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LIFTS OF 1-FORMS TO THE TANGENT BUNDLE OF HIGHER ORDER

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0. INTRODUCTION

Let M be a manifold. We denote by T^rM the tangent bundle of order r. A mapping

$$\mathscr{L}: \mathscr{X}^*(M) \to \mathscr{X}^*(T^rM)$$

where $\mathscr{X}^*(M)$ and $\mathscr{X}^*(T^rM)$ denote the modules of 1-forms on M and T^rM respectively, is called a *lift of 1-forms from M to T^rM* if the following conditions hold: (i) $\mathscr{L}$ is linear over $\boldsymbol{R}$, (ii) $\mathscr{L}$ is local, (iv) $\mathscr{L}$ is natural, and (ix) $\mathscr{L}$ is regular.

All the λ-lifts defined by A, Morimoto [4], [5], [6] are lifts in the sense of the proposed definition. In this paper we shall define a new lift of 1-forms from M to T^rM called *the $\square$-lift*. The main theorem of this paper says that if $\dim M \geqq 2$, then every lift $\mathscr{L}$ of 1-forms from M to T^rM is a linear combination (with constant coefficients) of the $\square$-lift and the λ-lifts for $\lambda = 0, \ldots, r$. (If $\dim M = 1$, then every lift $\mathscr{L}$ of 1-forms from M to T^rM is a linear combination (with constant coefficients) of the λ-lifts for $\lambda = 0, \ldots, r$.)

In this paper the differentiability means always the differentiability of class C^∞.

1. PRELIMINARIES

Let M be a manifold of dimension n and r be a natural number. Denote by $T^rM = J^r_0(\boldsymbol{R}, M)$ the set of r-jets at 0 of mappings $\boldsymbol{R} \to M$. This bundle is called the *tangent bundle of order* r. We denote by π: $T^rM \to M$ the bundle projection defined by

$$\pi(j^r_0\gamma) = \gamma(0) .$$

If φ: $M \to N$ is a differentiable mapping, then the induced mapping $T^r\varphi$: $T^rM \to T^rN$ is defined by the formula;

$$T^r\varphi(j^r_0\gamma) = j^r_0(\varphi \circ \gamma)$$

If (U, x^i) is a chart on M, then the induced chart $(\pi^{-1}(U), x^{i,\nu})$ is given by

$$(1.1) \qquad x^{i,\nu}(j^r_0\gamma) = \frac{1}{\nu!} D_\nu(x^i \circ \gamma)(0)$$

for $i = 1, \ldots, n = \dim M$ and $\nu = 0, \ldots, r$.

If f is a differentiable function on M, then for $\lambda = 0, \ldots, r$ the λ-lift of f to the bundle T^rM is the differentiable function $f^{(\lambda)}$ on T^rM defined by the formula:

$$(1.2) \qquad f^{(\lambda)}(j_0^r\gamma) = \frac{1}{\lambda!} D_\lambda(f \circ \gamma)(0).$$

The λ-lifts of functions have the the following properties:

$$(1.3) \qquad (af + bg)^{(\lambda)} = af^{(\lambda)} + bg^{(\lambda)},$$

$$(1.4) \qquad (fg)^{(\lambda)} = \sum_{v=1}^{\lambda} f^{(v)} g^{(\lambda - v)}.$$

Formulas (1.1) and (1.2) imply immediately

$$(1.5) \qquad x^{i,v} = (x^i)^{(v)}.$$

The family of functions $f^{(\lambda)}$, where f is a function on M and $\lambda = 0, \ldots, r$ is an important family of functions on T^rM because we have the following proposition:

Proposition 1.1. *If V and W are vector fields on T^rM such that*

$$V(f^{(\lambda)}) = W(f^{(\lambda)})$$

for every functions f on M and $\lambda = 0, \ldots, r$, then $V = W$.

The proof is an easy verification (see [1] or [3]).

