$, $\{b_1, c_1\} \subseteqq \{p, s\}$, $d_1 = q$ and so $(a_1, b_1) \neq (c_1, d_1)$. From cases 1 and 2 it follows that $p \in \{a_1, b_1, c_1\}$ and so $((a_1, b_1), (c_1, d_1)) \notin ((Q \times Q) \times (Q \times Q))^2$. Thus we have $((a_1, b_1), (c_1, d_1)) = (p, s), (p, q)) \in B \cap C \subseteqq C$, which is impossible.

Proof of Theorem 3. I. Let $\mathscr{V}$ be a tolerance modular variety of semigroups. It follows from Lemma 6 that

$$(7) \qquad \mathscr{V} \cap \mathscr{W}(xy = yx) \subseteqq \mathscr{W}(xyz^n = xy)$$

for a positive integer $n \geqq 2$. This implies

(8) $\mathscr{V} \subseteqq \mathscr{W}(x^2x^n = x^2)$

and consequently we have

(9) $\mathscr{V} \subseteqq \mathscr{W}(x^nx^n = x^n)$.

Let S be a semigroup from $\mathscr{V}$. Let $a, b \in S$ and by U denote the subsemigroup of S generated by a, b. We have $U = \langle a \rangle \cup U^1bU^1$. If $\langle a \rangle \cap U^1bU^1 = \emptyset$, then the two-element semilattice is a homomorphic image of U, which is impossible because according to (7) every semilattice from $\mathscr{V}$ is trivial. Consequently we have $\langle a \rangle \cap \cap U^1bU^1 \neq \emptyset$ and so $a^m \in U^1bU^1$ for a positive integer m. It follows from (8) that $a^2 \in U^1bU^1$. Hence we have

(10) $a^2 \in S^1bS^1$

for all $a, b \in S$ from $\mathscr{V}$.

Let $a, b, c \in S$. We shall show that $ab \in S^1cS^1$. By way of contradiction we assume that $ab \notin S^1cS^1$. By I we denote the intersection of all ideals of S. It follows from (10) that $x^2 \in I$ for every $x \in S$ and so $I \neq \emptyset$. Clearly $a, b, ab \notin I$ and $a \neq ab \neq \neq b \neq a$. We shall show that $ab \neq ba$. Suppose that $ab = ba$. By U we denote the subsemigroup of S generated by a, b. According to (7), we obtain $U \in \mathscr{W}(xyz^n = = xy)$ and so $ab = abd^n \in I$ for some $d \in U$, a contradiction. We have $ab \neq ba$. Put $J = \{ba, aba, bab\} \cup I$ and $T = \{a, b, ab\} \cup J$. Evidently T is a subsemigroup of S and J is an ideal of T. Thus we have $T \in \mathscr{V}$. It is easy to show that the semigroup P from Lemma 7 is isomorphic to the Rees quotient $T/J \in \mathscr{V}$, which implies $P \in \mathscr{V}$, a contradiction. Therefore for all $a, b, c \in S$ from $\mathscr{V}$ we have

$$ab \in S^1cS^1 .$$

This implies that S^2 is a simple subsemigroup of S. According to (9) and Theorem 2.55 (Munn W. D.) of [4], we have

(11) S^2 is a completely simple semigroup

for every semigroup S from $\mathscr{V}$.

It is well known that every completely simple semigroup is a union of groups and so, by (11), S^2 is a union of groups. Therefore $ab \in (ab)^2 S^1$ for every pair of elements $a, b \in S$ and so, by (8), we have $ab = (ab)^{n+1}$. We obtain

(12) $\mathscr{V} \subseteqq \mathscr{W}((xy)^{n+1} = xy)$.

