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BIREGULAR AND UNIFORM IDENTITIES OF ALGEBRAS

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In this paper we consider algebras of a given type $\tau: F \to N$ where F is a set of fundamental operation symbpls, N is the set of positive integers and F is finite.

An identity $\varphi = \psi$ of type τ is called regular (see [12]) if the sets of variables occuring in φ and ψ coincide. For a term φ we denote by $F(\varphi)$ the set of all fundamental operation symbols occuring in φ.

In [14] biregular and uniform identities were defined, namely: an identity $\varphi = \psi$ of type τ is called biregular if it is regular and $F(\varphi) = F\psi)$.

An identity $\varphi = \psi$ is called uniform if $F(\varphi) = F(\psi) = F$ or $F(\varphi) = F(\psi) \neq F$ and $\varphi = \psi$ is regular.

For a variety K of type τ we denote by $\mathrm{Id}(K)$ the set of all identities of type τ satisfied in K.

We denote by K_R, K_B and K_U the varieties of type τ defined by all regular, all biregular and all uniform identities from $\mathrm{Id}(K)$, respectively.

In this paper we give some representation theorems of algebras from K_B and K_U. Our results are of the form: If there exist unary terms $q_1(x), q_2(x), \ldots, q_s(x)$ satisfying some conditions then an algebra $\mathfrak{A}$ belongs to K_U(to K_B) iff $\mathfrak{A}$ is isomorphic to a subdirect product of some algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_r$ where $\mathfrak{A}_1$ belongs to K (to K_R) and the structures of $\mathfrak{A}_2, \ldots, \mathfrak{A}_r$ are also described.

In each case we construct an equational base of K_U and K_B by means of an equational base of K and K_R (see theorems 1 – 4, sections 1 – 3).

In section 4 we define a construction called the absorbing sum of a semilattice ordered system (which is useful in section 5, where we apply our results to important varieties of groups, lattices and Boolean algebras).

In section 6 we give an example of a variety K with three unary fundamental operation symbols which is defined by one identity and such that K_B is not finitely based.

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0. PRELIMINARIES

Let $\tau: F \to N$ be a type of algebras, where F is the set of all fundamental operation symbols and N is the set of positive integers (see [3]). If K is a variety of algebras of type τ we denote by $\mathrm{Id}(K)$ the set of all identities of type τ satisfied in K. If E is a set of identities of type τ we denote by $C(E)$ the set of all identities provable from E by means of Birkhoff's derivation rules (see [3], p. 93) and we denote by $V(E)$ the variety of type τ defined by E. If φ is a term of type τ we denote by $\mathrm{Var}(\varphi)$ the set of all variables occuring in φ and by $F(\varphi)$ – the set of all fundamental operation symbols occuring in φ. In many papers identities of some special forms were considered. Let us quote some definitions:

Definition 1. An identity $\varphi = \psi$ of type τ is called *regular* if $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$ (see [12]).

Definition 2. An identity $\varphi = \psi$ of type τ is called *biregular* if $F(\varphi) = F(\psi)$ and $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$ (see [14]).

Definition 3. An identity $\varphi = \psi$ of type τ is called *uniform* if $F(\varphi) = F(\psi) = F$ or $F(\varphi) = F(\psi) \neq F$ and $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$ (see [14]).

For example the identities $(x + y) . z = x . z + y . z$ and $(x + y) + z = x + (y + z)$ are biregular and $x + (x . y) = x + (x . x)$ is uniform if $F = \{+, \cdot\}$.

Let K be a variety of type τ. We denote by $R(K)$, $B(K)$ and $U(K)$ the sets of all identities from $\mathrm{Id}(K)$ described by definitions 1–3, respectively.

Regular identities and the varieties $V(R(K))$ were considered in a lot of papers, (see e.g. [6], [8]–[13] and [17]).

The results obtained for uniform identities can be applied to biregular identities. In fact we have

(i) $B(K) = U(R(K))$.

So in every section we describe first the variety $V(U(K))$ and then the variety $V(B(K))$.

For a term φ the notation $\varphi(x_{i_1}, \ldots, x_{i_n})$ will mean that $\mathrm{Var}(\varphi) = \{x_{i_1}, \ldots, x_{i_n}\}$. In the sequel we make use of the following facts:

(ii) If E is a set of regular, biregular, uniform identities of type τ, then every identity from $C(E)$ is regular, biregular, uniform, respectively (see [14], so each of $R(K)$, $B(K)$, $U(K)$ is an equational theory (see [18]).

(iii) Let K be a variety of algebras without nullary operations and K satisfies the condition

(0.1) there exists a term $\varphi(x, y)$ such that the identity $\varphi(x, y) = x$ belongs to $\mathrm{Id}(K)$. Then every algebra from $V(R(K))$ is the sum of a semilattice ordered system of algebras from K (see [13], theorem 3 and [12]).

The class of all sums of semilattice ordered systems of algebras from K will be denoted by K_S.

(iv) If K satisfies the assumption (0.1), F is finite and K is finitely based then $V(R(K))$ is finitely based (see [6]).

(v) If K is an equational class of unary algebras of type τ ($\tau(F) = \{1\}$), then an algebra $\mathfrak{A}$ belongs to $V(R(K))$ iff $\mathfrak{A}$ is the sum of a system $\mathscr{A} = \langle I, \{\mathfrak{A}_i\}_{i \in I}\rangle$ of disjoint algebras, where $\mathfrak{A}_i \in K$ (see [13] and [17]).

(vi) If K is an equational class of unary algebras of type τ, F is finite and K is finitely based, then $V(R(K))$ is finitely based (see [17]).

For a class K of unary algebras we denote by K_{S_u} the class of all sums of systems of disjoint algebras (see [17]).

We shall denote by $K_1 \vee K_2 \vee \ldots \vee K_n$ the join of varieties $K_1, \ldots, K_n$ and by $K_1 \otimes K_2 \otimes \ldots \otimes K_n$ the class of all algebras isomorphic to a subdirect products of algebras $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$, where $\mathfrak{A}_i \in K_i$ for $i = 1, \ldots, n$. Finally we denote $K_R = V(R(K))$, $K_U = V(U(K))$, $K_B = V(B(K))$.

1. SUBDIRECT DECOMPOSITION OF K_U AND K_B

From now on we shall consider algebras of type τ: $F \to N$, where $F = \{f_1, \ldots, f_m\}$ $(1 \leqq m \leqq \aleph_0)$, $\tau(f_i) = n_i > 0$, $i = 1, \ldots, m$.

We denote by $T(\{F\})$ the set of all identities $\varphi = \psi$ of type τ, where $F(\varphi) = F(\psi) = F$.

Let K be a variety of type τ. Put $E^* = T(\{F\}) \cup B(K)$, $K^* = V(E^*)$.

Lemma 1. *The set E^* is an equational theory.*

In fact, since $B(K)$ is an equational theory so E^* is closed under Birkhoff's derivation rules (see [3], p. 93).

Let K be a variety of type τ satisfying the following condition:

(1.1) There exists a term $q(x)$ of type τ such that the identity $q(x) = x$ belongs to $\mathrm{Id}(K)$ and $F(q(x)) = F$.

Let B be an equational base of K and B^* be an equational base of K^*.

For a fixed term $q(x)$ from the condition (1.1) we denote by B' the set of identities of type τ defined by the following conditions $(a_1)-(a_6)$.

(a_1) The identity $q(q(x)) = q(x)$ belongs to B'.

(a_2) Each of the identities $q(f_i(x_1, \ldots, x_{n_i})) = f_i(x_1, \ldots, x_{k-1}, q(x_k), x_{k+1}, \ldots, x_{n_i})$, $i = 1, \ldots, m$; $1 \leqq k \leqq n_i$ belongs to B'.

(a_3) If an identity $\varphi_1 = \varphi_2$ belongs to B, then the identity $q(\varphi_1) = q(\varphi_2)$ belongs to B'.

(a_4) If an identity $\varphi_1 = \varphi_2$ belongs to B^* and $F(\varphi_1) = F(\varphi_2) = F$, then the identities $q(\varphi_1) = \varphi_1$ and $q(\varphi_2) = \varphi_2$ belong to B'.

(a_5) If an identity $\varphi_1 = \varphi_2$ belongs to B^*,

$$F(\varphi_1) = F(\varphi_2) \neq F\,, \quad \mathrm{Var}(\varphi_1) = \mathrm{Var}(\varphi_2)\,,$$

then $(\varphi_1 = \varphi_2) \in B'$.

(a_6) B' contains only identities required in $(a_1)-(a_5)$.