From Proposition 1.1 we can obtain:

Proposition 1.2. (A. Morimoto [4], [6]) *If X is a vector field on M and $\lambda = 0, \ldots, r$, then there exists one and only one vector field $X^{(\lambda)}$ on T^rM such that for every function f on M and every $\mu = 0, \ldots, r$ we have*

$$(1.6) \qquad X^{(\lambda)}(f^{(\mu)}) = (Xf)^{(\lambda + \mu - r)}.$$

The vector field $X^{(\lambda)}$ on T^rM is called *the λ-lift of X from M* to T^rM. The r-lift is called the *complete lift of vector field* and we will write X^C instead of $X^{(r)}$. The family of vector fields X^C, where X is a vector field on M, is important because we have (see [3]):

Proposition 1.3. *If t and t' are differentiable fields of tensors of type (ε, p) on T^rM, where $\varepsilon = 0, 1$, such that for every vector fields $X_1, \ldots, X_p$ on M we have*

$$t(X_1^C, \ldots, X_p^C) = t'(X_1^C, \ldots, X_p^C)$$

then $t = t'$.

Now we can define the λ-lift of 1-forms from M to the tangent bundle T^rM. Namely we have:

Proposition 1.4. (Morimoto [4], [6]) *If ω is an 1-form on M and $\lambda = 0, \ldots, r$, then there exists one and only one 1-form $\omega^{(\lambda)}$ on T^rM such that for each vector field X on M and each $\mu = 0, \ldots, r$ the following formula holds*

$$(1.7) \qquad \omega^{(\lambda)}(X^{(\mu)}) = (\omega X)^{(\lambda + \mu - r)}.$$

Observe that according to Proposition 1.3 formula (1.7) determines uniquely the λ-lift of an 1-form ω. By using Propositions 1.2 and 1.4 it is not difficult to show the following formulas (see [4], [6]):

$$(1.8) \qquad (fX)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} X^{(\lambda-\mu)} ,$$

$$(1.9) \qquad (f\omega)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} \omega^{(\lambda-\mu)} .$$

If (U, x^i) is a chart on M, then we have

$$(1.10) \qquad \frac{\partial}{\partial x^{i,\nu}} = \left(\frac{\partial}{\partial x^i}\right)^{(r-\nu)} ,$$

$$(1.11) \qquad \mathrm{d}x^{i,\lambda} = (\mathrm{d}x^i)^{(\nu)} .$$

If $\omega = \omega_i \, \mathrm{d}x^i$, then

$$(1.11) \qquad \omega^{(\lambda)} = \sum_{i=0}^{n} \sum_{\nu=0}^{\lambda} \omega_i^{(\nu)} \, \mathrm{d}x^{i,\lambda-\nu} .$$

2. LIFTS OF 1-FORMS

We propose the following definition of lifts of 1-forms from M to the tangent bunde T^rM of order r.

Definition 2.1. We denote by $\mathscr{X}^*(M)$ and $\mathscr{X}^*(T^rM)$ the modules of 1-forms on M and T^rM respectively. A mapping $\mathscr{L}\colon \mathscr{X}^*(M) \to \mathscr{X}^*(T^rM)$ is called a *lift of 1-forms from M to T^rM if the following conditions hold*:

(a) $\mathscr{L}$ is linear over $\boldsymbol{R}$, that is, for every 1-forms ω, ω' on M and every real numbers a, b

$$\mathscr{L}(a\omega + b\omega') = a\, \mathscr{L}(\omega) + b\, \mathscr{L}(\omega') .$$

(b) $\mathscr{L}$ is local, that is, if U is an open subset of M and ω, ω' are 1-forms on M such that $\omega \mid U = \omega' \mid U$, then

$$\mathscr{L}\omega \mid \pi^{-1}(U) = \mathscr{L}\omega' \mid \pi^{-1}(U) .$$

(c) $\mathscr{L}$ is natural, that is, if $\varphi\colon U \to V$ is a diffeomorphism of open subsets U, V of M and ω is an 1-form on M, then

$$\mathscr{L}(\varphi^*\omega) = (T^r\varphi)^* (\mathscr{L}\omega)$$

where * denotes the pull-back of an 1-form by a diffeomorphism.

