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ON WEAKLY CONVERGENT NETS IN SPACES OF NON-NEGATIVE MEASURES

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The first part of this paper is concerned with several characterizations of the weakly convergent nets of real-valued, non-negative measures with the range of definition on the σ-algebra which is generated by the pavings of open Baire or Borel sets of some abstract topological space. The methods used in this section were developed by F. Topsøe in [11], [12] and the assertions proved here represent some sharping of thats in [11].

In the second section of the paper we derive necessary and sufficient conditions which the w-convergent net must satisfy in order to be weakly tight and using a result form [12] we show that the only types of regular and Hausdorff topological spaces in which each w-convergent net is weakly tight are the locally compact ones.

The whole theory is developed from the set-theoretical point of view. This approach shows quite well (from the view point of the classical probability theory) the relation between the weak convergence and the set-wise convergence which plays also an important role in the statistical testing.

0. PRELIMINARIES

Let us suppose that X is a non-empty abstract set and let $\mathscr{G}$ be a non-empty class of subsets of X which is closed under the formation of finite unions and intersections. Let us denote by $\mathscr{F}$ the class of sets defined by the relation $\mathscr{F} = \{F \subset X: F = G^c, G \in \mathscr{G}\}$ and consider a class $\mathscr{K} \subset \mathscr{F}$ (which is closed under the formation of finite unions and intersections) and has the property that $KF \in \mathscr{K}$ when $K \in \mathscr{K}$ and $F \in \mathscr{F}$. If $\mathscr{G}$ separates the sets in $\mathscr{K}$, that is, if to each couple K_1, K_2 of disjoint sets from $\mathscr{K}$ we can find two disjoint sets $G_1, G_2 \in \mathscr{G}$ such that $G_1 \supset K_1$, $G_2 \supset K_2$, we can say that $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ is a space. In the case when our considerations will not involve the system $\mathscr{K}$ we shall speak shortly about the space $X, \mathscr{G}$. If $\mathscr{G}$ forms an open topology on X, then under $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ we shall understand the space where $\mathscr{G}, \mathscr{F}$ and $\mathscr{K}$, are the open, closed and closed compact subsets of X, respectively. In this case the assumption that $\mathscr{G}$ separates the sets in $\mathscr{K}$ will be dropped. Each space $X, \mathscr{G}$ determines a set algebra $\mathscr{E}$ generated by $\mathscr{G}$. If $X, \mathscr{G}$ is a topology, we shall automatically assume that $\mathscr{E}$ is the Borel σ-algebra generated by $\mathscr{G}$.

A measure is a non-negative real-valued set function defined on a set algebra. If $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ is a space and μ is a measure on $\mathscr{E}(\mathscr{E}$ is generated by $\mathscr{G})$, then μ is said to be *regular with respect to* (wrt) $\mathscr{F}$ (or wrt $\mathscr{K}$) if for each $E \in \mathscr{E}$, $\mu E = \sup_{F \subset E} \mu F$ where $F \in \mathscr{F}$ (or $F \in \mathscr{K}$, respectively). The measure μ on $\mathscr{E}$ is called *σ-smooth* if for each countable class $\{F_n\} \subset \mathscr{F}$ which filters downvards to the set $F \in \mathscr{F}$ $\lim \mu F_n = \mu F$. If this relation holds for each subclass $\{F_\beta\} \subset \mathscr{F}$ which filters downvards to the set $F \in \mathscr{F}$ then μ is said to *be τ-smooth.*

If $X, \mathscr{G}$ is a regular topological space (or if $X, \mathscr{G}$ is a G_δ space, that is, if to each $G \in \mathscr{G}$ there are $\{G_n\} \subset \mathscr{G}$ and $\{F_n\} \subset \mathscr{F}$ with $G = \bigcup G_n = \bigcup F_n$, $G \supset F_n \supset G_n$ for all $n = 1, 2, \ldots$) and if μ is regular wrt $\mathscr{F}$, then μ is τ-smooth (or σ-smooth) iff $\lim \mu F_\beta = 0$ whenever $\{F_\beta\} \subset \mathscr{F}$ is a (countable) class filtering downvards to the empty set.

We shall denote the system of all finite measures on $\mathscr{E}$ by $\mathscr{M}^+(\mathscr{E})$. $\mathscr{M}^+(\mathscr{E}, \mathscr{F})$ and $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ are the subsets of $\mathscr{M}^+(\mathscr{E})$ which consist of all $\mathscr{F}$-regular and $\mathscr{K}$-regular measures in $\mathscr{M}^+(\mathscr{E})$. The subsets of all τ-smooth (σ-smooth) measures in $\mathscr{M}^+(\mathscr{E})$, $\mathscr{M}^+(\mathscr{E}, \mathscr{F})$ and $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ will be denoted by $\mathscr{M}^+_\tau(\mathscr{E})$, $\mathscr{M}^+_\tau(\mathscr{E}, \mathscr{F})$ and $\mathscr{M}^+_\tau(\mathscr{E}, \mathscr{K})$ ($\mathscr{M}^+_\sigma(\mathscr{E})$, $\mathscr{M}^+_\sigma(\mathscr{E}, \mathscr{F})$ and $\mathscr{M}^+_\sigma(\mathscr{E}, \mathscr{K})$), respectively.

Let $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ be a space and γ be a finite real-valued set function defined on $\mathscr{K}$. γ is said to be *tight* wrt $\mathscr{K}$ if $\gamma K_1 - \gamma K_2 = \sup_{K \subset K_1 - K_2} \gamma K$ whenever $K_1 \supset K_2$, $K_1, K_2 \in \mathscr{K}$ and it is called *τ-smooth* (*σ-smooth*) if $\lim \gamma K_\alpha = \gamma K$ for each (countable) class $\{K_\alpha\} \subset \mathscr{K}$ filtering downvards to K, $K \in \mathscr{K}$.

Theorem 0.1. *If $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ is a space and if γ is a tight set function on $\mathscr{K}$, then γ can be extended to a measure which is defined on $\mathscr{E}$ and regular* wrt $\mathscr{K}$.

If moreover γ is σ-smooth and if $\mathscr{K}$ is closed under the formation of countable intersections, then γ can be extended to a σ-smooth measure on the σ-algebra generated by $\mathscr{G}$ and this extension is regular wrt $\mathscr{K}$.

The extensions are uniquely determined by the values of γ on $\mathscr{K}$.

Proof. See [11], theorem 1.

We note that the assumption "$\mathscr{G}$ separates the sets in $\mathscr{K}$" is not necessary in 0.1 and clearly we can consider $\mathscr{F}$ instead of $\mathscr{K}$.

The w-topology is the weakest topology on $\mathscr{M}^+(\mathscr{E})$ for which the mapping $\mu \to \mu X$ is continuous and all mappings $\mu \to \mu G$ are lower semicontinuous for every $G \in \mathscr{G}$. In other terms, if $\mu \in \mathscr{M}^+(\mathscr{E})$ and if $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ is a net, then $\{\mu_\alpha\}$ converges to μ in the w-topology and we write $\mu_\alpha \to_w \mu$ iff $\lim \mu_\alpha X = \mu X$ and $\underline{\lim}\, \mu_\alpha G \geqq \mu G$ for all $G \in \mathscr{G}$. In this connection we shall say that the net $\{\mu_\alpha\}$ is w-convergent in $\mathscr{M}^+(\mathscr{E})$ and that μ is the w-limit point of $\{\mu_\alpha\}$.