Theorem 1. *If K is a variety of type τ satisfying* (1.1), *then $K_U = K \vee K^* = K \otimes K^*$. Moreover, if B is an equational base of K, B^* is an equational base of K^*, then B' is an equational base of K_U.*

Proof. Obviously $K \otimes K^* \subseteq K \vee K^*$. Further $U(K) \subseteq \mathrm{Id}(K) \cap (B(K) \cup T(\{F\}))$, so $K \vee K^* \subseteq K_U$. We shall show that $B' \subseteq U(K)$, so $K_U \subseteq V(B')$. In fact the identities required in $(a_1)-(a_4)$ belong to $U(K)$ by (1.1). Consider (a_5). By Lemma 1 the identity $\varphi_1 = \varphi_2$ belongs to $B(K) \setminus T(\{F\})$, so it belongs to $B(K) \subseteq U(K)$.

To complete the proof it is enough to show that every algebra $\mathfrak{A} = (A; F^{\mathfrak{A}})$ from $V(B')$ is decomposable into a subdirect product of two algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, where $\mathfrak{A}_1 \in K$ and $\mathfrak{A}_2 \in K^*$. On $\mathfrak{A}$ let us define two relations P_1 and P_2 by putting for $a, b \in A$:

$$a\, P_1\, b \quad \text{iff} \quad q(a) = q(b)$$

$$a\, P_2\, b \quad \text{iff} \quad a = b \quad \text{or} \quad q(a) = a \quad \text{and} \quad q(b) = b\,.$$

Obviously P_1 and P_2 are equivalences. We shall show that they are congruences on $\mathfrak{A}$. Denote $q^1(x) = q(x)$, $q^{n+1}(x) = q(q^n(x))$. If $a_k\, P_1\, b_k$ $(k = 1, \ldots, n_i)$, then we we have by (a_1), (a_2):

$$q(f_i(a_1, \ldots, a_{n_i})) = q^{n_i}(f_i(a_1, \ldots, a_{n_i})) = f_i(q(a_1), \ldots, q(a_{n_i})) =$$
$$= f_i(q(b_1), \ldots, q(b_{n_i})) = q^{n_i}(f_i(b_1, \ldots, b_{n_i})) = q(f_i(b_1, \ldots, b_{n_i}))\,.$$

Thus $f_i(a_1, \ldots, a_{n_i})\, P_1\, f_i(b_1, \ldots, b_{n_i})$.

Let $a_k\, P_2\, b_k$ for $k = 1, \ldots, n_i$. If $a_k = b_k$ for $k = 1, \ldots, n_i$, then $f_i(a_1, \ldots, a_{n_i}) = f_i(b_1, \ldots, b_{n_i})$. So $f_i(a_1, \ldots, a_{n_i})\, P_2\, f_i(b_1, \ldots, b_{n_i})$.

Otherwise there exists $p \in \{1, \ldots, n_i\}$ such that $q(a_p) = a_p$ and $q(b_p) = b_p$. Then by (a_2)

$$q(f_i(a_1, \ldots, a_{n_i})) = f_i(a_1, \ldots, a_{p-1}, q(a_p), a_{p+1}, \ldots, a_{n_i}) = f_i(a_1, \ldots, a_{n_i})\,.$$

Similarly $q(f_i(b_1, \ldots, b_{n_i})) = f_i(b_1, \ldots, b_{n_i})$, so $f_i(a_1, \ldots, a_{n_i})\, P_2\, f_i(b_1, \ldots, b_{n_i})$.

By (a_3), $(\mathfrak{A}/P_1) \in K$. By (a_4) and (a_5) $\mathfrak{A}/P_2$ satisfies all identities from B^*, so all identities from E^* and consequently $(\mathfrak{A}/P_2) \in K^*$.

We prove that $P_1 \cap P_2 = \omega$, where ω is the diagonal of $A \times A$.

If $a\, P_1\, b$ and $a\, P_2\, b$, then $a = b$ or $a = q(a) = q(b) = b$.

Now by Birkhoff's theorem (see [1], [2]) $\mathfrak{A}$ is isomorphic to a subdirect product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, where

$$\mathfrak{A}_1 = (\mathfrak{A}/P_1)\,; \quad \mathfrak{A}_2 = (\mathfrak{A}/P_2)\,, \quad \mathfrak{A}_1 \in K\,, \quad \mathfrak{A}_2 \in K^*\,.$$

This completes the proof.

Corollary 1. *If K is a variety of type τ satisfying* (1.1), *K is finitely based and K^* is finitely based, then K_U is finitely based.*

Corollary 2. *If K satisfies* (1.1), *$T(\{F\})$ and $B(K) \setminus T(\{F\})$ have finite equational bases, K is finitely based, then K_U is finitely based.*

In fact, by Lemma 1 $E^* = T(\{F\}) \cup (B(K) \setminus T(\{F\}))$ is an equational theory. So E^* has a finite base and we can apply Corollary 1.

Let K be a variety of type τ satisfying (1.1). Let D be an equational base of K_R and B^* be an equational base of K^*. Denote by B'' the set of identities of type τ defined by $(a_1), (a_2), (a_4)-(a_6)$ and the following condition (a_3^R):

(a_3^R) If an identity $\varphi_1 = \varphi_2$ belongs to D, then the identity $q(\varphi_1) = q(\varphi_2)$ belongs to B'.

Theorem 2. *If K is a variety of type τ satisfying (1.1), then $K_B = K_R \vee K^* = K_R \otimes K^*$. Moreover, if D is an equational base of K_R, B^* is an equational base of K^*, then B' is an equational base of K_B.*

Proof. First let us observe that we have always

$$(K_R)^* = K^*$$

since $B(K) = B(R(K))$. Further K_R also satisfies (1.1). Applying Theorem 1 to K_R we have by (i)

$$K_B = (K_R)_U = K_R \vee (K_R)^* = K_R \otimes (K_R)^* = K_R \vee K^* = K_R \otimes K^* .$$

Applying the second statement of Theorem 1 we must substitute K by K_R i.e. we must substitute (a_3) by (a_3^R). Then we infer that B is an equational base of $(K_R)_U = K_B$.

Corollary 3. *If K satisfies (1.1) and both K_R and K^* are finitely based, then K_B is finitely based.*

Corollary 4. *If K satisfies (1.1), K_R is finitely based and each of the sets $T(\{F\})$ and $B(K) \setminus T(\{F\})$ has a finite equational base, then K_B is finitely based.*

Corollary 5. *If K satisfies (1.1) and (0.1), then $K_B = K_S \otimes K^*$. Moreover, if K and K^* are finitely based, then K_B is finitely based.*

This follows from (iii), (iv) and Corollary 3.

Corollary 6. *If K is a variety of unary algebras satisfying (1.1), then $K_B = K_{S_u} \otimes K^*$. Moreover, if K is finitely based and K^* is finitely based, then K_B is finitely based.*

This follows from (v), (vi) and Corollary 3.

Remark 1. An identity $\varphi = \psi$ is called non-trivializing (see [15]) if it is of the form $x = x$ or neither φ nor ψ is a single variable. So if $F = \{f_1\}$, then an identity $\varphi = \psi$ is uniform iff it is non-trivializing. In this case the first statement of Theorem 1 coincides with the result from [15]. If $F = \{f_1\}$, then it is easy to see that K^* is finitely based, namely it can be defined by a single identity

$$(1.2) \qquad f_1(x_1, \ldots, x_{n_1}) = f_1(y_1, \ldots, y_{n_1}) .$$

So every algebra from K^* is an algebra with one constant operation. Thus the first statement of Theorem 1 gives a simple representation. Moreover, the assumption that K^* is finitely based in above corollaries is satisfied.

Corollary 7. If K is a variety of type $\tau: \{f_1\} \to N$, where $\tau(f_1) > 1$, K satisfies (0.1), then $K_B = K_S \otimes K^*$, where K^* satisfies (1.2). Moreover, if K is finitely based, then K_B is finitely based.

In fact, it is enough to put $q(x) = \varphi(x, x)$, where φ is the term from (0.1) and to apply Corollary 5.

Example 1. Let K be the variety of groups with one fundamental operation $\cdot$ satisfying $x^n = y^n$. Then $K_B = K_S \otimes K^*$, where K^* is described by $x \,.\, y = u \,.\, v$. In fact, it is enough to put $\varphi(x, y) = x \,.\, y^n$ and apply Corollary 7.

Remark 2. The class K_S in Example 1 was described previously by A. H. Clifford (see [4] and [5]).

2. VARIETIES OF ALGEBRAS WITH TWO FUNDAMENDAL OPERATIONS

In this section we assume that $F = \{f_1, f_2\}$. Let us denote by T_0 the system of the following identities

$$(2.1) \qquad f_1(x_1, \ldots, x_{k-1}, f_2(y_1, \ldots, y_{n_2}), x_{k+1}, \ldots, x_{n_1}) = \\ = f_2(z_1, \ldots, z_{j-1}, f_1(u_1, \ldots, u_{n_1}), z_{j+1}, \ldots, z_{n_2}),$$

where $1 \leqq k \leqq n_1$, $1 \leqq j \leqq n_2$.

Lemma 2. $T(\{F\}) = C(T_0)$.