By $\mathscr{U}$ we denote the class of all semigroups S from $\mathscr{V}$ such that $S = S^2$. According to (12), every subsemigroup of S from $\mathscr{U}$ belongs to $\mathscr{U}$. It is easy to show that $\mathscr{U}$ is closed under homomorphic images and direct products and so, by Birkhoff's Theorem [16], $\mathscr{U}$ is a variety of semigroups. It follows from (11) and Lemma 3 that $\mathscr{U} \subseteqq \mathscr{W}((uvu)^k = u^k)$ for a positive integer k. Putting $u = xy$ and $v = z(xy)^2 z$ we obtain $\mathscr{V} \subseteqq \mathscr{W}((xyz(xy)^2 zxy)^k = (xy)^k) = \mathscr{W}((xyzxy)^{2k} = (xy)^k) \subseteqq$

$\subseteq \mathscr{W}((xyzxy)^{2kn} = (xy)^{kn})$. Hence we have

$$\mathscr{V} \subseteq \mathscr{W}((xyzxy)^n = (xy)^n)\,.$$

Indeed, using (12), we obtain $(xyzxy)^n = (xyzxy)^{2kn} = (xy)^{kn} = (xy)^n$.

Now, we shall prove that

$$ab = a^n ab \tag{14}$$

for every pair of elements a, b of a semigroup S from $\mathscr{V}$.

If $a \in S^2$, then according to (12), we have $a = a^n a$ and so (14) is true. Now we can suppose that $a \in S \setminus S^2$. By U we denote the subsemigroup of S generated by a and $g = (ba)^n$. It follows from (9) that $g^2 = g$ and according to (13), we have

$$(gdg)^n = g \tag{15}$$

for all $d \in U$. We shall show that

$$ag = a^n ag\,. \tag{16}$$

Case 1. $gU \cap agU \neq \emptyset$. Then $gd \in agU$ for some $d \in U$ and so, by (15), we have $g = (gdg)^n \in agU$. Thus $g \in a^n gU$ and, by (9), we have $g = a^n g$, which implies (16).

Case 2. $agU \cap a^2U \neq \emptyset$. Then $agd \in a^2U$ for some $d \in U$ and so, by (15) and (8), we obtain $ag = a(gdg)^n \in a^2U \subseteq a^nU$. Consequently $ag = a^n ag$ (see (9)).

Case 3. $a^2U \cap gU \neq \emptyset$. Then $gd \in a^2U$ for some $d \in U$ and so $g \in a^2U \subseteq a^nU$. This implies $g = a^n g$ and so (16) is fulfilled.

Case 4. gU, agU and a^2U are pairwise disjoint subsets of U. It is easy to prove that $\{\{a\}, gU, agU, a^2U\}$ is a decomposition of U, the corresponding equivalence ϱ is a congruence on U and the quotient semigroup U/ϱ is isomorphic to the semigroup Q from Lemma 8. Since $U, U/\varrho \in \mathscr{V}$, we have $Q \in \mathscr{V}$, which contradicts Lemma 8.

Therefore (16) is true and from this and (12) we have $ab = a(ba)^n b = agb = a^n agb = a^n a(ba)^n b = a^n ab$. Consequently (14) is proved and so we have

$$\mathscr{V} \subseteq \mathscr{W}(x^n xy = xy)\,.$$

Dually we can get

$$\mathscr{V} \subseteq \mathscr{W}(xyy^n = xy)\,.$$

Hence we have

$$\mathscr{V} \subseteq \mathscr{W}((xyx)^n = x^n)\,.$$

Indeed, using (13) and (9), we obtain

$$(xyx)^n = (x^n(xyx)\,x^n)^n = (x^n x^n)^n = x^n\,.$$

II. Let S be a semigroup satisfying

$$S \in \mathscr{W}((xy)^{n+1} = xy) \cap \mathscr{W}((xyx)^n = x^n) \tag{17}$$

for a positive integer n. It is easy to show that

(18) $$S \in \mathscr{W}((xy)^n (xy)^n = (xy)^n).$$

Further, we have

(19) $$S \in \mathscr{W}(x^n xy = xy) \cap \mathscr{W}(xyy^n = xy).$$

Indeed, $xyx = (xyx)^n xyx = x^n xyx$ and so

$$xy = (xy)^{n+1} = x^n (xy)^{n+1} = x^n xy.$$

Let $a \in S$ and $e, f \in E(S)$. We shall show that

(20) $$(eaf)^n = (ef)^n.$$

According to (17), we have $(eaf)^n e = (eafe)^n = e$ and $f(eaf)^n = (feaf)^n = f$. Then $(eaf)^n (ef)^n (eaf)^n = (ef)^n$ and so, by (17) and (18), we obtain $(eaf)^n = ((eaf)^n \cdot (ef)^n (eaf)^n)^n = (ef)^n$.