The s-topology is defined as the weakest topology for which all the mappings $\mu \to \mu E$, where $E \in \mathscr{E}$, are continuous. If $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ is a net and if $\mu \in \mathscr{M}^+(\mathscr{E})$ then $\{\mu_\alpha\}$ converges to μ in the s-topology iff $\lim \mu_\alpha E = \mu E$ for all $E \in \mathscr{E}$. In this connection we say, that $\{\mu_\alpha\}$ s-converges to μ and μ is the s-limit point of $\{\mu_\alpha\}$.

Our main aim in this section is to derive some characterizations of w-convergent and s-convergent nets of non-negative measures. The convergence conditions we shall give were at first used by F. Topsøe [11], [12] for description of w-convergent and s-compact nets of Radon measures defined on a Hausdorff topological space. We will show that they are useful also for description of convergent nets connected with regular topological spaces and G_δ spaces. We shall use the results obtained later for the study of the weakly uniformly tight nets in spaces of non-negative measures.

Theorem 1.1. *Let us consider the space* $(X, \mathcal{G}, \mathcal{F}, \mathcal{K})$ *and the set* $\mathcal{M}^+(\mathcal{E}, \mathcal{K})$ *with the w-topology. The net* $\{\mu_\alpha\} \subset \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ *is w-convergent in* $\mathcal{M}^+(\mathcal{E}, \mathcal{K})$ *iff the conditions*

i) $\overline{\lim}\, \mu_\alpha X < \infty$;

ii) $\inf_{G \supset K} \underline{\lim}\, \mu_\alpha G = \inf_{G \supset K} \overline{\lim}\, \mu_\alpha G$ *for all* $K \in \mathcal{K}$;

iii) $\inf_{K} \sup_{F \supset K^c} \overline{\lim}\, \mu_\alpha F = 0$

are fulfilled.

Proof. Let $\{\mu_\alpha\} \subset \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ be a w-convergent net with the limit point $\mu \in \in \mathcal{M}^+(\mathcal{E}, \mathcal{K})$. Of course i) is true. In order to prove ii) we fix any $K \in \mathcal{K}$ and $\varepsilon > 0$. As

$$\mu E = \sup_{K \subset E} \mu K = \inf_{K^c \supset E} \mu K^c \quad \text{for all} \quad E \in \mathcal{E}$$

we can find $\dot{K} \in \mathcal{K}$ such that $K \subset \dot{K}^c$ and $\mu K > \mu \dot{K}^c - \varepsilon$. $\mathcal{G}$ separates the sets in $\mathcal{K}$. Hence there are $G \in \mathcal{G}$ and $F \in \mathcal{F}$ with $K \subset G \subset F \subset \dot{K}^c$ such that

$$\mu K \leqq \mu G \leqq \underline{\lim}\, \mu_\alpha G \leqq \overline{\lim}\, \mu_\alpha G \leqq \overline{\lim}\, \mu_\alpha F \leqq \mu F \leqq \mu \dot{K}^c < \mu K + \varepsilon .$$

Since ε can be made arbitrarily small we can conclude that

ii*) $$\mu K = \inf_{G \supset K} \underline{\lim}\, \mu_\alpha G = \inf_{G \supset K} \overline{\lim}\, \mu_\alpha G .$$

The proof of iii) follows easily from the relation

$$\lim \mu_\alpha X = \mu X = \sup_K \mu K = \sup_K \inf_{G \supset K} \underline{\lim}\, \mu_\alpha G \leqq \lim \mu_\alpha X .$$

In order to prove the reverse implication we shall assume that we have a net $\{\mu_\alpha\} \subset \subset \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ satisfying i)–iii). Let μ be a set function defined by the relation ii*). We will show that μ is tight wrt $\mathcal{K}$ and defines due to the theorem 0.1 uniquely a measure $\bar{\mu} \in \mathcal{M}(\mathcal{E}, \mathcal{K})$. Let us have two sets $K_1 \supset K_2$ in $\mathcal{K}$ and some $\varepsilon > 0$. We choose $G_2 \supset K_2$ in $\mathcal{G}$ with $\mu K_2 > \overline{\lim}\, \mu_\alpha G_2 - \varepsilon$ and put $K = K_1 - G_2$. Then for every $G \supset K$

$$\overline{\lim}\, \mu_\alpha G \geqq \overline{\lim}\, \mu_\alpha G \cup G_2 - \overline{\lim}\, \mu_\alpha G_2 \geqq \mu K_1 - \mu K_2 - \varepsilon .$$

Consequently $\mu K \geqq \mu K_1 - \mu K_2 - \varepsilon$ and as ε can be made arbitrarily small $\sup_{K \subset K_1 - K_2} \mu K \geqq \geqq \mu K_1 - \mu K_2$.

Using the fact that $\mathcal{G}$ separates the sets in $\mathcal{K}$ and $\underline{\lim}\, \mu_\alpha G_1 \cup G_2 \geqq \underline{\lim}\, \mu_\alpha G_1 +$

$+ \underline{\lim}\, \mu_\alpha G_2$ if $G_1, G_2 \in \mathscr{G}$ are disjoint, it is not difficult to verify the reverse inequality $\sup_{K \subset K_1 - K_2} \mu K \leqq \mu K_1 - \mu K_2$. Thus the tightness of μ is established and the set function $\bar{\mu}$ defined by the relation

$$\bar{\mu} E = \sup_{K \subset E} \inf_{G \supset K} \underline{\lim}\, \mu_\alpha G \quad \text{for all} \quad E \in \mathscr{E}$$

is a measure in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$. For each $G \in \mathscr{G}$, $\bar{\mu} G \leqq \underline{\lim}\, \mu_\alpha G$ and iii) implies

$$\overline{\lim}\, \mu_\alpha X = \overline{\lim}\, \mu_\alpha X - \inf_K \sup_{F \subset K^c} \overline{\lim}\, \mu_\alpha F \leqq \sup_K \inf_{G \supset K} \overline{\lim}\, \mu_\alpha G =$$

$$= \sup_K \inf_{G \supset K} \underline{\lim}\, \mu_\alpha G \leqq \underline{\lim}\, \mu_\alpha X .$$

This means, that $\lim \mu_\alpha X$ exists, is equal to $\bar{\mu} X$, and $\mu_\alpha \to_w \bar{\mu}$.

The proof that μ defined by ii*) is a tight set function resembles the proof given by F. Topsøe in [11], lemma 2. We have nowhere used the assumption $\{\mu_\alpha\} \subset \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$. Whence 1.1 can be reformulated in the form of

Corollary 1.2. *The net* $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ *w-converges to any* $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ *iff the assumptions* i)–iii) *of the theorem* 1.1 *hold.*

Remark 1.3. Let $X, \mathscr{G}$ be a compact Hausdorff topological space such that $\mathscr{M}^+(\mathscr{E}) \not\subseteq \mathscr{M}^+(\mathscr{E}, \mathscr{K})$. Such a space exists due to J. Dieudonné (see [9], sec. 1, theorem 3.5). If μ is a measure in $\mathscr{M}^+(\mathscr{E})$ not contained in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ then the sequence $\{\mu_n\} \subset \mathscr{M}^+(\mathscr{E})$ defined by the relation $\mu_n = \mu$ for each natural n satisfies i)–iii). By the proof of 1.2 $\mu_n \to_w \bar{\mu}$ where $\bar{\mu}$ is defined by the relation ii*). We see that $\{\mu_n\}$ has two w-limit points, μ and $\bar{\mu}$.