Proof. Obviously $T_0 \subseteq T(\{F\})$, so $C(T_0) \subseteq T(\{F\})$. Let us denote:

$$(2.2) \qquad c \equiv f_1(f_2(y_1, \ldots, y_{n_2}), x_2, \ldots, x_{n_1}),$$

where $\varphi \equiv \psi$ means that φ and ψ have the same structure.

To prove that $T(\{F\}) \subseteq C(T_0)$ it is enough to show that for every term φ such that $F(\varphi) = \{f_1, f_2\}$ we have:

$$(2.3) \qquad (\varphi = c) \in C(T_0).$$

We prove this by induction on the complexity of φ.

If φ is obtained in the second step of constructing terms, then φ is one of the form written on the left or on the right hand side of (2.1) (up to choice of variables).

Thus, by (2.1) and (2.2), we get (2.3).

Let us assume that (2.3) holds for all terms obtained in k'th step for $2 \leqq k < n$ and φ is obtained in n'th step. Then we have

$$(2.4) \qquad \varphi \equiv f_1(\psi_1, \ldots, \psi_{n_1})$$

or

$$(2.5) \qquad \varphi \equiv f_2(\chi_1, \ldots, \chi_{n_2}).$$

Consider case (2.4) – the proof in case (2.5) is analogous. Since $F(\varphi) = F$, there must exist $s \in \{1, \ldots, n_1\}$ such that $f_2 \in F(\psi_s)$. If $F(\psi_s) = \{f_2\}$, then $\varphi \equiv f_1(\psi_1, \ldots \ldots, \psi_{s-1}, f_2(\mu_1, \ldots, \mu_{n_2}), \psi_{s+1}, \ldots, \psi_{n_1})$ and by (2.1) we get (2.3). If $F(\psi_s) = F$, then by induction hypothesis we have $(\varphi_s = c) \in C(T_0)$, so by (2.4) we get:

$$(\varphi = f_1(\psi_1, \ldots, \psi_{s-1}, c, \psi_{s+1}, \ldots, \psi_{n_1})) \in C(T_0) . \tag{2.6}$$

By (2.2) the identity $f_1(\psi_1, \ldots, \psi_{s-1}, c, \psi_{s+1}, \ldots, \psi_{n_1}) = f_1(\psi_1, \ldots, \psi_{s-1}, f_2(z_1, \ldots, z_{j-1}, f_1(u_1, \ldots, u_{n_1}), z_{j+1}, \ldots, z_{n_2}), \psi_{s+1}, \ldots, \psi_{n_1})$ belongs to $C(T_0)$. Thus by (2.1) and (2.6) we get (2.3).

It is interesting to note that if $|F| > 2$ then $T(\{F\})$ need not have a finite equational base (see Remark 5 in section 6).

From Lemma 2 we get a useful Corollary which simplifies Corollaries 1 – 7.

Corollary 8. *If* $|F| = 2$, $B(K) \setminus T(\{F\})$ *has a finite equational base, then* K^* *is finitely based.*

In fact, $\mathrm{Id}(K^*) = E^* = B(K) \cup T(\{F\}) = (B(K) \setminus T(\{F\}) \cup T(\{F\})$ by Lemma 1. So if E is an equational base of $B(K) \setminus T(\{F\})$, then $E \cup T_0$ is an equational base of K^*.

Example 2. Let K be a variety with two unary operations f_1, f_2 satisfying the identities:

$$f_1(f_2(x)) = x = f_2(f_1(x)) . \tag{2.7}$$

Since identities (2.7) are regular, so $K_U = K_B$. We want to describe algebras from K_B. By Lemma 2 the identity:

$$f_1(f_2(x)) = f_2(f_1(y)) \tag{2.8}$$

forms a base of $T(\{F\})$. By Birkhoff's derivation rules it can be easily shown, that the only identities from $B(K) \setminus T(\{F\})$ are of the form $\varphi = \varphi$, so the set $\emptyset$ is a base of $B(K) \setminus T(\{F\})$. By Corollary 8 K^* is defined by (2.8). By Corollaries 8 and 3 K_B is finitely based. By Theorem 2 every algebra from K_B is isomorphic to a subdirect product of an algebra from K and an algebra from K^*. The structure of every algebra from K is clear, namely we have two $1 - 1$ mappings such that each of them is the converse of the other one. If $\mathfrak{A} = (A; f_1, f_2) \in K^*$, then by (2.8) $A = A_0 \cup f_1(A) \cup \cup f_2(A)$, where A_0 consists of all elements being values neither of f_1 nor of f_2 and there is an element $e \in f_1(A) \cap f_2(A)$ such that $f_2(x) = e$ for $x \in f_1(A)$ and $f_1(y) = e$ for $y \in f_2(A)$.

3. GENERAL REPRESENTATION THEOREMS

In this section we assume that $F = \{f_1, \ldots, f_m\}$, $m > 1$. For a family $\boldsymbol{H}$ of subsets of F we denote by $T(\boldsymbol{H})$ the set of all identities $\varphi = \psi$ of type τ such that there exist $H_1, H_2 \in \boldsymbol{H}$ with $H_1 \subseteq F(\varphi)$ and $H_2 \subseteq F(\psi)$. For $Z \subseteq F$ we denote $T_1(Z) = T(\{\{f_i\} : f_i \in Z\})$.

Let

$$\boldsymbol{S} = \{F_1, \ldots, F_s\} , \quad 1 \leqq s \leqq \binom{m}{k} \tag{3.1}$$

be a family of k-element subsets of F where $1 \leqq k < m$. For $Z \subseteq F$ put $Z' = F \setminus Z$. Denote

$$S^0 = \{F_i \cup \{f_j\} : i \in \{1, \ldots, s\},\ j \in F'_i\} .$$

Let K be a variety of type τ. We denote by $R_{F_i}(K)$ the set of all identities $\varphi = \psi$ from $R(K)$ for which $F(\varphi), F(\psi) \subseteq F_i$.

The variety defined by $R_{F_i}(K) \cup T_1(F'_i)$ will be denoted by K_i. We denote by K^0 the variety defined by $T(S^0) \cup B(K)$ and we denote by K_{s+1} the variety defined by $T(S) \cup B(K)$.

One can easily prove.

Lemma 3. *Each of the sets* $R_{F_i}(K) \cup T_1(F'_i)$, $B(K) \cup T(S)$, $B(K) \cup T(S^0)$ *is an equational theory.*

Let K be a variety satisfying the following condition:

(3.2) There exist terms $q_1(x), \ldots, q_s(x)$ of type τ such that $(q_i(x) = x) \in \mathrm{Id}(K)$ and $F(q_i(x)) = F_i$ $(i = 1, \ldots, s)$.

Let B_i be an equational base of K_i $(i = 1, \ldots, s + 1)$. For fixed $q_i(x)$ $(i = 1, \ldots, s)$ from (3.2) we define a set B^0 of identities of type τ by the following conditions $(b_1) - (b_7)$:

(b_1) $q_i(q_i(x)) = q_i(x)$ belongs to B^0 $(i = 1, \ldots, s)$.
(b_2) $q_i(f_j(x_1, \ldots, x_{n_j})) = f_j(x_1, \ldots, x_{p-1}, q_i(x_p), x_{p+1}, \ldots, x_{n_j})$ belongs to B^0 $(i = 1, \ldots, s;\ j = 1, \ldots, m;\ p = 1, \ldots, n_j)$.
(b_3) If $(\varphi = \psi) \in B_i$, then $(q_i(\varphi) = q_i(\psi)) \in B^0$ $(i = 1, \ldots, s)$.
(b_4) If $(\varphi = \psi) \in B_{s+1} \cap T(S)$, where $F_{i_1} \subseteq F(\varphi)$, $F_{i_2} \subseteq F(\psi)$ for $F_{i_1}, F_{i_2} \in S$, then the identities $q_{i_1}(\varphi) = \varphi$ and $q_{i_2}(\psi) = \psi$ belong to B^0.
(b_5) If $\varphi = \psi$ belongs to $B_{s+1} \cap (B(K) \setminus T(S))$, then $\varphi = \psi$ belongs to B^0.
(b_6) $q_i(q_j(x)) = q_j(q_i(x))$ belongs to B^0 for each $i, j \in \{1, \ldots, s\}$.
(b_7) B^0 contains only identities from $(b_1)-(b_6)$.

Lemma 4. *If* $F = \{f_1, \ldots, f_m\}$, $m > 1$, K *satisfies* (3.2) *for a family* S *from* (3.1), *then* $K^0 = K_1 \vee \ldots \vee K_{s+1} = K_1 \otimes \ldots \otimes K_{s+1}$. *Moreover, if* B_i *is an an equational base of* K_i $(i = 1, \ldots, s + 1)$, *then* B^0 *is an equational base of* K^0.