Now, we shall prove the following proposition:

(21) For $a, b, c, d \in S$ there exists $g \in E(S)$ such that $ab = agb$ and $cd = cgd$.

First, we shall show that (21) is fulfilled for $a, b, c, d \in E(S)$. Put $e = (ba)^n$ and $f = (dc)^n$. Using (17) and (18) we obtain $e^2 = e$, $ab = (ab)^{n+1} = a(ba)^n b = aeb$, $f^2 = f$ and $cd = cfd$. Put $g = (ef)^n$. Then, by (18) and (17), we have $g^2 = g$, $ge = (ef)^n e = (efe)^n = e$ and dually $fg = f$. Therefore using (20) and (17) we can get $ab = aeb = ageb = ag(ba)^n b = agb(ab)^n = agb(agb)^n = agb$ and dually $cd = cgd$.

Now, we shall suppose that $a, b, c, d \in S$. It follows from (17) that $a^n, b^n, c^n, d^n \in E(S)$ and so $a^n b^n = a^n g b^n$, $c^n d^n = c^n g d^n$ for some $g \in E(S)$. According to (19), we have $ab = a^{n+1} b^{n+1} = a^{n+1} g b^{n+1} = agb$ and $cd = cgd$.

Let $A, B \in \mathrm{Tol}(S)$ We shall show that

(22) $$ABAB \subseteq AB.$$

Let $(u, v) \in ABAB$. Then $(u, v) = (a_1, a_2) \cdot (b_1, b_2) \cdot (c_1, c_2) \cdot (d_1, d_2)$, where (a_1, a_2), $(c_1, c_2) \in A$ and $(b_1, b_2), (d_1, d_2) \in B$. According to (21) there exist $g, h \in E(S)$ such that $u = a_1 b_1 c_1 d_1 = a_1 g b_1 c_1 h d_1$ and $v = a_2 b_2 c_2 d_2 = a_2 g h_2 c_2 h d_2$. Using (17) and (20) we obtain $g b_1 c_1 h = g b_1 c_1 h (g b_1 c_1 h)^{2n} = g b_1 c_1 h (g b_1 c_2 h)^{2n} = g b_1 c_1 h (g b_1 c_2 h)^{2n-1} g b_1 c_2 h$ and dually $g b_2 c_2 h = g b_1 c_2 h (g b_1 c_2 h)^{2n-1} g b_2 c_2 h$. Consequently we have $(u, v) = (a_1 g b_1 c_1 h, a_2 g b_1 c_2 h) \cdot (g b_1 c_2 h, g b_1 c_2 h)^{2n-1} \cdot (g b_1 c_2 h d_1, g b_2 c_2 h d_2) \in AB$.

Let $A, B, C \in \mathrm{Tol}(S)$ and $A \subseteq C$. We shall prove the following inclusions:

(23) $$AB \cap C \subseteq A(B \cap C),$$

(24) $$BA \cap C \subseteq (B \cap C) A,$$

(25) $$ABA \cap C \subseteq A(B \cap C) A \quad \text{and}$$

(26) $$BAB \cap C \subseteq (B \cap C) A(B \cap C).$$

Inclusion (23). Let $(u, v) = (a_1, a_2) . (b_1, b_2) \in C$, where $(a_1, a_2) \in A$ and $(b_1, b_2) \in \in B$. According to (21) we have $(u, v) = (a_1g, a_2g) . (gb_1, gb_2)$ for some $g \in E(S)$. Evidently $(a_1g, a_2g) \in A$ and $(gb_1, gb_2) \in B$. Further, by (20) and (21), we have $(gb_1, gb_2) = (ga_1g, ga_2g)^{2n} (b_1, b_2) = (ga_1g, ga_2g)^{2n-1} (ga_1gb_1, ga_2gb_2) = (ga_1g, ga_2g)^{2n-1} (gu, gv) \in C$. Consequently $(u, v) \in A(B \cap C)$.

Inclusion (24). It is dual to (23).