This example shows, that $\mathscr{M}^+(\mathscr{E})$ need not be a Hausdorff space wrt the w-topology. In spite of it the theorem 1.1 guarantees that if $\mathscr{G}$ separates sets in $\mathscr{K}$, $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ provided with the w-topology is a Hausdorff space (compare with Topsøe, [12] theorem 11.2).

Theorem 1.4. *Let us consider the space* $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ *and the set* $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ *with the w-topology. Let us assume moreover that X is a topological space with at last one from these properties*: α) *Hasudorff's property,* β) *regularity,* γ) *complete regularity,* δ) *X is a* G_δ *space. Then the net* $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ *is w-convergent in* $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ *iff the conditions* i)–iii) *of* 1.1 *hold.*

Proof. Considering α) or γ) we see, that the assertion can be proved in the same way as 1.1. We have only to keep in mind, using the theorem 0.1, that now $\mathscr{E}$ is a σ-algebra.

Let us treat the proof of 1.4 under the assumption β). We fix an arbitrary closed set $F \in \mathscr{F}$. By the regularity of X we can find two classes $\{G_\beta\} \subset \mathscr{G}$, $\{F_\beta\} \subset \mathscr{F}$ filtering downwards with $\bigcap_\beta G_\beta = \bigcap_\beta F_\beta = F$ and such that $F_\beta \supset G_\beta \supset F$ for each β. Whence for each β

$$\mu F \leqq \mu G \leqq \underline{\lim}\, \mu_\alpha G_\beta \leqq \overline{\lim}\, \mu_\alpha F_\beta \leqq \mu F_\beta .$$

Since μ is regular wrt the paving of closed compact sets μ is τ-smooth and

$$\mu F = \inf_\beta \underline{\lim}\, \mu_\alpha G_\beta = \inf_\beta \overline{\lim}\, \mu_\alpha G_\beta = \inf_\beta \mu G_\beta .$$

It is easy to see that this is equivalent to the relation

$$\mu F = \inf_{G \supset F} \underline{\lim}\, \mu_\alpha G = \inf_{G \supset F} \overline{\lim}\, \mu_\alpha G .$$

If we take into account only $F \in \mathscr{K}$, then ii) is established.

In order to prove the converse implication we shall assume that $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ is a net satisfying i)–iii). By iii) and from the proof of 1.1 it follows that $\lim \mu_\alpha X$ exists and

$$\lim \mu_\alpha X = \sup_K \inf_{G \supset K} \underline{\lim}\, \mu_\alpha G .$$

Let $\{G_\beta\} \subset \mathscr{G}$ be a net with $G_\beta \uparrow X$ and $\varepsilon > 0$ be a fixed number. Now we can find a $K \in \mathscr{K}$ such that

$$\lim \mu_\alpha X < \inf_{G \supset K} \underline{\lim}\, \mu_\alpha G + \varepsilon .$$

Since $K \subset \bigcup_\beta G_\beta$ is a compact set there is any $G_{\beta_0} \in \{G_\beta\}$ such that G_{β_0} contains K and

$$\lim \mu_\alpha X < \underline{\lim}\, \mu_\alpha G_{\beta_0} + \varepsilon .$$

This inequality holds for all $\beta \geqq \beta_0$. Thus $\overline{\lim}\, \mu_\beta G_\beta^c < \varepsilon$ for all $\beta \geqq \beta_0$ and we can conclude that the function $\lim \mu_\alpha$ is τ-smooth at $\emptyset$ wrt $\mathscr{F}$.

Let μ be the set function on $\mathscr{K}$ defined by the relation ii*) and let $K_1 \supset K_2$ be in $\mathscr{K}$. The inequality $\sup_{K \subset K_1 - K_2} \mu K \geqq \mu K_1 - \mu K_2$ can be proved in the same way as in 1.1. In order to prove the reverse inequality (and hence the tightness of μ) we need to know that μ is additive on $\mathscr{K}$. Clearly μ is subadditive on $\mathscr{K}$. If $K_1, K_2 \in \mathscr{K}$ are disjoint, then the class of all sets $F_1 F_2 \in \mathscr{F}$ such that $F_1 \supset G_1 \supset K_1$, $F_2 \supset$ $\supset G_2 \supset K_2$ for some $G_1, G_2 \in \mathscr{G}$ filters downwards toward the empty set. From the relation

$$\mu K_1 + \mu K_2 \leqq \underline{\lim}\, \mu_\alpha G_1 + \underline{\lim}\, \mu_\alpha G_2 \leqq \underline{\lim}\, (\mu_\alpha G_1 + \mu_\alpha G_2) =$$

$$= \underline{\lim}\, (\mu_\alpha G_1 \cup G_2 + \mu_\alpha G_1 G_2) \leqq \underline{\lim}\, \mu_\alpha G_1 \cup G_2 + \overline{\lim}\, \mu_\alpha F_1 F_2$$

and from the τ-smoothness of $\overline{\lim}\, \mu_\alpha$ at $\emptyset$ it follows that

$$\mu K_1 + \mu K_2 \leqq \inf_{G \supset K_1 \cup K_2} \underline{\lim}\, \mu_\alpha G = \mu K_1 \cup K_2 .$$

The additivity of μ is established. The verification of the inequality $\sup_{K \subset K_1 - K_2} \mu K \leqq$ $\leqq \mu K_1 - \mu K_2$ is left to the reader. Thus μ is tight wrt a compact paving $\mathscr{K}$ and it can be uniquely extended to a Borel measure (see the theorem 0.1). The rest of the proof is analogous to that one in 1.1. The G_δ case is now clear.

Corollary 1.5. *The net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$, where $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ is the space from* 1.4, *w-converges to any* $\mu \in \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ *iff the assumptions* i)–iii) *of the theorem* 1.1 *hold.*

It is convenient to note that if $X \in \mathcal{K}$ then every net $\{\mu_\alpha\} \subset \mathcal{M}^+(\mathcal{E})$ w-converging to some $\mu \in \mathcal{M}^+(\mathcal{E})$ has at the same time a w-limit point $\bar{\mu} \in \mathcal{M}^+(\mathcal{E}, \mathcal{K})$. If moreover $(X, \mathcal{G}, \mathcal{F}, \mathcal{K})$ is one of the spaces considered above in 1.1 or 1.4, then $\bar{\mu}$ is uniquely determined by the values of the function $\underline{\lim}\, \mu_\alpha$ on $\mathcal{G}$. The proofs of the following theorems can be easily derived from that in 1.1 and 1.4.