Proof. Obviously $K_1 \otimes \ldots \otimes K_{s+1} \subseteq K_1 \vee \ldots \vee K_{s+1}$. We shall show that $K_i \subseteq K^0$ $(i = 1, \ldots, s + 1)$. So we must show that

$$B(K) \cup T(S^0) \subseteq R_{F_i}(K) \cup T_1(F'_i)\,, \quad (i = 1, \ldots, s) \tag{3.3}$$

and

$$B(K) \cup T(S^0) \subseteq B(K) \cup T(S)\,. \tag{3.4}$$

Let

$$\varphi_1 = \varphi_2 \tag{3.5}$$

belong to $B(K) \cup T(S^0)$.

We prove (3.3). If (3.5) is biregular and $F(\varphi_1) = F(\varphi_2) \subseteq F_i$, then (3.5) belongs to $R_{F_i}(K)$. If (3.5) is biregular and $F(\varphi_1) = F(\varphi_2) \nsubseteq F_i$, then there exists $f_j \in F'_i \cap \cap F(\varphi_1) \cap F(\varphi_2)$. So (3.5) belongs to $T_1(F'_i)$. If (3.5) belongs to $T(S^0)$, then there exist F_{i_1}, f_{j_1} and F_{i_2}, f_{j_2} such that $f_{j_1} \in F'_{i_1}$, $f_{j_2} \in F'_{i_2}$ and $(F_{i_1} \cup \{f_{j_1}\}) \subseteq F(\varphi_1)$, $(F_{i_2} \cup \{f_{j_2}\}) \subseteq F(\varphi_2)$. Both sets $F_{i_1} \cup \{f_{j_1}\}$, $F_{i_2} \cup \{f_{j_2}\}$ are $k + 1$-element, so there must exist f_{p_1}, f_{p_2} such that $f_{p_1} \in (F_{i_1} \cup \{f_{j_1}\}) \setminus F_i$ and $f_{p_2} \in (F_{i_2} \cup \{f_{j_2}\}) \setminus F_i$. Hence $f_{p_1}, f_{p_2} \in F'_i$ and consequently (3.5) belongs to $T_1(F'_i)$.

Proof of (3.4) is obvious.

Thus $K_1 \vee \ldots \vee K_{s+1} \subseteq K^0$.

It is easy to see that $B^0 \subseteq B(K) \cup T(S^0)$, so $K^0 \subseteq V(B^0)$. This follows from the fact that by Lemma 3 for $1 \leqq i \leqq s$ we have:

$$B_i = (B_i \cap R_{F_i}(K)) \cup (B_i \cap T_1(F'_i)) .$$

Similarly $B_{s+1} = (B_{s+1} \cap T(S)) \cup (B_{s+1} \cap (B(K) \setminus T(S)))$.

To complete the proof it is enough to show that any algebra $\mathfrak{A} = (A; f_1, \ldots, f_m)$ from $V(B^0)$ is isomorphic to a subdirect product of algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_{s+1}$, where $\mathfrak{A}_i \in K_i$ $(i = 1, \ldots, s + 1)$. In $\mathfrak{A}$ we define $s + 1$ relations $P_1, \ldots, P_{s+1}$ putting for $a, b \in A$

$$a\, P_i\, b \Leftrightarrow q_i(a) = q_i(b)$$

for $i = 1, \ldots, s$ and

$$aP_{s+1}b \quad \text{iff} \quad a = b \quad \text{or} \quad q_i(a) = a \quad \text{and} \quad q_j(b) = b$$

for some $i, j \in \{1, \ldots, s\}$.

Obviously every P_i is an equivalence. The proof that any P_i has the substitution property follows from (b_1) and (b_2) and is similar to that in Theorem 1.

By (b_3) we have $(\mathfrak{A}/P_i) \in K_i$ $(i = 1, \ldots, s)$ and $(\mathfrak{A}/P_{s+1}) \in K_{s+1}$ by (b_4) and (b_5).

We prove that $P_1 \cap P_2 \cap \ldots \cap P_{s+1} = \omega$

Let aP_ib for $i = 1, \ldots, s + 1$, then either $a = b$ or for some $i, j \in \{1, \ldots, s\}$ we have $a = q_i(a) = q_i(b) = q_i(q_j(b)) = q_j(q_i(b)) = q_j(q_i(a)) = q_j(a) = q_j(b) = b$ by (b_6). Thus by Birkhoff's subdirect decomposition theorem $\mathfrak{A}$ is isomorphic to $(\mathfrak{A}/P_1) \otimes \ldots \otimes (\mathfrak{A}/P_{s+1})$ and $V(B^0) \subseteq K_1 \otimes \ldots \otimes K_{s+1}$.

Corollary 9. *If K satisfies (3.2) and each of the varieties K_i $(i = 1, \ldots, s + 1)$ is finitely based, then K^0 is finitely based.*

Let $Z \subseteq F$. We denote by $T^0_1(Z)$ the set of the following identities:

$$(3.6) \qquad f_u(x_1, \ldots, x_{n_u}) = f_v(y_1, \ldots, y_{n_v}), \quad (f_u, f_v \in Z)$$

$$(3.7) \qquad f_j(x_1, \ldots, x_{p-1}, f_u(y_1, \ldots, y_{n_u}), x_{p+1}, \ldots, x_{n_j}) = \\ = f_u(y_1, \ldots, y_{n_u}) \quad \textit{for} \quad f_u \in Z \quad \textit{and} \quad f_j \in F .$$

Lemma 5. *For every $Z \subseteq F$ the set $T^0_1(Z)$ is an equational base of $T_1(Z)$.*

Proof. Let us fix $f_w \in Z$. By easy induction we can prove that for any term φ such that there exists $f_u \in Z \cap F(\varphi)$ we have $(\varphi = f_w(y_1, \ldots, y_{n_w})) \in C(T_1^0(Z))$.

Corollary 10. *If $R_{F_i}(K)$ is finitely based for some $i \in \{1, \ldots, s\}$, then K_i is finitely based.*

In fact, $\mathrm{Id}(K_i) = R_{F_i}(K) \cup T_1(F'_i)$.

Corollary 11. *If $T(S)$ and $B(K) \setminus T(S)$ are finitely based, then K_{s+1} is finitely based.*

This follows from Lemma 3.

Corollary 12. *If K satisfies (3.2) and each of the sets $R_{F_i}(K)$ $(1 \leqq i \leqq s)$, $T(S)$, $B(K) \setminus T(S)$ is finitely based, then K^0 is finitely based.*

For a family $\{K_i\}_{i \in I}$ of varietes of type τ we shall denote by $\bigvee_{i \in I} K_i$ the join of this family and by $\bigotimes_{i \in I} K_i$ we shall denote the class of all algebras isomorphic to a subdirect product of a family $\{\mathfrak{A}_i\}_{i \in I}$ of algebras, where $\mathfrak{A}_i \in K_i$.

Let $F = \{f_1, \ldots, f_m\}$, $(m > 1)$ and let L be a proper subset of F. As usual $[L)$ will denote the principal filter generated in 2^F by L.

We put

$$[L)^* = \begin{cases} [L) \setminus \{F\} & \text{if } L \neq \emptyset \\ [L) \setminus \{F, \emptyset\} & \text{if } L = \emptyset . \end{cases}$$

So we have

$$(3.8) \qquad [L)^* = S_1 \cup S_2 \cup \ldots \cup S_r ,$$

where $S_n = \{\tilde{F} : \tilde{F} \in [L)^* \wedge |\tilde{F}| = m - n\}$, $1 \leqq n \leqq r$, $r = m - 1$ if $L = \emptyset$, and $r = m - |L|$ if $L \neq \emptyset$.

Let K be a variety of type τ. For $\tilde{F} \in [L)^*$ we denote by $K_{\tilde{F}}$ the variety of type τ defined by $R_{\tilde{F}}(K) \cup T_1(\tilde{F}')$, where $R_{\tilde{F}}(K)$ consists of all identities $\varphi = \psi$ from $R(K)$ such that $F(\varphi)$, $F(\psi) \subseteq \tilde{F}$. Further we denote by K^r the variety defined by $(B(K) \setminus T(S_r)) \cup T(S_r)$.

Assume that K satisfies the following condition

(3.9) for each $\tilde{F} \in [L)^*$ there exists a term $q_{\tilde{F}}(x)$ such that $(q_{\tilde{F}}(x) = x) \in \mathrm{Id}(K)$ and $F(q_{\tilde{F}}(x)) = \tilde{F}$.

Lemma 6. *If K satisfies (3.9), then $K^* = \bigotimes_{\tilde{F} \in [L)^*} K_{\tilde{F}} \otimes K^r$. Moreover, if each of the sets $R_{\tilde{F}}(K)$, $B(K) \setminus T(S_r)$, $T(S_r)$ has a finite equational base, then K^* is finitely based.*

Proof. To prove the first part of Lemma 6 it is enough to show by induction that the following statement holds for $1 \leqq n \leqq r$:

$$(3.10) \qquad K^* = \bigotimes_{\tilde{F} \in S_1} K_{\tilde{F}} \otimes \bigotimes_{\tilde{F} \in S_2} K_{\tilde{F}} \otimes \ldots \otimes \bigotimes_{\tilde{F} \in S_n} K_{\tilde{F}} \otimes K^n ,$$

where K^n is defined by $(B(K) \setminus T(S_n)) \cup T(S_n)$.