Inclusion (25). Let $(u, v) = (a_1, a_2) . (b_1, b_2) . (c_1, c_2) \in C$, where $(a_1, a_2), (c_1, c_2) \in \in A$ and $(b_1, b_2) \in B$. By (21) there exists $g \in E(S)$ such that $b_1c_1 = b_1gc_1$ and $b_2c_2 = b_2gc_2$. Put $(x, y) = (a_1, a_2) . (b_1(gc_1g)^2, b_2(gc_1g)^2)$. Using (17) and (20) it is easy to show that $(x, y) \in AB$ and $(x, y) = (u, v) ((gc_1g)^{2n+1}, (gc_2g)^{2n-1} (gc_1g)^2) \in \in C$. By (23) we have $(x, y) \in AB \cap C \subseteqq A(B \cap C)$ and so $(u, v) = (x, y) . ((gc_1g)^{3n-2} c_1, (gc_1g)^{3n-2} c_2) \in A(B \cap C) A$.

Inclusion (26). Let $(u, v) = (b_1, b_2) . (a_1, a_2) . (d_1, d_2) \in C$, where (b_1, b_2), $(d_1, d_2) \in B$ and $(a_1, a_2) \in A$. Clearly $(u^n, v^n) \in C$. It follows from (19) that $(u, v) = (b_1^n, b_2^n) (u, v) (d_1^n, d_2^n)$ and so, by (18) and (20), we have $(u^n, v^n) = (b_1^n d_1^n, b_2^n d_2^n)^n \in B$. Hence

$$(27) \qquad (u^n, v^n) \in B \cap C\,.$$

According to (17), (21) and (22), there exists $g \in E(S)$ such that $(u, v) = (u, v) . (g, g) . (u^n, v^n) \in BABA(u^n, v^n) \subseteqq BA(u^n, v^n)$. This means that $(u, v) = (x, y) . (u^n, v^n)$, where $(x, y) \in BA$. According to (21), there exists $h \in E(S)$ such that $(u, v) = (xh, yh) . (u^n, v^n)$. Using (20), (18) and (27) it is easy to see that $(xh, yh) \in BA$ and $(xh, yh) = (x, y) \;\; (hu^nh, hv^nh)^{2n} = (xhv^n, yhv^n) \;\; (hu^nh, hv^nh)^{2n-1} = (u, v) . (hu^nh, hv^nh)^{2n-1} \in C$. Then, by (24), we have $(xh, yh) \in BA \cap C \subseteqq \subseteqq (B \cap C) A$. Consequently, by (27), $(u, v) = (xh, yh) . (u^n, v^n) \in (B \cap C) A(B \cap C)$.

We are now ready to complete the proof. We shall show that the lattice $\mathrm{Tol}\,(S)$ is modular. Let $A, B, C \in \mathrm{Tol}\,(S)$ and $A \subseteqq C$. It follows from (1), (2), (22), (23), (24), (25) and (26) that $(A \vee B) \wedge C = (A \cup B \cup AB \cup BA \cup ABA \cup BAB) \cap C \subseteqq \subseteqq A \cup (B \cap C) \cup A(B \cap C) \cup (B \cap C) A \cup A(B \cap C) A \cup (B \cap C) A(B \cap C) = A \vee \vee (B \wedge C) \subseteqq (A \vee B) \wedge C$.

Corollary 2. *Every principal tolerance trivial variety of semigroups is tolerance modular.*

The proof follows from Theorem 1 and Theorem 3.

Corollary 3. *Let $\mathscr{V}$ be a variety of semigroups and $\mathscr{V} \cap \mathscr{Z} = \emptyset$. Then $\mathscr{V}$ is principal tolerance trivial if and only if $\mathscr{V}$ is tolerance modular.*

The proof follows from Lemma 2, Theorem 1 and Theorem 3.

Corollary 4. *Let $\mathscr{V}$ be a variety of commutative semigroups and $\mathscr{V} \cap \mathscr{Z} = \emptyset$. Then $\mathscr{V}$ is tolerance trivial (congruence modular) if and only if $\mathscr{V}$ is tolerance modular.*

The proof follows from Corollary 3, Theorem 2 and Corollary 1.

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