Theorem 1.6. *If* $X, \mathcal{G}$ *is a* G_δ *space then* $\{\mu_\alpha\} \subset \mathcal{M}^+(\mathcal{E})$ *is w-convergent to any* $\mu \in \mathcal{M}^+(\mathcal{E}, \mathcal{F})$ *iff the conditions*

i) $\overline{\lim}\, \mu_\alpha X < \infty$;

ii) $\inf_{G \supset F} \underline{\lim}\, \mu_\alpha G = \inf_{G \supset F} \overline{\lim}\, \mu_\alpha G$ *for all* $F \in \mathcal{F}$;

iv(σ)) $\inf_n \overline{\lim}\, \mu_\alpha F_n = 0$ *if* $\{F_n\} \subset \mathcal{F},\ F_n \downarrow \emptyset$

hold. μ *is uniquely determined by the values of* $\overline{\lim}\, \mu_\alpha$ *on* $\mathcal{F}$.

Theorem 1.7. *If* $X, \mathcal{G}$ *is a regular topological space then* $\{\mu_\alpha\} \subset \mathcal{M}^+(\mathcal{E})$ *is w-convergent to any* $\mu \in \mathcal{M}_\tau^+(\mathcal{E}, \mathcal{F})$ *iff* i) *with* ii) *from* 1.6 *hold and*

iv(τ)) $\inf_\beta \overline{\lim}\, \mu_\alpha F_\beta = 0$ *whenever* $\{F_\beta\} \subset \mathcal{F},\ F_\beta \downarrow \emptyset$.

μ *is uniquely determined by the values of* $\overline{\lim}\, \mu_\alpha$ *on* $\mathcal{F}$.

Theorem 1.8. *Let us consider the space* $(X, \mathcal{G}, \mathcal{F}, \mathcal{K})$ *and the space* $\mathcal{M}^+(\mathcal{E})$ *with the s-topology. The net* $\{\mu_\alpha\} \subset \mathcal{M}^+(\mathcal{E})$ *is s-convergent to* $\mu \in \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ *iff it is w-convergent to* μ *and*

v) $\inf_{\dot{K} \subset K^c} \overline{\lim}\, \mu_\alpha K^c - \dot{K} = 0$ *for all* $K \in \mathcal{K}$.

Proof. Assume that $\mu_\alpha \to_w \mu$ where $\mu \in \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ and that v) holds. Using v) we can prove that $\overline{\lim}\, \mu_\alpha K^c = \sup_{\dot{K} \subset K^c} \overline{\lim}\, \mu_\alpha \dot{K}$ for all $K \in \mathcal{K}$. Choose some $K \in \mathcal{K}$ and some $\varepsilon > 0$. Then we can find $\dot{K} \subset K^c$ in $\mathcal{K}$ with $\overline{\lim}\, \mu_\alpha K^c < \overline{\lim}\, \mu_\alpha \dot{K} + \varepsilon$. It follows from the relation

$$\overline{\lim}\, \mu_\alpha K^c - \varepsilon < \overline{\lim}\, \mu_\alpha \dot{K} \leqq \mu \dot{K} \leqq \inf_{G \supset \dot{K}} \underline{\lim}\, \mu_\alpha G \leqq \underline{\lim}\, \mu_\alpha K^c$$

that $\lim \mu_\alpha K^c$ exists and this holds for all $K \in \mathcal{K}$. For an arbitrary $E \in \mathcal{E}$ we have

$$\mu E = \sup_{K \subset E} \mu K = \sup_{K \subset E} \lim \mu_\alpha K \leqq \underline{\lim}\, \mu_\alpha E$$

and using the equality $\lim \mu_\alpha X = \mu X$ it is easy to verify the reverse inequality. Hence $\mu_\alpha \to_s \mu$.

The proof of the reverse implication is left to the reader.

Remark 1.9. In the proof of 1.8 we did not use the assumption that $\mathcal{G}$ separates sets in $\mathcal{K}$. Clearly $\mathcal{E}$ can be a σ-algebra as well as an algebra generated by $\mathcal{G}$.

Using 1.2 and 1.8 we can establish for a space $(X, \mathcal{G}, \mathcal{F}, \mathcal{K})$

Corollary 1.10. *The net* $\{\mu_\alpha\} \subset \mathcal{M}^+(\mathcal{E})$ *is s-convergent to a point in* $\mathcal{M}^+(\mathcal{E}, \mathcal{K})$

iff $\inf_{\dot{K} \subset K^c} \overline{\lim}\, \mu_\alpha K^c - \dot{K} = 0$ *and* $\overline{\lim}\, \mu_\alpha F \leqq \underline{\lim}\, \mu_\alpha G < \infty$ *whenever* $F \subset G$, $G \in \mathscr{G}$, $F \in \mathscr{F}$.

By 1.5 and 1.8 this assertion is true also for that kind of topological spaces which are considered in 1.4. The theorems 1.6 and 1.8 are leading to

Corollary 1.11. *If* $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ *is a* G_δ *space or if* $\mathscr{G}$ *separates the sets in* $\mathscr{F}$ *then* $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ *s-converges to a point in* $\mathscr{M}_\sigma^+(\mathscr{E}, \mathscr{F})$ *iff the smoothness condition* iv(σ)) *holds,* $\overline{\lim}\, \mu_\alpha F \leqq \underline{\lim}\, \mu_\alpha G < \infty$ *for* $F \subset G$, $G \in \mathscr{G}$, $F \in \mathscr{F}$ *and* $\inf_{F \subset G} \overline{\lim}\, \mu_\alpha G - - F = 0$.

We leave to the reader the formulation of the τ-smooth analogy of 1.10 (using 1.7 and 1.8) for the regular topological spaces. If $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ is an arbitrary net with $\overline{\lim}\, \mu_\alpha X < \infty$, then it contains a subnet $\{\mu_{\alpha_\beta}\}$ with the property $\lim \mu_{\alpha_\beta} E$ exists for all $E \in \mathscr{E}$. In other words every bounded net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ contains a subnet which satisfies the conditions i) and ii) of 1.1. Now it is easy to observe that every bounded net that satisfies the condition iii) in 1.1 contains a w-convergent subnet with a limit in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$. From the foregoing propositions we can derive a number of similar criteria (see Topsøe [11]).

2. SOME GLOBAL CHARACTERISTICS OF CONVERGENT NETS

We shall call the net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ *weakly σ-equicontinuous* if for every decreasing sequence $\{F_n\} \subset \mathscr{F}$ with the empty intersection and for each $\varepsilon > 0$ we find α_0 and n_0 such that $\sup_{\alpha \geqq \alpha_0} \mu_\alpha F_{n_0} < \varepsilon$. If the foregoing property holds for every class $\{F_\beta\} \subset \mathscr{F}$ filtering downvards to the empty set we shall call the net *weakly τ-equicontinuous*. The net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ such that to every $\varepsilon > 0$ we can find a $K \in \mathscr{K}$ and any α_0 satisfying the relation $\sup_{\alpha \geqq \alpha_0} \mu_\alpha K^c < \varepsilon$ will be called *weakly tight* wrt $\mathscr{K}$.