We use Lemma 4.

Let $n = 1$. We put $S = S_1$, $k = m - 1$, $s = |S_1|$ in Lemma 4. Then $(S_1)^0 = F$, $K^0 = K^*$. By Lemma 4 $K^* = \bigotimes_{\tilde{F} \in S_1} K_{\tilde{F}} \otimes K_{s+1}$, where $K_{s+1} = K^1$.

Assume that Lemma 6 holds for $n \geqq 1$.

Put in Lemma 4 $S = S_{n+1}$, $k = m - (n + 1)$, $s = |S_{n+1}|$. Then $S_{n+1}^0 = S_n$, $K^0 = K^n$. Applying Lemma 4 for $S = S_{n+1}$ we get

$$(3.11) \qquad K^n = \bigotimes_{\tilde{F} \in S_{n+1}} K_{\tilde{F}} \otimes K^{n+1}\,.$$

Using (3.11) and (3.10) we get equality (3.10) for $n + 1$.

To prove the second part of Lemma 6 observe that if $B(K) \setminus T(S_r)$, $T(S_r)$ and each $R_{\tilde{F}}(K)$ for $\tilde{F} \in S_r$ has a finite equational base, then by Corollary 12 K^{r-1} is finitely based. Further we use induction but from the end to the begining.

From Theorem 1, Lemma 6 and Corollary 1 we get

Theorem 3. *Let L be a proper subset of $F = \{f_1, \ldots, f_m\}$, $(m > 1)$ and let K be a variety satisfying the following condition*

(3.12) *for every $\tilde{F} \in [L) \setminus \{\emptyset\}$ there exists a term $q_{\tilde{F}}(x)$ such that $(q_{\tilde{F}}(x) = x) \in \operatorname{Id}(K)$ and $F(q_{\tilde{F}}(x)) = \tilde{F}$.*

Then

$$K_U = K \otimes \bigotimes_{\tilde{F} \in [L)^*} K_{\tilde{F}} \otimes K^r\,.$$

Moreover, if K is finitely based and

(3.13) *each of the sets $R_{\tilde{F}}(K)$ for $\tilde{F} \in [L)^*$, $B(K) \setminus T(S_r)$, $T(S_r)$ has a finite equational base,*

then K_U is finitely based.

From Theorem 2, Lemma 6 and Corollary 3 we get

Theorem 4. *If $m > 1$, a variety K satisfies (3.12), then $K_B = K_R \otimes \bigotimes_{\tilde{F} \in [L)^*} K_{\tilde{F}} \otimes K^r$. Moreover, if K_R is finitely based and (3.13) holds, then K_B is finitely based.*

Corollary 13. *Let K be a variety satisfying (3.12) and (0.1). Then $K_B = K_S \otimes \bigotimes_{\tilde{F} \in [L)^*} K_{\tilde{F}} \otimes K^r$. If K is finitely based and (3.13) holds, then K_B is finitely based.*

Proof. This follows from Theorem 4, (iii) and (iv) from section 0.

Corollary 14. *If K is a variety of unary algebras satisfying (3.12), then $K_B = K_{S_u} \otimes \bigotimes_{\tilde{F} \in [L)^*} K_{\tilde{F}} \otimes K^r$. Moreover, if K is finitely based and (3.13) holds, then K_B is finitely based.*

This follows from (v) and (vi).

Corollary 15. *Let $f \in F$ and K satisfy (3.12) for $L = \{f\}$. Then $K_U = K \otimes \bigotimes_{\tilde{F} \in [L)^*} K_{\tilde{F}} \otimes K^{m-1}$, where K^{m-1} is defined by $T_1(\{f\}) \cup (B(K) \setminus T_1(\{f\}))$.*

If K is finitely based, each of the sets $R_{\tilde F}(K)$, $\tilde F \in [\{f\})^*$ and $B(K) \setminus T_1(\{f\})$ has a finite equational base, then K_U is finitely based.

This follows from Theorem 3 and Lemma 5.

Corollary 16. *Let $f \in F$ and K satisfy* (3.12) *for $L = \{f\}$. Then*

$$K_B = K_R \otimes \bigotimes_{\tilde F \in [L)^*} K_{\tilde F} \otimes K^{m-1} \,.$$

If K_R is finitely based and each of the sets $R_{\tilde F}(K)$, $B(K) \setminus T_1(\{f\})$ has a finite equational base, then K_B is finitely based.

This follows from Theorem 4.

Corollary 17. *If K satisfies* (0.1) *for some term $\varphi(x, y)$ such that $F(\varphi(x, y)) = \{f\}$, where $f \in F$, then the statements of Corollary* 15 *hold.*

In fact, let $\tilde F \in [\{f\})$ and $\tilde F = \{f, f_{i_1}, f_{i_2}, \ldots, f_{i_p}\}$. Then we put $q_{\tilde F}(x) = \varphi(x, f_{i_1}(f_{i_2}(\ldots f_{i_p}(x, \ldots, x), x, \ldots, x), \ldots), x, \ldots, x)$.

Corollary 18. *If the assumptions of Corollary* 17 *hold, then*

$$K_B = K_S \otimes \bigotimes_{\tilde F \in [\{f\})^*} K_{\tilde F} \otimes K^{m-1} \,.$$

If K is finitely based and each of the sets $R_{\tilde F}(K)$, $B(K) \setminus T_1(\{f\})$ has a finite equational base, then K_B is finitely based.

The proof is analogous to that of Corollary 17 and we use Corollary 13.

Corollary 19. *If K satisfies* (3.12) *for $L = \emptyset$, then*

$$K_U = K \otimes \bigotimes_{\tilde F \in 2^F \setminus \{\emptyset, F\}} K_{\tilde F} \otimes K^{(m-1)} \,,$$

where $K^{(m-1)}$ is defined by $T_1(F)$. If K is finitely based and each of $R_{\tilde F}(K)$ has a finite equational base, then K_U is finitely based.

Proof. This follows from Theorem 3 and Lemma 5 since by Lemma 5, $T_1(F)$ has a finite equational base and $B(K) \setminus T_1(F)$ has a finite base namely $\emptyset$.

Corollary 20. *If K satisfies the assumptions of Corollary* 19, *then*

$$K_B = K_R \otimes \bigotimes_{\tilde F \in 2^F \setminus \{\emptyset, F\}} K_{\tilde F} \otimes K^{(m-1)} \,.$$

If K_R is finitely based and each of the sets $R_{\tilde F}(K)$ has a finite equational base, then K_B is finitely based.

Corollary 21. *If K satisfies the condition*

(3.14) *for every $f_j \in F$ there exists a term $q_{\{f_j\}}(x)$ such that $F(q_{\{f_j\}}(x)) = \{f_j\}$ and $(q_{\{f_j\}}(x) = x) \in \mathrm{Id}(K)$,*

then the statements of Corollary 19 *hold.*

In fact if $\tilde{F} \in 2^F \setminus \{\emptyset\}$ and $\tilde{F} = \{f_{j_1}, \ldots, f_{j_p}\}$, then put $q_{\tilde{F}}(x) = q_{j_1}(q_{j_2}(\ldots(q_{j_p}(x))\ldots))$.

Corollary 22. *If K satisfies* (3.14), *then the statements of Corollary* 20 *hold.*

Corollary 23. *If K satisfies* (3.14) *and* (0.1), *then* $K_B = K_S \otimes \bigotimes_{\tilde{F} \in 2^F \setminus \{\emptyset, F\}} K_{\tilde{F}} \otimes K^{(m-1)}$. *If K is finitely based and each of the sets $R_{\tilde{F}}(K)$ has a finite equational base, then K_B is finitely based.*

Proof. Use Corollary 22, (iii) and (iv).

4. THE $\tilde{F}$-ABSORBING SUM OF A SEMILATTICE ORDERED SYSTEM OF ALGEBRAS

Note that if a variety K satisfies all identities from $T_1(W)$ for some $W \subseteq F$ then in any algebra $\mathfrak{A} \in K$ the realizations of all $f_j \in W$ are equal to the same constants c and whenever c is an argument of some $f_i \in F$, then the value of f_i is equal to c. Such element c will be called an *absorbing element* of $\mathfrak{A}$ (see [10]).

Theorems 3 and 4 and Corollaries after them can be applied for important varieties of algebras namely those of groups, rings, lattices and Boolean Algebras and we can find representations of algebras from K_U and K_B when K is one of these class. However for this aim we need one more construction and one more theorem which gives a description of algebras from K_F. If F and $\tilde{F}$ are two sets with $\tilde{F} \subset F$, we agree that every identity $\varphi = \psi$ of type $\tilde{\tau} = \tau/\tilde{F}$ is also an identity of type τ.