(d) $\mathscr{L}$ is regular, that is, if K is an open subset of $\boldsymbol{R}^k$ and

$$\omega\colon K \times M \to T^*M$$

is a differentiable mapping such that for every $t \in K$ ω_t is an 1-form on M, then the mapping

$$K \times T^rM \ni (t, p) \to (\mathscr{L}\omega_t)(p) \in T^*(T^rM)$$

is differentiable.

Now we have the following theorem:

Theorem 2.2. *For every* $\lambda = 0, \ldots, r$ *the mapping*

$$\mathscr{X}^*(M) \ni \omega \to \omega^{(\lambda)} \in \mathscr{X}^*(T^r M)$$

is a lift of 1*-forms from* M *to* $T^r M$.

Proof. The conditions (a), (b) and (d) are evident. We need only to show that the above mapping is natural. In order to do this, let ω be an 1-form on M and $\varphi: U \to V$ be a diffeomorphism of two open subsets of M. Now by using (1.7) and the formula

$$(\varphi^* \omega)(X) = \omega(\varphi_* X) \circ \varphi$$

where X is a vector field on M and $\varphi_* X$ is the image of X by φ, we obtain

$$\begin{aligned} (\varphi^*\omega)^{(\lambda)}(X^C) &= ((\varphi^*\omega)(X))^{(\lambda)} = ((\omega(\varphi_* X) \circ \varphi)^{(\lambda)} = \\ &= (\omega(\varphi_* X))^{(\lambda)} \circ T^r\varphi = \omega^{(\lambda)}(\varphi_* X)^C \circ T^r\varphi = \omega^{(\lambda)}((T^r\varphi)_* X^C) \circ T^r\varphi = \\ &= ((T^r\varphi)^* \omega^{(\lambda)})(X^C). \end{aligned}$$

In the above calculation we have used two facts proved in [1], namely, we have used that for every function f on M and every vector field X on M two formulas hold

$$(f \circ \varphi)^{(\lambda)} = f^{(\lambda)} \circ T^r\varphi, \quad (\varphi_* X)^C = (T^r\varphi)_* X^C.$$

The proof of Theorem 2.2 is now finished.

Now we shall define a new lift of 1-forms. For any $r \geqq 1$ we consider the projection

$$\pi_1^r: T^r M \ni j_0^r \gamma \to j_0^1 \gamma = \gamma(0) \in TM.$$

Let ω be an 1-form on M. We consider the vertical lift ω^V of ω to the tangent bundle TM, $\omega^V: TM \to \boldsymbol{R}$ is the differentiable function on TM given by $\omega^V(v) = \omega_{\pi(v)}(v)$. Immediately from the definition of ω^V we obtain

$$(\varphi^*\omega)^V = \omega^V \circ \mathrm{d}\varphi.$$

We define

$$\omega^\square = \mathrm{d}(\omega^V \circ \pi_1^r) \tag{2.1}$$

$\omega^\square$ is an 1-form on $T^r M$ and it is called *the* $\square$*-lift of* ω *from* M *to* $T^r M$. This definition implies immediately the following formula

$$\omega^\square = \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} x^{i,1} \frac{\partial w_i}{\partial x^j} \mathrm{d}x^{j,0} + \omega_j \mathrm{d}x^{j,1} \right\}. \tag{2.2}$$

From (1.11) and (2.2) we obtain that if ω is a closed 1-form on M, then $\omega^\square = \omega^{(1)}$.

Now we prove:

Theorem 2.3. *The mapping* $(\ )^\square: \mathscr{X}^*(M) \ni \omega \to w^\square \in \mathscr{X}^*(T^r M)$ *is a lift of* 1*-forms from* M *to* $T^r M$.