Theorem 2.1. *Let* $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ *be a space,* $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ *be a net w-convergent to some* $\mu \in \mathscr{M}^+(\mathscr{E})$. *If* $\mu \in \mathscr{M}_\sigma^+(\mathscr{E})$ *then* $\{\mu_\alpha\}$ *is weakly σ-equicontinuous. If* X *is a* G_δ *space,* $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{F})$ *and if* $\{\mu_\alpha\}$ *is weakly σ-equicontinuous then* $\mu \in \mathscr{M}_\sigma^+(\mathscr{E})$.

We omit the proof (see the theorem 1.6) and show that both the assumptions $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{F})$ and X is G_δ are in 2.1 necessary. The necessity of $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{F})$ follows easily from the remark 1.3. To prove the second proposition we recall a counter-example of A. D. Alexandroff [1].

Example 2.2. Let $X = \langle 0, 1 \rangle \cup \boldsymbol{N}$ and $\mathscr{F}$ be the closed topology generated by the finite unions of sets

$$\langle a, b \rangle\,, \quad \text{where} \quad b < 1\,, \quad \langle a, 1 \rangle \cup \boldsymbol{N}\,, \quad \text{where} \quad a \in \langle 0, 1 \rangle \quad \text{and}$$
$$\boldsymbol{N} - \{1, 2, \ldots, n\}\,, \quad \text{where} \quad n \in \boldsymbol{N}\,.$$

X provided with the closed topology $\mathscr{F}$ is a normal separable locally compact T_1

space which is not T_2. Let δ_x be the one-point measure concentrated in $x \in X$. Then the net $\{\delta_{x_n}\}$, where $x_n = 1 - 1/n$ for $n = 1, 2, \dots$, w-converges to the measure μ which is defined by the relations $\mu E = 0$ if $E \cap N = \emptyset$ and $\mu E = 1$ if $E \cap N \neq \emptyset$. $\mathscr{E}$ is the Borel σ-algebra generated by $\mathscr{G}$, E is taken in $\mathscr{E}$. Clearly $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{F})$ but μ is not σ-smooth since for the sequence $\{F_m\} \subset \mathscr{F}$ defined by $F_m = \{m, m + 1, \dots\}$ we have $F_m \downarrow \emptyset$ and $\lim \mu F_m = 1$. At the same time it is easy to see $\{\delta_{x_n}\}$ is weakly σ-equicontinuous.

If X is a locally compact separable topological space then $\mathscr{M}^+_\sigma(\mathscr{E}, \mathscr{F}\} = \mathscr{M}^+(\mathscr{E}, \mathscr{K})$. 2.2 shows that the assumption about the σ-smoothness is essential for the validity of this equation.

Theorem 2.3. *Let us consider a space $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ and a net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ w-converging to some $\mu \in \mathscr{M}^+(\mathscr{E})$. If $\mu \in \mathscr{M}^+_\tau(\mathscr{E})$ then $\{\mu_\alpha\}$ is weakly τ-equicontinuous. If X is a regular topological space, if $\{\mu_\alpha\}$ is weakly τ-equicontinuous and $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{F})$ then $\mu \in \mathscr{M}^+_\tau(\mathscr{E})$.*

The proof of this assertion follows from 1.7. By 2.2 the assumption about the regularity of X cannot be dropped.

Theorem 2.4. *Let $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ be a space and $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ be a net w-converging to any $\mu \in \mathscr{M}^+(\mathscr{E})$. Consider the conditions*

i) *$\{\mu_\alpha\}$ is weakly tight* wrt $\mathscr{K}$;

ii) *for each $\varepsilon > 0$ there is any α_0 and $K \in \mathscr{K}$ such that* $\sup_{\alpha \geqq \alpha_0} \mu_\alpha F < \varepsilon$ *for each $F \subset K^C$.*

Either of them implies that $\{\mu_\alpha\}$ has a w-limit point in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ and if $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{F})$ then they are equivalent.

The first part of the proof follows easily from 1.2. We omit the simple proof of the second part. It is natural to ask under what assumptions the conditions i) and ii) in 2.4 are necessary for the w-convergence to $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{K})$.

Theorem 2.5. *Let us consider a space $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ and any net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ w-converging to $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{K})$. Then $\{\mu_\alpha\}$ is weakly tight* wrt $\mathscr{K}$ *iff*

$$\text{(T)} \qquad \mu K_0 = \inf_{K^c \supset K_0} \sup_{K \subset \dot{K}^c} \underline{\lim}\, \mu_\alpha K \quad \textit{for each} \quad K_0 \in \mathscr{K}\,.$$

Proof. Let us suppose that $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ is weakly tight wrt $\mathscr{K}$ and $\mu_\alpha \to_w \mu$, where $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{F})$. Fix any K_0, $\dot{K} \in \mathscr{K}$, $K_0 \subset \dot{K}^c$ and $\varepsilon > 0$. To K_0 and $\dot{K}$ we can find $G \in \mathscr{G}$ and $F \in \mathscr{F}$ such that $K_0 \subset G \subset F \subset \dot{K}^c$. Since $\{\mu_\alpha\}$ is weakly tight there is $K_\varepsilon \in \mathscr{K}$ and α_0 with $\mu_\alpha X < \mu_\alpha K_\varepsilon + \varepsilon$ for all $\alpha \geqq \alpha_0$. Hence $\mu_\alpha F < \mu_\alpha F K_\varepsilon + \varepsilon$ for each $\alpha \geqq \alpha_0$ and $\underline{\lim}\, \mu_\alpha F \leqq \underline{\lim}\, \mu_\alpha F K_\varepsilon + \varepsilon$. $FK_\varepsilon \in \mathscr{K}$, $FK_\varepsilon \subset \dot{K}^c$ thus

$$\mu K_0 \leqq \mu G \leqq \underline{\lim}\, \mu_\alpha G \leqq \underline{\lim}\, \mu_\alpha F \leqq \underline{\lim}\, \mu_\alpha F K_\varepsilon + \varepsilon \leqq \sup_{K \subset \dot{K}^c} \underline{\lim}\, \mu_\alpha K + \varepsilon$$

and the last relation is true for each $\dot{K} \in \mathscr{K}$ such that $K_0 \subset \dot{K}^c$. Hence

$$\mu K_0 \leqq \inf_{\dot{K}^c \supset K_0} \sup_{K \subset \dot{K}^c} \underline{\lim}\, \mu_\alpha K + \varepsilon$$

for every $\varepsilon > 0$. The rest of the proof of (T) can be left to the reader.

To prove the reverse implication we use the same arguments as Topsøe [12], theorem 9.3. For each fixed $K_0 \in \mathscr{K}$

$$\mu K_0 = \inf_{\dot{K}^c \supset K_0} \sup_{K \subset \dot{K}^c} \underline{\lim}\, \mu_\alpha K \leqq \sup_K \underline{\lim}\, \mu_\alpha K$$

Now it suffices to a given $\varepsilon > 0$ choose any K_0, $K \in \mathscr{K}$ such that $\mu X < \mu K_0 + \varepsilon$ and $\mu K_0 < \underline{\lim}\, \mu_\alpha K + \varepsilon$. Since $\lim \mu_\alpha X = \mu X$ we can easily find α_0 with

$$\sup_{\alpha \geqq \alpha_0} \mu_\alpha K^c < \overline{\lim}\, \mu_\alpha K^c + \varepsilon < \mu K_0^c + 2\varepsilon < 3\varepsilon$$

the weak tightness of $\{\mu_\alpha\}$ wrt $\mathscr{K}$ is established.