Let $\tilde{F} \subseteq F$, $\tilde{F} \neq \emptyset \neq F \setminus \tilde{F}$. Further let

$$\mathscr{A} = ((I; \leqq); \{\mathfrak{A}_i\}_{i \in I}; \{h_i^j\}_{i,j \in I, i \leqq j})$$

be a semilattice ordered system of algebras $\mathfrak{A}_i = (A_i; \tilde{F}^{\mathfrak{A}_i})$ of type $\tilde{\tau}$ (see [12]). We assume that $\mathscr{A}$ satisfies the following condition:

(4.1) There exists in the semilattice I the greatest index ι and the algebra $\mathfrak{A}_\iota$ is 1-element.

Let us denote $A_\iota = \{c\}$.

We shall say that an algebra $\mathfrak{A} = (A; F^{\mathfrak{A}})$ of type $\tau: F \to N$ is the $\tilde{F}$-absorbing sum of a semilattice ordered system $\mathscr{A}$ of algebras $\mathfrak{A}_i$ if the following conditions are satisfied:

(1°) The algebra $(A; \tilde{F})$ is the sum of a semilattice ordered system $\mathscr{A}$.

(2°) For every $f_j \in \tilde{F}' = F \setminus \tilde{F}$ and $a_1, \ldots, a_{n_j} \in A$ we have $f_j(a_1, \ldots, a_{n_j}) = c$.

Let K be a variety of type τ and let $\tilde{F}$ be a non-empty proper subset of F such that for some $f_i \in \tilde{F}$ we have $n_i > 1$. We denote by $K(\tilde{F})$ the variety of type $\tilde{\tau}$ defined by all identities $(\varphi = \psi) \in \mathrm{Id}(K)$ such that $F(\varphi), F(\psi) \subseteq \tilde{F}$.

Theorem 5. *If there exists a term $f(x, y)$ of type $\tilde{\tau}$ such that the identity*

(4.2) $$f(x, y) = x$$

belongs to $\mathrm{Id}(K)$ *then an algebra* $\mathfrak{A} = (A; F^{\mathfrak{A}})$ *belongs to* $K_{\tilde{F}}$ *iff* $\mathfrak{A}$ *is the* $\tilde{F}$*-absorbing sum of a semilattice ordered system of a family* $\{\mathfrak{A}_i\}_{i\in I}$ *of algebras of type* $\tilde{\tau}$, *where* $\mathfrak{A}_i \in K(\tilde{F})$. *Moreover, if* $K(\tilde{F})$ *is finitely based,* F *is finite then* $K_{\tilde{F}}$ *is finitely based.*

Proof. Let $\mathfrak{A} \in K_{\tilde{F}}$. So $\mathfrak{A}$ satisfies $R(K(\tilde{F}))$. By (iii) the algebra $(A; \tilde{F})$ is the sum of a semilattice ordered system of algebras $\mathfrak{A}_i$, where $\mathfrak{A}_i \in K(\tilde{F})$. We have $T_1(\tilde{F}') \subseteq \mathrm{Id}(K_{\tilde{F}})$. Let us denote by c the absorbing element in A determined by all $f_j \in \tilde{F}'$. We denote by A_ι the component to which c belongs. Recall that for any $f_j \in \tilde{F}$ we have:

$$\text{if} \quad a_k \in A_{i_k}, \quad k = 1, \ldots, n_j, \quad \text{then}$$

$$f_j(a_1, \ldots, a_{n_j}) \in A_t, \quad \text{where} \quad t = \mathrm{l.u.b.}(i_1, \ldots, i_{n_j}).$$

However if c is an argument of f_j then the value of f_j is equal to c. So ι must be the greatest index in I. The identity (4.2) belongs to $\mathrm{Id}(K(\tilde{F}))$, so for $x \in A_\iota$ we have $f(x, c) = x$. However we have also $f(x, c) = c$. Thus $A_\iota = \{c\}$.

The implication $\Leftarrow$ follows from the fact that the sum of a semilattice ordered system preserves all regular identities satisfied in any component (see [12], Theorem 1) and from the fact that the realization of any term containing some $f_j \in \tilde{F}'$ must be equal to c, which is easy to see. So all identities from $T_1(\tilde{F}')$ must be satisfied in an $\tilde{F}$-absorbing sum of a semilattice ordered system of algebras $\mathfrak{A}_i$ belonging to $K(\tilde{F})$.

Finally $R_{\tilde{F}}(K) = R(K(\tilde{F}))$ and (4.2) belongs to $\mathrm{Id}(K(\tilde{F}))$. But $K(\tilde{F})$ is finitely based so by (iv) $R(K(\tilde{F})) = R_{\tilde{F}}(K)$ is finitely based. By Lemma 5 $T_1(\tilde{F}')$ is finitely based. We have $\mathrm{Id}(K_{\tilde{F}}) = R_{\tilde{F}}(K) \cup T_1(\tilde{F}')$. So $K_{\tilde{F}}$ is finitely based.

5. APPLICATIONS

Theorem 6. *Let* K *be the variety of all groups,* $F = \{\cdot, {}^{-1}\}$. *Then*

1° $K_U = K \otimes K_{\{-1\}} \otimes K^1$,

2° $K_B = K_S \otimes K_{\{-1\}} \otimes K^1$,

3° $K_{\{-1\}}$ *is a variety of algebras in which the operation* ${}^{-1}$ *is an involution and the value of* $x \,.\, y$ *is an absorbing element,*

4° K^1 *is the variety of algebras in which* $\cdot$ *is a semigroup operation and the value of* x^{-1} *is an absorbing element,*

5° *Both* K_U *and* K_B *are finitely based.*

Proof. Put $q(x) = x \,.\, x \,.\, x^{-1}$, $q_{\{-1\}}(x) = (x^{-1})^{-1}$. By Corollary 15 we have 1° where $K_{\{-1\}}$ is defined by $R_{\{-1\}}(K) \cup T_1(\{\cdot\})$ and K^1 is defined by

$$T_1(\{^{-1}\}) \cup (B(K) \setminus T_1(\{^{-1}\})).$$

But $R_{\{-1\}}(K)$ is finitely based since it is enough to derive it from

$$(5.1) \qquad (x^{-1})^{-1} = x\,.$$

Similarly $B(K) \setminus T_1(\{^{-1}\})$ has a finite equational base. In fact, this set contains only identities $\varphi = \psi$ where φ and ψ contain only the symbol $\cdot$ so φ and ψ can differ only in brackets (see [7] Lemma 7.1.1. p. 91). So we can accept

$$(5.2) \qquad (x \,.\, y) \,.\, z = x \,.\, (y \,.\, z)$$

as a base of $B(K) \setminus T(\{^{-1}\})$.

Since K is finitely based, so by the second statement of Corollary 15 K_U is finitely based.

By Lemma 5 it is now easy to see that $K_{\{-1\}}$ is defined by (5.1) and the identities $x \,.\, y = u \,.\, v = (x \,.\, y)^{-1} = x \,.\, y^{-1} = x^{-1} \,.\, y$.

So we get 3°. Similarly in every algebra from K^1 the operation $\cdot$ satisfies (5.2) and x^{-1} defines an absorbing element since K^1 satisfies

$$x^{-1} = y^{-1} = (x \,.\, y)^{-1} = x \,.\, y^{-1} = x^{-1} \,.\, y \,.$$

So we get 4°. Put $\varphi(x, y) = x \,.\, y \,.\, y^{-1}$. Then by Corollary 13 we get 2° and K_B is finitely based.

Theorem 7. *Let K be the variety of all abelian groups where $F = \{\cdot, {}^{-1}\}$. Then*

1° $K_U = K \otimes K_{\{-1\}} \otimes K_a^1$

2° $K_B = K_S \otimes K_{\{-1\}} \otimes K_a^1$

3° *K_a^1 is the variety of algebras in which $\cdot$ is a commutative semigroup operation and x^{-1} determines an absorbing element,*

4° *Both K_U and K_B are finitely based.*

The proof is similar to the proof of Theorem 7.

One should only remember that in the variety of abelian groups to each term φ such that $F(\varphi) = \{\cdot\}$ there corresponds the unique canonical form.

Remark 3. The case of lattices was solved in [16] (see Corollary 24 below). In [16] it was derived from representation theorems of more general varieties those of bisemilattices. Here we derive Corollary 24 from results of this paper.