Proof. The conditions (a), (b) and (d) of Definition 2.1 are evident. To show the condition (c), let ω be an 1-form on M and $\varphi: U \to V$ be a diffeomorphism of open subsets of M. Now, the formula $(\varphi^*\omega)^V = \omega^V \circ d\varphi$ implies

$$(\varphi^*\omega)^{\square} = d((\varphi^*\omega)^V \circ \pi_1^r) = d(\omega^V \circ d\varphi \circ \pi_1^r) =$$
$$= d(\omega^V \circ \pi_1^r \circ T^r\varphi) = (T^r\varphi)^* (d(\omega^V \circ \pi_1^r)) = (T^r\varphi)^* (\omega^{\square})$$

because $d\varphi \circ \pi_1^r = \pi_1^r \circ T^r\varphi$. The proof is now complete.

From (2.1) we obtain immediately:

Corollary 2.4. *For any* 1*-form* ω *on* $M \omega^{\square}$ *is a closed* 1*-form on* T^rM.

From (1.7) and (1.11) we have immediately:

Corollary 2.5. *If* ω *is a closed* 1*-form on* M, *then for every* $\lambda = 0, \ldots, r$, $\omega^{(\lambda)}$ *is a closed* 1*-form on* T^rM.

To end this paragraph we formulate the following proposition which it is easy to verify.

Proposition 2.6. *If* $\mathscr{L}$ *is a lifting of* 1*-forms from* M *to* T^rM, *then for any* 1*-form* ω *and any vector field* X *on* M *we have*

$$\mathscr{L}(L_X\omega) = L_{X^C}(\mathscr{L}\omega).$$

3. THE MAIN THEOREM

In this section we formulate and we prove the main theorem of this paper.

Theorem 3.1. *Let* M *be a manifold such that* $\dim M \geqq 2$. *If* $\mathscr{L}$ *is a lift of* 1*-forms from* M *to the tangent bundle of order* r, *then* $\mathscr{L}$ *is a linear combination with constant real coefficients of* $\square$*-lift and* λ*-lifts for* $\lambda = 0, \ldots, r$, *that is, there exist real numbers* $a_{\square}, a_0, \ldots, a_r$ *such that for every* 1*-form* ω *on* M *the following formula holds*:

$$\mathscr{L}\omega = \sum_{\mu=0}^{\lambda} a_{(\mu)}\omega^{(\mu)} + a_{\square}\omega^{\square}. \tag{3.1}$$

The proof of this theorem is based on the following sequence of lemmas and propositions. At first, we formulate an elementary lemma which is an immediate consequence of Euleur's identity.

Lemma 3.2. *Let* $f: \boldsymbol{R}^q \to \boldsymbol{R}$ *be a differentiable function.*

(i) *If* f *satisfies the condition*

$$\sum_{j=1}^{q} v^j \frac{\partial f}{\partial v^j} = 0$$

then f *is constant.*

(ii) *If f satisfies the condition*

$$\sum_{j=1}^{q} v^j \frac{\partial f}{\partial v^j} + f = 0$$

then f is identically zero on $\boldsymbol{R}^q$.

Lemma 3.3. *Let (U, x^i) be a chart on M and x_0 be a point of U. If ω is a closed 1-form on M, then there exists a vector field X on M such that*

$$(3.2) \qquad \omega = L_X(\mathrm{d}x^1)$$

in some neighborhood of x_0.

Proof. Let $\omega = \omega_i \, \mathrm{d}x^i$. Since ω is closed, thus $\partial_i \omega_j = \partial_j \omega_i$ for $i, j = 1, \ldots, n$. If $X = X^i(\partial/\partial x^i)$, then the condition (3.2) is equivalent to the following one:

$$(3.3) \qquad \omega_j = \partial X^1 / \partial x^j \, .$$

Now we define $X^j = 0$ for $j \neq 1$ and X^1 is a solution of (3.3). The condition $\partial_j \omega_i = \partial_i \omega_j$ implies the existence of solutions of (3.3).