Theorem 2.6. *If $X, \mathscr{G}$ is an arbitrary topological space such that there are classes $\{K_\beta\} \subset \mathscr{K}$, $\{G_\beta\} \subset \mathscr{G}$, $G_\beta \uparrow X$ and $G_\beta \subset K_\beta \subset X$ for each β then every w-convergent net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$ with the w-limit point $\mu \in \mathscr{M}^+_\tau(\mathscr{E})$ is weakly tight.*

We omit the simple proof (see 2.3 and 2.4) and note that each topological space satisfying the hypothesis of 2.6 must be locally compact. It is a characteristic property of locally compact regular spaces that to each couple of sets $K \subset G$ where $K \in \mathscr{K}$ and $G \in \mathscr{G}$ we can find any $\dot{K} \in \mathscr{K}$ and $\dot{G} \in \mathscr{G}$ with $K \subset \dot{G} \subset \dot{K} \subset G$ (see [2] sec. 3.3 theorem 3.3.2). This makes us possible to prove several interesting propositions.

Theorem 2.7. *Let us suppose that $X, \mathscr{G}$ is a locally compact regular topological space and $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E})$. Then the following conditions are equivalent:*

i) *there exists a $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ such that $\mu_\alpha \to_w \mu$;*

ii) $\overline{\lim}\, \mu_\alpha X < \infty$, $\inf_K \sup_{\dot{K} \subset K^c} \overline{\lim}\, \mu_\alpha \dot{K} = 0$ *and*

$$\inf_{\dot{K}^c \supset K} \underline{\lim}\, \mu_\alpha \dot{K}^c = \inf_{\dot{K}^c \supset K} \overline{\lim} \mu_\alpha \dot{K}^c \quad \textit{for each} \quad K \in \mathscr{K}\,.$$

Proof. Suppose that i) holds. Since μ is a smooth function and $X, \mathscr{G}$ is due to the local compactness a completely regular space, the system $\mathscr{G}_\mu$ of all open sets G_μ with $\mu\, \partial G = 0$ (∂G_μ is the boundary of G_μ) forms a base for $\mathscr{G}$. Fix any $G_\mu \in \mathscr{G}_\mu$ and $\varepsilon > 0$. If we denote by $\{\mu_\alpha^0\}$ and μ^0 the restrictions of $\{\mu_\alpha\}$, μ from $\mathscr{E}$ to $\mathscr{E} \cap G_\mu = \{E_\mu : E_\mu = E \cap G_\mu,\ E \in \mathscr{E}\}$ we see, that G_μ can be considered as a locally compact topological space, μ^0, $\{\mu_\alpha^0\} \subset \mathscr{M}^+(\mathscr{E} \cap G_\mu, \mathscr{K}_\mu)$ where $\mathscr{K}_\mu \subset \mathscr{K}$ are closed compact subsets of G_μ and $\mu_\alpha^0 \to_w \mu^0$. By 2.6 there is a $K_\mu \in \mathscr{K}_\mu$ and α_0 with $\mu_\alpha^0 G_\mu < \mu_\alpha^0 K_\mu + \varepsilon$ for all $\alpha \geqq \alpha_0$.

Using this procedure, we can find to each $G_\mu \in \mathscr{G}_\mu$ a $K_\mu \in \mathscr{K}$ contained in G_μ with

$$\underline{\lim}\, \mu_\alpha G_\mu \leqq \underline{\lim}\, \mu_\alpha K_\mu + \varepsilon \leqq \sup_{K \subset G_\mu} \underline{\lim}\, \mu_\alpha K + \varepsilon\,.$$

Hence for each $G_\mu \in \mathscr{G}_\mu$ we can derive that

$$\mu G_\mu = \sup_{K \subset G_\mu} \underline{\lim}\, \mu_\alpha K = \sup_{K \subset G_\mu} \overline{\lim}\, \mu_\alpha K .$$

Since μ is τ-smooth and $\mathscr{G}_\mu$ forms a base for $\mathscr{G}$ it is easy to verify the analogous relation for each $G \in \mathscr{G}$. Particularly we obtain that

$$\mu \dot{K}^c = \sup_{K \subset \dot{K}^c} \underline{\lim}\, _{\mu\alpha} K = \sup_{K \subset \dot{K}^c} \overline{\lim}\, \mu_\alpha K \quad \text{for each} \quad \dot{K} \in \mathscr{K} .$$

Clearly this is equivalent with the third relation in ii). The remaining relations can be established now as in 1.1.

The proof of the reverse implication is analogous to that in 1.4. We have only to consider $\mathscr{K}$ instead of $\mathscr{F}$.

Theorem 2.8. *Let us suppose that $X, \mathscr{G}$ is a Hausdorff topological space. Then the following conditions are equivalent:*

i) *X is locally compact;*

ii) *every net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ which w-converges in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ satisfies the condition* (T) *from* 2.5.

The theorem 2.8 was established by F. Topsøe in [12], theorem 6.3.

Lemma 2.9. *Let $X, \mathscr{G}$ be a regular topological space and $[x]$ be the closure of the one-point set $\{x\}$, $x \in X$. Let Y consist of all sets of the form $[x]$, $x \in X$ and denote by f the mapping $x \to [x]$. Then*

i) *Y is a regular Hausdorff space in the quotient topology $\mathscr{G}'$ induced on Y by f and $\mathscr{G}$;*

ii) *f is a continuous, open and closed mapping;*

iii) *if $\{F_\beta\} \subset \mathscr{F}$ is a decreasing class then $F_\beta \downarrow \emptyset$ iff $f(F_\beta) \downarrow \emptyset$;*

iv) *$K \in \mathscr{K}$ iff $f(K) \in \mathscr{K}'$ ($\mathscr{K}'$ are all the compact subsets of Y);*

v) *the Borel σ-algebra of X is of the form $\mathscr{E} = \{E \subset X: E = f^{-1}(\dot{E}),\ \dot{E} \in \mathscr{E}'\}$ where $\mathscr{E}'$ is the Borel σ-algebra of Y;*

vi) *the spaces $\mathscr{M}^+(\mathscr{E}, \mathscr{F})$, $\mathscr{M}^+_\tau(\mathscr{E}, \mathscr{F})$ and $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ are homeomorphic with $\mathscr{M}^+(\mathscr{E}', \mathscr{F}')$, $\mathscr{M}^+_\tau(\mathscr{E}', \mathscr{F}')$, $\mathscr{M}^+(\mathscr{E}', \mathscr{K}')$, respectively,* wrt *the w-topology.*

We omit the proof (see [5]). Notice that $\mathscr{K}$ denotes the system of all closed compact subsets of X! From 2.5, 2.8 and 2.9 we can conclude

Theorem 2.10. *Let $X, \mathscr{G}$ be a regular or Hausdorff topological space. Then the following conditions are equivalent:*

i) *X is locally compact;*

ii) *every net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ which w-converges in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ is weakly tight.*