Corollary 24. *Let K be a variety of lattices (not necessarily of all lattices) where $F = \{+, \cdot\}$.*

Then

1° $K_U = K \otimes K_{\{+\}} \otimes K_{\{\cdot\}} \otimes K^{(1)}$,

2° $K_B = K_S \otimes K_{\{+\}} \otimes K_{\{\cdot\}} \otimes K^{(1)}$,

3° *The variety $K^{(1)}$ is defined by $x + y = u \,.\, v$ i.e. in every algebra from $K^{(1)}$ the values of $x + y$ and $x \,.\, y$ are equal to the same constant.*

4° *Each algebra from $K_{\{+\}}$ is a join semilattice with respect to $+$ and the value of $x \,.\, y$ is the unit.*

5° *Each algebra from $K_{\{\cdot\}}$ is a meet semilattice with respect to $\cdot$ and the value of $x + y$ is the 0-element.*

6° *Both K_U, K_B are finitely based if K is finitely based.*

Proof. Put $q_{\{+\}}(x) \equiv x + x$, $q_{\{\cdot\}}(x) \equiv x \,.\, x$. We have

$$(5.4) \qquad (x + x \,.\, y = x) \in \mathrm{Id}(K)\,.$$

So by Corollaries 21 and 23 we have 1° and 2°. The variety $K^{(1)}$ is defined by $T_1(\{+, \cdot\})$. So 3° follows from Lemma 5. We have $R_{\{+\}}(K) = R(K(\{+\}))$ and $K(\{+\})$ is the variety of $\{+\}$ join semilattices. This follows from the fact that $R(K(\{+\})$ contains all join semilattice identities and cannot contain any other identity. In fact the variety of join semilattices is equationally complete and $K(\{+\})$ is not degenerate. So $R_{\{+\}}$ is implied by

$$x + x = x\,, \quad x + y = y + x\,, \quad x + (y + z) = (x + y) + z$$

and is finitely based. In any algebra from $K_{\{+\}}$ the value of $x \,.\, y$ is an absorbing element so it must be the unit and we get 4°.

Similarly we prove that $R_{\{\cdot\}}$ is finitely based and 5° holds.

Since $R_{\{+\}}$ and $R_{\{\cdot\}}$ are finitely based and (5.4) holds so by the second statements of Corollaries 21 and 23 we get 6°.

Theorem 8. *Let K be the variety of Boolean Rings where $F = \{+, \cdot\}$. Then*

1° $K_U = K \otimes K_{\{+\}} \otimes K_{\{\cdot\}} \otimes K^{(1)}$,

2° $K_B = K_S \otimes K_{\{+\}} \otimes K_{\{\cdot\}} \otimes K^{(1)}$.

3° $K^{(1)}$ *satisfies the condition* 3° *from Corollary* 24.

4° *Each algebra from $K_{\{+\}}$ is the $\{+\}$-absorbing sum of a semilattice ordered system of algebras from $K(\{+\})$, where $K(\{+\})$ is the variety of Boolean Groups.*

5° *The variety $K_{\{\cdot\}}$ satisfies the condition* 5° *from Corollary* 24.

6° *Both K_U, K_B are finitely based.*

Proof. Put $q_{\{+\}}(x) \equiv 3x$, $q_{\{\cdot\}}(x) \equiv x^2$. The identity

$$(5.5) \qquad x + 2y = x$$

belongs to $\mathrm{Id}(K)$.

Hence by Corollaries 21 and 23 we get 1° and 2°.

Proofs of 3° and 5° are similar to 3° and 5° from Corollary 24.

Obviously $R_{\{\cdot\}}$ is finitely based since it is implied by

$$x \,.\, x = x\,, \quad x \,.\, y = y \,.\, x\,, \quad x \,.\, (y \,.\, z) = (x \,.\, y) \,.\, z\,.$$

By (5.5) and Theorem 5 we get 4°.

By (5.5) and (iv) $K_{\{+\}}$ is finitely based. Since K is finitely based so by (5.5), Corollaries 21 and 23 we get 6°.

Theorem 9. *Let K be the variety of Boolean Algebras where $F = \{+, \cdot, '\}$. Then*

1° $K_U = K \otimes K_{\{+,\cdot\}} \otimes K_{\{+,'\}} \otimes K_{\{\cdot,'\}} \otimes K_{\{+\}} \otimes K_{\{\cdot\}} \otimes K_{\{'\}} \otimes K^{(2)}$.

2° $K_B = K_S \otimes K_{\{+,\cdot\}} \otimes K_{\{+,'\}} \otimes K_{\{\cdot,'\}} \otimes K_{\{+\}} \otimes K_{\{\cdot\}} \otimes K_{\{'\}} \otimes K^{(2)}$.

3° *Each algebra from $K_{\{+,\cdot\}}$ is the $\{+,\cdot\}$-absorbing sum of a semilattice ordered system of distributive lattices.*

4° *Each algebra from $K_{\{+,'\}}$ is the $\{+,'\}$-absorbing sum of a semilattice ordered system of Boolean Algebras with fundamental operations $+,'$.*

5° *Each algebra from $K_{\{\cdot,'\}}$ is the $\{\cdot,'\}$-absorbing sum of a semilattice ordered system of Boolean Algebras with fundamental operations $\cdot,'$.*

6° *Each algebra from $K_{\{+\}}$ is a join semilattice with respect to $+$, where $x\,.\,y = u'$ defines the unit.*

7° *Each algebra from $K_{\{\cdot\}}$ is a meet semilattice with respect to $\cdot$, where $x + y = u'$ defines the 0-element.*

8° *In each algebra from $K_{\{'\}}$ the operation x' is an involution and the operations $x + y$ and $x\,.\,y$ define the absorbing element.*

9° *In each algebra from $K^{(2)}$ the values of all operations $+$, $\cdot$ and $'$ are equal to the same constant.*

10° *Both K_U and K_B are finitely based.*

Proof. Put $q_{\{+\}}(x) \equiv x + x$, $q_{\{\cdot\}}(x) \equiv x\,.\,x$, $q_{\{'\}}(x) \equiv (x')'$. The following identities belong to $\mathrm{Id}(K)$:

(5.6) $$x + xy = x$$

(5.7) $$x + (x' + y')' = x$$

(5.8) $$x\,.\,(x'\,.\,y')' = x\,.$$

By (5.6) and Corollaries 21 and 23 we get 1° and 2°.

The variety of distributive lattices is equationaly complete so $R_{\{+,\cdot\}} = R(K(\{+,\cdot\}))$ contains only regular identities of distributive lattices. By (5.6) and Theorem 5 we get 3°. By (5.6) and (iv) the set $R_{\{+,\cdot\}}$ has a finite equational base.

The variety of Boolean Algebras where $F = \{+,'\}$ is equationaly complete so $R_{\{+,'\}} = R(K(\{+,'\}))$ contains only regular identities of Boolean Algebras containing symbols $+$ and $'$ only. By (5.7) and Theorem 5 we get 4°. But the variety $K(+,')$ is finitely based since the theory $\mathrm{Id}(K(\{+,'\}))$ is equivalent to $\mathrm{Id}(K(\{+,\cdot,'\}))$ By (5.7) and (iv) the set $R_{\{+,'\}}$ has a finite equational base.

The proof of 5° and the proof that $R_{\{\cdot,'\}}$ has a finite equational base is similar.

The proof of 6° and the proof that $R_{\{+\}}$ has a finite equational base is similar to that of 4° in Corollary 24.

The proof of 7° and the proof that $R_{\{\cdot\}}$ has a finite equational base is also similar.

$R_{\{'\}} = R(K(\{'\}))$ contains only regular identities with the symbol $'$, which are true in Boolean Algebras. It is easy to see that the identity

$$(x')' = x$$

forms a base of $R_{\{'\}}(K)$. The values of $x + y$ and $x\,.\,y$ determine the absorbing element by the definition of $K_{\{'\}}$.

The proof of 9° and 10° is obvious by Corollaries 21 and 23.

We shall give an example of a variety K such that K is finitely based and K_B is not. Let us consider a type $\tau_0\colon F_0 \to N$ where $F_0 = \{f_1, f_2, f_3\}$ and $\tau_0(f_k) = 1$ for $k = 1, 2, 3$. We define K to be the variety of type τ_0 defined by the single identity

$$f_1(f_2(f_3(x))) = x\,. \tag{6.1}$$

Put $\gamma(x) \equiv f_1(f_2(f_3(x)))$. For a term φ of type τ_0 we denote by φ^* the term obtained from φ by substituting in it every subterm $\gamma(\psi)$ by ψ as many times as possible. E.g. $(f_1(f_1(f_2(f_3(f_2(f_3(f_2(x))))))))^* \equiv f_2(x)$.