Proposition 3.4. *If $\mathscr{L}$ is a lift of 1-forms from M to the tangent bundle of order r, then there exist real numbers $c_0, \ldots, c_r$ such that for every closed 1-form ω on M we have*

$$\mathscr{L}\omega = \sum_{\mu=0}^{r} c_\mu \omega^{(\mu)} \, .$$

Proof. Let (U, x^i) be a chart on M. Then there exist differentiable functions $c_{j,\mu}$ ($j = 1, \ldots, n$, $\mu = 0, \ldots, r$) defined on $T^rM \mid U$ such that

$$(3.4) \qquad \mathscr{L}(\mathrm{d}x^1) = \sum_{j=1}^{n} \sum_{\mu=0}^{r} c_{j,\mu} \, \mathrm{d}x^{j,\mu} = \sum_{j=1}^{n} \sum_{\mu=0}^{r} c_{j,\mu} (\mathrm{d}x^j)^{(\mu)} \, .$$

Using Proposition 2.5 from the formula

$$\delta_q^1 \, \mathrm{d}x^k = L_{x^k \partial/\partial x^q} \, \mathrm{d}x^1$$

we obtain

$$\delta_q^1 \mathscr{L}(\mathrm{d}x^k) = L_{(x^k \partial/\partial x^q)^C} \, \mathscr{L}(\mathrm{d}x^1) \, .$$

Next according to Lemma 1.4 from [6] we obtain

$$(x^k \, \partial/\partial x^q)^C = \sum_{\nu=0}^{r} x^{k,\nu} \, \partial/\partial x^{q,\nu} \, .$$

By (3.4) it implies

$$(3.5) \qquad \delta_q^1 \mathscr{L}(\mathrm{d}x^k) = \sum_{j=1}^{n} \sum_{\mu=0}^{r} \left(\sum_{\nu=0}^{r} x^{k,\nu} \frac{\partial c_{j,\mu}}{\partial x^{q,\nu}} + \delta_j^k c_{q,\mu} \right) \mathrm{d}x^{j,\mu} \, .$$

For $k = 1$, from (3.4) and (3.5) we have

$$(3.6) \qquad \delta_q^1 c_{j,\mu} = \sum_{\nu=0}^{r} x^{1,\nu} \frac{\partial c_{j,\mu}}{\partial x^{q,\nu}} + \delta_j^1 c_{q,\mu} \, .$$

If we set $k = q + 1$ and $j = 1$ in (3.5) then

$$(3.7) \qquad \sum_{v=0}^{r} x^{q,v} \frac{\partial c_{1,\mu}}{\partial x^{q,v}} = 0 .$$

On the other hand, if we set $q = 1$ and $j = 1$ in (3.6), then we obtain

$$(3.8) \qquad \sum_{v=0}^{r} x^{1,v} \frac{\partial c_{1,\mu}}{\partial x^{1,v}} = 0 .$$

Now (3.7) and (3.8) imply

$$\sum_{q=1}^{n} \sum_{v=0}^{r} x^{q,v} \frac{\partial c_{1,\mu}}{\partial x^{q,v}} = 0 .$$

According to Lemma 3.2 it implies $c_{1,\mu}$ is constant.

Next we observe that from (3.6) for $j = 1$ and $q \neq 1$ we obtain

$$c_{q,\mu} = - \sum_{v=0}^{r} x^{1,v} \frac{\partial c_{1,\mu}}{\partial x^{1,v}} = 0 .$$

We set $c_v = c_{1,v}$. Now from (2.5) we have

$$(3.9) \qquad \mathscr{L}(\mathrm{d}x^k) = \sum_{\mu=0}^{r} c_\mu \, \mathrm{d}x^{1,\mu} .$$

If X is a vector field on U, then Proposition 2.6 and formula (3.9) imply

$$\mathscr{L}(L_X \, \mathrm{d}x^1) = L_{X^c}(\mathscr{L}(\mathrm{d}x^1)) = L_{X^c}\left(\sum_{\mu=0}^{r} c_\mu \, \mathrm{d}x^{1,\mu}\right) =$$
$$= \sum_{\mu=0}^{r} c_\mu L_{X^c}(\mathrm{d}x^{1,\mu}) = \sum_{\mu=0}^{r} c_\mu (L_X \, \mathrm{d}x^1)^{(\mu)} .$$

According to Lemma 3.3 every 1-form ω on M can be written in the form $\omega = L_X(\mathrm{d}x^1)$, thus from the last formula we obtain Proposition 3.4. The proof is finished.