We shall say that the set $\mathscr{M} \subset \mathscr{M}^+(\mathscr{E})$ is *uniformly tight* wrt $\mathscr{K}$ if to each $\varepsilon > 0$ we can find any $K \in \mathscr{K}$ such that $\mu K^c < \varepsilon$ for each $\mu \in \mathscr{M}$. It is easy to see that every sequence $\{\mu_n\} \subset \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ that is weakly tight wrt $\mathcal{K}$ is uniformly tight wrt $\mathcal{K}$. Although by 2.10 the only type of regular topological spaces admitting that each w-convergent net in $\mathcal{M}^+(\mathcal{E}, \mathcal{K})$ is weakly tight wrt $\mathcal{K}$ is the locally compact one, there is a large class of spaces with the property that each sequence $\{\mu_n\} \subset \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ which is w-convergent in $\mathcal{M}^+(\mathcal{E}, \mathcal{K})$ is uniformly tight. Topsøe [12] sec. 7 obtained, by a simple modification of the proof from Le-Cam [6], theorem 4, this interesting result:

Theorem 2.11. *If $X, \mathcal{G}$ is such a regular or Hausdorff topological space that each $K \in \mathcal{K}$ has a countable base of neighbourhoods in $\mathcal{G}$ then each sequence $\{\mu_n\} \subset \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ which w-converges in $\mathcal{M}^+(\mathcal{E}, \mathcal{K})$ is uniformly tight* wrt $\mathcal{K}$.

Proof. See [12] sec. 7.

We note that $K \in \mathcal{K}$ has a countable base of neighbourhoods if there is a sequence $\{G_n\} \subset \mathcal{G}$, $G_1 \supseteq G_2 \supseteq \ldots \supseteq K$ such that if $G \in \mathcal{G}$ contains K then $G \supseteq G_n \supseteq K$ for all n starting from any n_0. 2.11 extends the classical Prohorov's result which was obtained for complete metric spaces.

$X, \mathcal{G}$ in 2.11 is of course first countable whence it is an image of any locally compact Hausdorff space in a quotient mapping (see [2] theorem 3.3.20). Thus it is natural to ask about some deeper relation between the local compactness and 2.11.

Remark 2.12. Each σ-compact subset F_σ of a regular T_1 space $X, \mathcal{G}$ is a completely regular (even a normal) Hausdorff space in the induced topology $\mathcal{G} \cap F_\sigma$. To each sequence $\{\mu_n\}_{0 \leqq n} \subset \mathcal{M}^+(\mathcal{E}, \mathcal{K})$, $\mu_n \to_w \mu_0$ we can find a σ-compact set $F_\sigma \subset X$ such that $\mu_n X = \mu_n F_\sigma$ for all $n = 0, 1, \ldots$. Hence $\{\mu_n\}_{0 \leqq n}$ can be identified with a sequence $\{\mu_n^0\}_{0 \leqq n} \subset \mathcal{M}^+(\mathcal{E} \cap F_\sigma, \mathcal{K}_\sigma)$, $\mu_n^0 \to_w \mu_0^0$ which is related to a completely regular (even a normal) Hausdorff space.

Let $X, \mathcal{G}$ be a completely regular Hausdorff space. $X, \mathcal{G}$ is called complete in the sense of Čech if it is a G_δ set in its Stone-Čech compactification βX. $X, \mathcal{G}$ is locally complete in the sense of Čech if each $x \in X$ has a neighbourhood which is G_δ in βX. It is well known that each locally compact Hausdorff space and each complete metric space is complete in the sense of Čech. If $X, \mathcal{G}$ is paracompact and locally complete in the sense of Čech then it is complete in the sense of Čech (see [2] sec. 5.5.8). Each G_δ subset of X can be considered as a countable intersection of locally compact topological spaces and this motivates the following considerations. We note that $\mathcal{G}^\wedge$, $\mathcal{F}^\wedge$, $\mathcal{K}^\wedge$ are the systems of all open, closed and closed compact subsets of βX. Next we identify X with a dense subset of βX.

Lemma 2.13. *Let $X, \mathcal{G}$ be a completely regular Hausdorff space. Then each $\mu \in \mathcal{M}^+(\mathcal{E}, \mathcal{K})$ has an extension $\hat{\mu} \in \mathcal{M}^+(\mathcal{E}^\wedge, \mathcal{K}^\wedge)$ where $\mathcal{E}^\wedge$ is the Borel σ-algebra of βX.*

Proof. If μ is in $\mathcal{M}^+(\mathcal{E}, \mathcal{K})$ then $\hat{\mu}$ defined by $\hat{\mu}\hat{E} = \mu\hat{E}X$ for each $\hat{E} \in \mathcal{E}^\wedge$ is a τ-smooth Borel measure. Since βX is a regular topological space $\hat{\mu}$ must be regular. Thus $\hat{\mu} \in \mathcal{M}^+(\mathcal{E}, \mathcal{K})$.

Lemma 2.14. *If $X, \mathscr{G}$ is complete in the sense of Čech and if $\{\mu_n\} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ is w-convergent in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ then $\{\mu_n\}$ is uniformly tight.*

Proof. Suppose that $\mu_n \to_w \mu_0$ where $\mu_0 \in \mathscr{M}^+(\mathscr{E}, \mathscr{K})$. Let $X = \bigcap \hat{G}_m$ where $\{\hat{G}_m\} \subset \mathscr{G}^\wedge$. If $\{\hat{\mu}_n\}_{0 \leqq n}$ are the extensions of $\{\mu_n\}_{0 \leqq n}$ from $\mathscr{E}$ to $\mathscr{E}^\wedge$ defined by 2.13 we see that $\hat{\mu}_n \to_w \hat{\mu}_0$. We can consider the restrictions of $\{\hat{\mu}_n\}_{0 \leqq n}$ from $\mathscr{E}^\wedge$ to $\mathscr{E}^\wedge \cap \hat{G}_m$, $m = 0, 1, \ldots$ as w-convergent sequences related to locally compact spaces $\hat{G}_m$, $\mathscr{G} \cap \cap \hat{G}_m$. By 2.6 we can find to a given $\varepsilon > 0$ $\{\hat{K}_m\} \subset \mathscr{K}^\wedge$ with $\hat{\mu}_n \hat{K}_m^c = \hat{\mu}_n \hat{G}_m - \hat{K}_m < < \varepsilon/2^m$ for all $n = 1, 2, \ldots$ and each $m = 1, 2, \ldots$. Since $K = \bigcap K_m \subset X$ is a compact set and $\mu K^c = \hat{\mu} \cup \hat{K}_m^c \leqq \sum \hat{\mu} \hat{K}_m^c < \varepsilon$ the proof is finished.