Lemma 7. *For every term φ of type τ_0 the term φ^* is uniquely determined i.e. its structure does not depend on an order of substituting $\gamma(\psi)$ by ψ.*

Proof. Let

$$\varphi \equiv f_{i_1}(f_{i_2}(\dots(f_{i_s}(x))\dots))\,.$$

We use induction on s. For $s = 1$ the statement is obvious. Assume that the statement is true for each $s' < s$. Let $i_k = 1$, $i_{k+1} = 2$, $i_{k+3} = 3$, $i_r = 1$, $i_{r+1} = 2$, $i_{r+2} = 3$ for some $k, r \in \{1, \dots, s\}$.Then $\{k, k+1, k+2\} = \{r, r+1, r+2\}$ or

$$\{k, k+1, k+2\} \cap \{r, r+1, r+2\} = \emptyset\,. \tag{6.2}$$

We can assume $k < r$. If in the first step we cancel the string $f_{i_k} f_{i_{k+1}} f_{i_{k+2}}$ then we get a term

$$\psi_1 \equiv f_{i_1}(\dots(f_{i_{k-1}}(f_{i_{k+3}}(\dots(f_{i_s}(x))\dots)))\dots)\,.$$

or

$$\psi_1 \equiv f_{i_{k+3}}(\dots(f_{i_s}(x))\dots)\,.$$

By the induction hypothesis ψ_1^* is unique. In particular we get ψ_1^* if we have (6.2) and we cancel the string $f_{i_r} f_{i_{r+1}} f_{i_{r+2}}$ in ψ_1.

Assume that (6.2) holds and in the first step we cancel in φ the string $f_{i_r} f_{i_{r+1}} f_{i_{r+2}}$. Then we obtain

$$\psi_2 \equiv f_{i_1}(\dots(f_{i_{r-1}}(f_{i_{r+3}}(\dots(f_{i_s}(x)\dots)\,.$$

Then ψ_2^* is unique. In particular we get ψ_2^* if we cancel $f_{i_k} f_{i_{k+1}} f_{i_{k+2}}$ in ψ_2.

But $(f_{i_1}(\dots(f_{i_{k-1}}(f_{i_{k+3}}(\dots(f_{i_{r-1}}(f_{i_{r+3}}(\dots(f_{i_s}(x)\dots)^*$ is unique by assumption so $\psi_1^* \equiv \psi_2^*$ and consequently φ^* is unique.

Let us say that an identity $\varphi = \psi$ has the property $(*)$ if $\varphi^* \equiv \psi^*$.

Lemma 8. *The set* $\mathrm{Id}(K)$ *consists exactly of all identities $\varphi = \psi$ which have the property* $(*)$.

In fact, denote by S the set of all identities $\varphi = \psi$ having the property $(*)$. If $(\varphi = \psi) \in S$ then using (6.1) we can prove

$$\varphi = \varphi^* \equiv \psi^* = \psi \quad \text{so} \quad (\varphi = \psi) \in \mathrm{Id}(K)\,.$$

Using Birkhoff's derivation rules we shall prove that $\mathrm{Id}(K) \subseteq S$. Denote by E the set consisting exactly of (6.1). It is enough to show that $E \subseteq S$ and S is closed under derivation rules.

1° Obviously $(6.1) \in S$.

2° Every identity $\varphi = \varphi$ belongs to S.

3° If $(\varphi = \psi) \in S$ then $\psi^* \equiv \varphi^*$ so $(\psi = \varphi) \in S$.

4° Similarly if $(\varphi = \psi) \in S$ and $(\psi = \chi) \in S$, then $(\varphi = \chi) \in S$.

5° If $(\varphi = \psi) \in S$ and $f_i \in F_0$ then $(f_i(\varphi))^* \equiv (f_i(\varphi^*))^* \equiv (f_i(\psi^*))^* \equiv (f_i(\psi))^*$ by Lemma 7. So $(f_i(\varphi) = f_i(\psi)) \in S$.

6° If $(\varphi(x) = \psi(x)) \in S$ then for every term φ_1 of type τ_0 we have $(\varphi(\varphi_1))^* \equiv \equiv (\varphi^*(\varphi_1))^* \equiv (\psi^*(\varphi_1))^* \equiv (\psi(\varphi_1))^*$, by Lemma 7.

Thus $\mathrm{Id}(K) \subseteq S$.

From Lemma 8 we get

Lemma 9. *An identity $\varphi = \psi$ belongs to $B(K) = U(K)$ iff $F(\varphi) = F(\psi)$, $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$ and $\varphi^* \equiv \psi^*$. Moreover if $F(\varphi) = F(\psi) \neq F_0$ then $\varphi \equiv \varphi^* \equiv \psi^* \equiv \psi$.*

Let us denote

$$\gamma_n(x) \equiv f_1(\underbrace{f_2(\dots(f_2}_{n\text{-times}}(f_3(x)\dots), \quad (n \geqq 1). \tag{6.3}$$

A term of the form (6.3) will be called n-medial.

Remark 4. By Lemma 9 if $(\varphi = \psi) \in B(K)$ and $\varphi \not\equiv \psi$ then at most one side can be an n-medial term.

Let $(\varphi = \psi) \in B(K)$. We put

$$m(\varphi = \psi) = \begin{cases} n & \text{if} \quad \varphi \equiv \gamma_n(x) \quad \text{and} \quad \varphi \not\equiv \psi\,, \\ n & \text{if} \quad \psi \equiv \gamma_n(x) \quad \text{and} \quad \varphi \not\equiv \psi\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

By Remark 4 the number $m(\varphi = \psi)$ is well defined. For $H \subseteq B(K)$ put

$$m(H) = \begin{cases} 0 & \text{if} \quad H = \emptyset\,, \\ \sup\{m(e)\colon e \in H\} & \text{if} \quad H \neq \emptyset\,. \end{cases}$$

Lemma 10. *For every $H \subseteq B(K)$ where $|H| < \aleph_0$ we have $m(C(H)) = m(H)$.*

Proof. Denote $m_0 = m(H)$. If $H = \emptyset$ then $(\varphi = \psi) \in C(H)$ iff $\varphi \equiv \psi$. Thus $m(\varphi = \psi) = 0$ and Lemma 10 holds. Let $m_0 > 0$. To prove Lemma 10 it is enough to show that if $S' \subseteq B(K)$, $m(S') \leqq m_0$ and an identity,

$$\varphi = \psi$$

is derived from S' by one of the Birkhoff's derivation rules then $m(\phi = \psi) \leqq m_0$.

1° If $\varphi \equiv \psi$ then $m(\varphi = \psi) = 0 < m_0$.

2° If $(\varphi = \psi) \in S'$ then $m(\psi = \varphi) = m(\varphi = \psi) \leqq m_0$.

3° If $(\varphi = \psi) \in S'$ and $(\psi = \chi) \in S'$ then $\varphi^* \equiv \psi^* \equiv \chi^*$ and using Remark 4 one can check that $m(\varphi = \chi) \in \{0, \max\{m(\varphi = \psi), m(\psi = \chi)\}\}$.

4° Let $(\varphi = \psi) \in S'$ and $f_i \in F_0$. If $\varphi \equiv \psi$ then $f_i(\varphi) \equiv f_i(\psi)$ and $m(f_i(\varphi) = f_i(\psi)) = 0 < m_0$. If $\varphi \not\equiv \psi$ then neither $f_i(\varphi)$ nor $f_i(\psi)$ is an n-medial term for some n so $m(f_i(\varphi) = f_i(\psi)) = 0 < m_0$.

5° If $(\varphi(x) = \psi(x)) \in S'$ and χ is a term of type τ_0 different from a variable, then $m(\varphi(\chi) = \psi(\chi)) = 0 < m_0$ in all cases.

Thus $m(C(H)) = m(H)$.

Theorem 10. *The variety K_B is not finitely based.*

Proof. Let $H \subseteq B(K)$ be a finite set of identities and $m_0 = m(H)$. Consider an identity

$$\gamma_1(\gamma_{m_0+1}(x)) = \gamma_{m_0+1}(x)\,. \tag{6.4}$$

Then $m(\gamma_1(\gamma_{m_0+1}(x)) = \gamma_{m_0+1}(x)) = m_0 + 1 > m_0$. But $(6.4) \in B(K)$ and $(6.4) \notin C(H)$ by Lemma 10. Thus no finite subset $H \subseteq B(K)$ is an equational base of K_B.

Remark 5. It was proved in section 2 Lemma 2 that if $|F| = 2$ then $T(\{F\})$ has a finite equational base. If $|F| = 1$, $F = \{f_1\}$ then $T(\{F\})$ has a base $\{f_1(x_1, \ldots, x_{n_1}) = f_1(y_1, \ldots, y_{n_1})\}$.

From Theorem 10 it follows that if $|F| = 3$ then $T(\{F\})$ need not be finitely based. In fact, take the variety K defined by (6.1). By Theorem 10 $U(K)$ is not finitely based. However K is finitely based and $B(K) \setminus T(\{F_0\})$ is finitely based. In fact by Lemma 9 it contains only identities of the form $\varphi = \varphi$, so we can accept $\emptyset$ as its base. Now as K is finitely based K_U is not finitely based and $B(K) \setminus T(\{F_0\})$ is finitely based so by Corollary 2 $T(\{F_0\})$ is not finitely based.

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