From Proposition 3.4 and corollary 2.5 we obtain:

Corollary 3.5. *Let $\mathscr{L}$ be a lift of 1-forms from M to T^rM. If ω is a closed 1-form on M, then $\mathscr{L}\omega$ is closed.*

If $\dim M = 1$, then every 1-form on M is closed. Thus from Proposition 3.4 we obtain

Theorem 3.6. *Let M be a manifold such that* $\dim M = 1$. *If $\mathscr{L}$ is a lift of 1-forms from M to T^rM, then $\mathscr{L}$ is a linear combination with real coefficients of the λ-lifts for $\lambda = 0, \ldots, r$.*

Next we prove the following proposition:

Proposition 3.7. *Let* $\dim M \geqq 2$. *If $\mathscr{L}^\sim$ is a lift of 1-forms from M to T^rM such that $\mathscr{L}^\sim(\omega) = 0$ for every closed 1-form ω, then there exists a real number b*

such that

$$(3.10)\qquad \mathscr{L}^{\sim}(\omega) = b(\omega^{\square} - \omega^{(1)})$$

for every 1*-form* ω *on* M, *where* $\omega^{\square}$ *is defined by* (2.1).

Proof. Let (U, x^i) be a chart on M. At first, we shall show the formula (3.10) for $w_0 = x^2\, dx^1$ (ω_0 is well-defined because $\dim M \geqq 2$).

There exist diferentiable functions $c_{i,v}$ such that

$$(3.11)\qquad \mathscr{L}(\omega_0) = \sum_{j=1}^{n} \sum_{v=0}^{r} c_{j,\lambda}\, dx^{j,v}.$$

Let $X = \partial_q = \partial/\partial x^q$. Since $L_X\omega_0$ is closed, thus from Proposition 2.6 we obtain

$$0 = \mathscr{L}(L_{\partial_q}\omega_0) = L_{\partial_q c}(\mathscr{L}\omega_0) = \sum_{j=1}^{n} \sum_{v=0}^{r} \frac{\partial c_{j,v}}{\partial x^{q,0}}\, dx^{j,v}$$

because $\partial_q^C = \partial_{q,0} = \partial/\partial x^{q,0}$. The last formula implies that $\partial c_{j,v}/\partial x^{q,0} = 0$, that is,

$$(3.12)\qquad c_{j,v} \text{ does not depend of } x^{q,0}.$$

Let $Y = x^k\, \partial_q$. Now for any k and $q \geqq 3$ the 1-form $L_Y\omega_0$ is closed, and from Proposition 2.6, we obtain

$$L_{Yc}(\mathscr{L}\omega_0) = \mathscr{L}(L_Y\omega_0) = 0.$$

It means that for $q \geqq 3$ and $k = 1, \ldots, n$ we have

$$\sum_{\lambda=0}^{r} x^{k,\lambda} \frac{\partial c_{i,v}}{\partial x^{q,\lambda}} + \delta_i^k c_{q,v} = 0.$$

Setting $k = q$ and summing with respect to k from 3 to n we obtain

$$\sum_{k=3}^{n} \sum_{\lambda=0}^{r} x^{k,\lambda} \frac{\partial c_{i,v}}{\partial x^{k,\lambda}} + c_{i,v} = 0 \quad \text{for} \quad i \geqq 3,$$

$$\sum_{k=3}^{n} \sum_{\lambda=0}^{r} x^{k,\lambda} \frac{\partial c_{i,v}}{\partial x^{k,\lambda}} = 0 \quad \text{for} \quad i \leqq 2.$$