Theorem 2.15. *If $X, \mathscr{G}$ is locally complete in the sense of Čech and if $\{\mu_n\} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ is w-convergent in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ then $\{\mu_n\}$ is uniformly tight.*

Proof. Let Let $\mu_n \to_w \mu$ where $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{K})$, $X = \bigcup_{x \in X} G(x)$ where $G(x) \in \mathscr{G}$ is an open neighbourhood of $x \in X$ and $G(x)$ is G_δ in βX. Since μ is τ-smooth the system of all μ-continuity sets contains a base for $\mathscr{G}$. As each open subset $G \in \mathscr{G}$ of an open set which is G_δ in βX is G_δ again we can assume that each $G(x)$ is a μ-continuity set (i.e. $\mu \partial G(x) = 0$). Using 2.1 and the τ-smoothness of all $\{\mu_n\}$ we can find to a given $\varepsilon > 0$

$$\{x_1, \ldots, x_m\} \subset X \quad \text{with} \quad \mu_n X < \mu_n G_\varepsilon + \varepsilon \quad \text{where}$$
$$G_\varepsilon = G(x_1) \cup \ldots \cup G(x_m)\,.$$

For each $i \in \{1, \ldots, m\}$ the restrictions μ_n^i of μ_n from $\mathscr{E}$ to $\mathscr{E} \cap G(x_i)$ are w-converging to the restrictions μ^i of μ from $\mathscr{E}$ to $\mathscr{E} \cap G(x_i)$. By 2.14 we can find $\{K_1, \ldots, K_m\} \subset \mathscr{K}$ such that $\mu_n^i\, G(x_i) - K_i = \mu_n\, G(x_i) - K_i < \varepsilon/m$ for all $n = 1, 2, \ldots$, $K_i \subset G(x_i)$ for each $i = 1, 2, \ldots, m$. If we put $K_\varepsilon = K_1 \cup \ldots \cup K_m$ we see that

$$\mu_n G_\varepsilon - K_\varepsilon = \mu_n(\bigcup G(x_i)) (\bigcap K_i^c) \leqq \mu_n \cup (G(x_i) - K_i) \leqq$$
$$\leqq \mu_n\, G(x_i) - K_i \leqq \varepsilon$$

for all $n = 1, 2, \ldots$ and $\mu_n X < \mu_n K_\varepsilon + 2\varepsilon$ for all n which establishes the theorem.

We conclude this section by some facts about the relation between the relative compactness wrt the w-topology and the uniform tightness in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$.

Theorem 2.16. *Consider the space $(X, \mathscr{G}, \mathscr{F}, \mathscr{K})$ and a set $\mathscr{M} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ which is relatively compact* wrt *the w-topology. Then the conditions*

i) *$\mathscr{M}$ is uniformly tight* wrt $\mathscr{K}$;

ii) *there exists a uniformly tight subset $\mathscr{M}_0 \subset \mathscr{M}$ which is dense in $\mathscr{M}$;*

iii) *there is $\mathscr{M}_0 \subset \mathscr{M}$ which is dense in $\mathscr{M}$ and every net $\{\mu_\alpha\} \subset \mathscr{M}_0$ which is w-convergent in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ is weakly tight*

are equivalent. If in addition each net $\{\mu_\alpha\} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ with $\lim \mu_\alpha = \mu$, $\mu \in \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ *contains a countable subnet then* i) *is equivalent with*

iv) *each w-convergent sequence $\{\mu_n\} \subset \mathscr{M}$ is weakly uniformly tight* wrt $\mathscr{K}$.

Proof. i) $\Rightarrow$ ii) $\Rightarrow$ iii) are trivial. We prove iii) $\Rightarrow$ i). Assume for the purpose of an indirect proof that there is any $\varepsilon > 0$ such that to each $K \in \mathscr{K}$ we find any $\mu_K \in \mathscr{M}$ with $\mu_K K^c > \varepsilon$. Since $\mathscr{M}_0$ is dense in wrt *the w-topology* $\mathscr{M}$ to a given μ_K we can relate $\mu_K^0 \in \mathscr{M}_0$ such that $\mu_K K^c < \mu_K^0 K^c + \varepsilon/2$. Put $\varepsilon' = \varepsilon/2$. $\mathscr{K}$ forms a direction wrt the inclusion hence $\{\mu_K^0\}$ is a net and it ought to contain a subnet $\{\mu_K^0\}_{K \in \mathscr{K}'}$ $(\mathscr{K}' \subset \mathscr{K})$ which is w-convergent in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$. By the assumption $\{\mu_K^0\}_{\dot{K} \in \mathscr{K}'}$ must be weakly tight wrt $\mathscr{K}$. Thus to ε' we can find $K_0 \in \mathscr{K}$ and $\dot{K}_0 \subseteq \mathscr{K}'$ such that $\sup_{\dot{K} \supseteq \dot{K}_0} \mu_{\dot{K}}^0 K_0^c < \acute{\varepsilon}$. Since $\{\mu_K^0\}_{K' \in \mathscr{K}'}$ is a subnet of $\{\mu_K^0\}$ we can find a $\dot{K} \in \mathscr{K}'$ with $\dot{K} \supseteq \dot{K}_0 \cup K_0 \supseteq \dot{K}_0$ and

$$\mu_{\dot{K}}^0 \dot{K}^c \leqq \mu_{\dot{K}}^0 \dot{K}_0^c K_0^c \leqq \mu_{\dot{K}}^0 K_0^c < \acute{\varepsilon}\,.$$

At the same time we have by the selection of $\{\mu_K\}$ the relation $\mu_{\dot{K}}^0 \dot{K}^c > \acute{\varepsilon}$ and this is a contradiction. The proof of i) $\Leftrightarrow$ iv) is similar.

Corollary 2.17. *If $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ is metrizable then, under the assuptions of theorem 2.11, each compact set $\mathscr{M} \subset \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ is uniformly tight.*

Theorem 2.18. *If X is a topological space which is locally complete in the sense of Čech and if $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ is provided with the w-topology then each relatively compact subset $\mathscr{M} \subseteq \mathscr{M}^+(\mathscr{E}, \mathscr{K})$ is uniformly tight* wrt $\mathscr{K}$.

Proof. If X is locally compact then 2.17 follows from 2.16 and 2.10. If X is complete in the sense of Čech, particularly if X is a complete metric space, then 2.17 can be proved analogically to 2.14. We can prove the general case by similar arguments as 2.15 using the fact that if $G \in \mathscr{G}$ is G_δ in βX and $\mathscr{M}$ is conditionally compact in $\mathscr{M}^+(\mathscr{E}, \mathscr{K})$ then for each $F \subset G$ $\mathscr{M}_0 \subset \mathscr{M}^+(\mathscr{E} \cap F, \mathscr{K}_0)$ which consists of the restrictions of $\mu \in \mathscr{M}$ from $\mathscr{E}$ to $\mathscr{E} \cap F$ is conditionally compact in $\mathscr{M}^+(\mathscr{E} \cap F, \mathscr{K}_0)$ and F is G_δ in βX.

The theorem 2.17 was established by J. Hoffmann-Jørgensen [4], theorem 4. Our foregoing considerations were based on the fact that each $x \in X$ has a neighbourhood $G \in \mathscr{G}$ such that each net (sequence) which is w-convergent in $\mathscr{M}^+(\mathscr{E} \cap \cap G, \mathscr{K}_0)$ ($\mathscr{K}_0$ are the closed compact subsets of G) is weakly tight. If we are working with nets the local compactness of G (whence of X) in the $\mathscr{G} \cap G$ topology cannot be droped. However if we work with sequences or relatively compact sets, the local compactness is not necessary and we can consider instead of the neighbourhoods which are G_δ in X those with the property that, say, each sequence from $\mathscr{M}^+(\mathscr{E} \cap G, \mathscr{K}_0)$ is weakly uniformly tight. More about this idea can be found in [4].

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