According to Lemma 3.2 it implies

$$(3.13)\qquad c_{i,v} = 0 \quad \text{for} \quad i \geqq 3,$$

$$(3.14)\qquad c_{1,v} \text{ and } c_{2,v} \text{ do not depend of } x^{k,\lambda} \text{ for } k \geqq 3.$$

From (3.11)–(3.14) we obtain

$$(3.15)\qquad \mathscr{L}\omega_0 = \sum_{v=0}^{r} \{c_{1,v}\, dx^{1,v} + c_{2,v}\, dx^{2,v}\},$$

$$c_{1,v} = c_{1,v}(x^{1,1}, \ldots, x^{1,r}, x^{2,1}, \ldots, x^{2,r}),$$

$$c_{2,v} = c_{2,v}(x^{1,1}, \ldots, x^{1,r}, x^{2,1}, \ldots, x^{2,r}).$$

Next by using $V = (x^1)^k\, \partial_2$ we observe that $L_V\omega_0$ is closed for any natural number k, and hence, by Proposition 2.5 we have

$$\sum_{\lambda=1}^{r} \sum_{\lambda_1+\ldots+\lambda_k=\lambda} x^{1,\lambda_1} \ldots x^{1,\lambda_k} \frac{\partial c_{i,v}}{\partial x^{2,\lambda}} +$$

$$+ \sum_{\lambda=1}^{r} \sum_{\lambda_1+\ldots+\lambda_{k-1}=\lambda-v} x^{1,\lambda_1} \ldots x^{1,\lambda_{k-1}} \delta_i^1 \frac{\partial c_{2,v}}{\partial x^{2,\lambda}} = 0$$

for $i = 1, 2$. If $i = 2$, then we obtain

$$\sum_{\lambda=1}^{r} \sum_{\lambda_1+\ldots+\lambda_k=\lambda} x^{1,\lambda_1} \ldots x^{1,\lambda_k} \frac{\partial c_{2,v}}{\partial x^{2,\lambda}} = 0 . \tag{3.17}$$

We fix v and we consider the system of linear equations (3.17) for $k = 1, \ldots, r$. where $\xi_\lambda = \partial c_{2,v}/\partial x^{2,\lambda}$ are unknown. Since the determinant W of this system is a polynomial of r variables $x^{1,1}, \ldots, x^{1,r}$, thus $W \neq 0$ on some dense and open subset $\mathscr{U}$ of R^r. Now $\xi_\lambda = 0$ on $\mathscr{U}$. Since $c_{2,v}$ is a differentiable functions of $x^{1,1}, \ldots, x^{1,r}$, thus

$$\xi_\lambda = \frac{\partial c_{2,v}}{\partial x^{2,\lambda}} = 0 .$$

It means that $c_{2,v}$ does not depend of $x^{2,1}, \ldots, x^{2,r}$. We can write

$$c_{2,v} = c_{2,v}(x^{1,1}, \ldots, x^{1,r}) . \tag{3.18}$$

Analogously, if we will use $V = (x^2)^k \partial_1$ instead of $(x^1)^k \partial_2$, then we obtain that $c_{1,v}$ does not depend of $x^{1,1}, \ldots, x^{1,r}$, that is,

$$c_{1,v} = c_{1,v}(x^{2,1}, \ldots, x^{2,r}) . \tag{3.19}$$

Now from (3.16) for $k = 1$ and $i = 1$ we obtain

$$\sum_{\lambda=1}^{r} x^{1,\lambda} \frac{\partial c_{1,v}}{\partial x^{2,\lambda}} + c_{2,v} = 0 . \tag{3.20}$$

According to (3.18) and (3.19) the last equality implies that $\partial c_{1,v}/\partial x^{2,\lambda}$ is constant, that is, $c_{1,v}$ is a linear function of $x^{2,1}, \ldots, x^{2,r}$. Hence there exist real numbers $a_{\lambda v}$ such that

$$c_{1,v} = \sum_{\lambda=1}^{r} a_{\lambda,v} x^{2,\lambda} \tag{3.21}$$

for $v = 0, \ldots, r$. From (3.20) and (3.21) we have also

$$c_{2,v} = -\sum_{\lambda=1}^{r} a_{\lambda,v} x^{1,\lambda} . \tag{3.22}$$

Now from (3.16) for $k = 2$ and $i = 1$ we obtain

$$\sum_{\lambda=1}^{r} \sum_{\mu=0}^{r} x^{1,\mu} x^{1,\lambda-\mu} a_{\lambda,v} = \sum_{\lambda=0}^{r} \sum_{\mu=1}^{r} x^{1,\lambda} x^{1,\mu} a_{\mu,\lambda+v} .$$

By the comparing coefficients of polynomes from the last equality we obtain $a_{\lambda,v} = 0$

provided $\lambda \neq 1$ or $\nu \neq 0$. We set $b = a_{1,0}$. Now from (3.15), (3.21) and (3.22) we obtain

$$\mathscr{L}(\omega_0) = b(x^{2,1}\,\mathrm{d}x^{1,0} - x^{1,1}\,\mathrm{d}x^{2,0})\,. \tag{3.23}$$

According to (1.11) and (2.2) we have

$$\omega_0^{(1)} = x^{2,0}\,\mathrm{d}x^{1,1} + x^{2,1}\,\mathrm{d}x^{1,0}\,,$$

$$\omega_0^{\sim} = x^{1,1}\,\mathrm{d}x^{2,0} + x^{2,0}\,\mathrm{d}x^{1,1}\,.$$

Now formula (3.23) implies

$$\mathscr{L}(\omega_0) = b(\omega_0^{(1)} - \omega_0^{\square})\,. \tag{3.24}$$

By using a local diffeomorphism φ which permutates coordinates, by the naturality condition (see Definition 2.1), from (3.24) we obtain for any numbers $i \neq j$

$$\mathscr{L}(x^i\,\mathrm{d}x^j) = b((x^i\,\mathrm{d}x^j)^{(1)} - (x^i\,\mathrm{d}x^j)^{\square})\,. \tag{3.25}$$

Now for an 1-form $\omega = f\,\mathrm{d}x^i$, where f is any function, we have

$$\omega = L_{(f\partial/\partial x^j)}x^j\,\mathrm{d}x^i\,, \quad j \neq i$$

and by Proposition 2.6 from (3.25) we obtain the formula (3.10) for $\omega = f\,\mathrm{d}x^i$. By the linearity of $\mathscr{L}$, the formula (3.10) holds for every 1-form ω, and the proof of our proposition is finished.

Proof of Theorem 3.1. Let $\mathscr{L}$ be a lift of 1-forms and let $c_0, \ldots, c_r$ be the real numbers as in Proposition 3.3. It means that for every closed 1-form ω the following equality holds:

$$\mathscr{L}\omega = \sum_{\mu=0}^{r} c_\mu \omega^{(\mu)}\,.$$

Now we define a new lift of 1-forms $\mathscr{L}'\colon \mathscr{X}^*(M) \to \mathscr{X}^*(T^rM)$ setting

$$\mathscr{L}'\omega = \mathscr{L}\omega - \sum_{\mu=0}^{r} c_\mu \omega^{(\mu)}$$

$\mathscr{L}'$ satisfies the assumption of Proposition 3.5. Thus, there exists a real number b such that

$$\mathscr{L}'\omega = b(\omega^{\square} - \omega^{(1)}) = \mathscr{L}\omega - \sum_{\mu=0}^{r} c_m \omega^{(\mu)}\,.$$

From the last formula we obtain Theorem 3.1 if we set

$$a_{\square} = b\,, \quad a_1 = c_1 - b\,, \quad a_i = c_i \quad \text{for} \quad i \neq 1\,.$$

The proof of Theorem 3.1 is now finished.

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