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# NONNEGATIVE NONINCREASING SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE 3RD ORDER

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**1.** Existence of nonnegative nonincreasing solutions of differential equations of the $n$-th order ($n \geqq 2$) was proved in [4] under the assumption that the right-hand side of the equation does not change its sign. New sufficient conditions for the existence of such solutions without the above assumption were found in [6].

In this paper we consider the problem

$$(1.1) \qquad u''' = f(t, u, u', u''),$$

$$(1.2) \qquad u(t) \geqq 0, \quad u'(t) \leqq 0, \quad u''(t) \geqq 0 \quad \text{for} \quad t \in R_+,$$

$$(1.3) \qquad \varphi(u(0), u'(0), u''(0)) = 0.$$

Here, for $n = 3$, we prove more general conditions for solvability of (1.1), (1.2), (1.3) than in [6].

We shall use the following notation:

$$R = (-\infty, \infty), \quad R_+ = \langle 0, \infty), \quad R_- = (-\infty, 0\rangle,$$

$$D^3 = R_+ \times R_- \times R_+, \quad D^3_r = R_+ \times \langle -r, 0\rangle \times R_+,$$

$C(J)$ is the set of all real continuous functions on $J$,

$L_{\text{loc}}(J)$ is the set of all real functions which are Lebesgue-integrable on each segment contained in $I$,

$AC^2(J)$ is the set of all real functions which are absolutely continuous with their second derivatives on $I$,

$\text{Car}_{\text{loc}}(J \times I)$ is the set of all functions $f\colon J \times I \to R$ satisfying the local Carathéodory conditions on each segment contained in $J$, i.e.

$$f(\cdot, x_1, x_2, x_3)\colon J \to R \quad \text{is measurable for every} \quad (x_1, x_2, x_3) \in I,$$

$$f(t, \cdot, \cdot, \cdot)\colon I \to R \quad \text{is continuous for almost every} \quad t \in J,$$

$$\sup \{|f(\cdot, x_1, x_2, x_3)|\colon \sum_{i=1}^{3} |x_i| \leqq \varrho\} \in L_{\text{loc}}(J) \quad \text{for any} \quad \varrho \in R_+.$$

In what follows we shall assume

$$(1.4)\qquad f \in \mathrm{Car}_{\mathrm{loc}}(R_+ \times D^3)\,, \quad f(t, 0, 0, 0) = 0\,,$$
$$f(t, x_1, x_2, 0) \leqq 0 \quad \text{on} \quad R_+ \times D^3\,,$$

$$(1.5)\qquad \varphi \in C(D^3)\,, \quad \varphi(0, 0, 0) < 0\,,$$
$$\varphi(x_1, x_2, x_3) > 0 \quad \text{for} \quad |x_2| > r\,, \quad r \in (0, \infty)\,.$$

We shall find solutions of the problem (1.1), (1.2), (1.3) in the set $AC^2(R_+)$.

Remark. a) In the special case $\varphi(x_1, x_2, x_3) = |x_2| - r$ the condition (1.3) reduces to $|u'(0)| = r$.

b) In [11] we proved existence theorems for (1.1)–(1.3) under the assumption $\varphi(x_1, x_2, x_3) > 0$ for $x_1 > r$.

c) Similar problems for differential systems were solved in [1, 2, 3, 7, 8, 9, 10].

**Theorem.** *Let the conditions* (1.4) *and* (1.5) *be fulfilled. Let there exist* $a_0, a \in (0, \infty)$, $a_0 < a$, $\alpha \in R_+$, $k_1, k_2 \in N$, *functions* $h_i \in L_{\mathrm{loc}}(\langle a, \infty))$, $i = 0, 1, 2$, *positive functions* $h \in L(\langle 0, a_0\rangle)$ *and* $\omega \in C(R_+)$ *satisfying*

$$(1.6)\qquad \int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = +\infty$$

*and*

$$(1.7)\qquad \int_0^{a_0} \frac{\mathrm{d}t}{H(t)} = +\infty\,, \quad \text{where} \quad H(t) = \int_0^t h(\tau)\,\mathrm{d}\tau\,,$$

*and a function* $\delta: \langle 0, a\rangle \times R_+ \to R_+$ *such that*

$$(1.8)\qquad \begin{cases} \delta(.\,, x) \in L(\langle 0, a\rangle) & \text{for any} \quad x \in R_+\,, \\ \delta \text{ is nondecreasing in its second argument}\,, \\ \lim_{x\to\infty} \int_0^a t\delta(t, x)\,\mathrm{d}t > r\,, \end{cases}$$

*and the following inequalities are satisfied:*

*on the set* $\langle 0, a\rangle \times D_r^3$ *the inequality*

$$(1.9)\qquad f(t, x_1, x_2, x_3) \leqq -\delta(t, x_1)\,,$$

*on the set* $\langle 0, a_0\rangle \times D_r^3$ *the inequality*

$$(1.10)\qquad f(t, x_1, x_2, x_3) \geqq -h(t)\,(1 + x_3)^2$$

*and on the set* $(a, \infty) \times D_r^3$ *the inequality*

$$(1.11)\qquad f(t, x_1, x_2, x_3) \leqq \Big[h_0(t) + \sum_{i=1}^2 h_i(t)\,|x_i|^{k_i} + \alpha x_3\Big]\,\omega(x_3)\,.$$

*Then the problem* (1.1), (1.2), (1.3) *has at least one solution.*

Remark. The assumptions (1.8) and (1.9) are essential and they cannot be omitted.

For example, the problem

$$u'''(t) = 0\,, \quad u(t) \geqq 0\,, \quad u'(t) \leqq 0\,, \quad u''(t) \geqq 0\,, \quad u'(0) = -r\,,$$
$$\text{for} \quad t \in R_+\,,$$

has no solution though all assumptions of Theorem except (1.8) and (1.9) are fulfilled.

**Corollary.** *Let the conditions* (1.4), (1.5), (1.8) *and* (1.9) *be fulfilled. Let there exist a function* $h \in L_{loc}(R_+)$ *such that*

$$|f(t, x_1, x_2, x_3)| \leqq h(t)\,(1 + \sum_{i=1}^{3} |x_i|)$$

*on the set* $R_+ \times D_r^3$. *Then the problem* (1.1), (1.2), (1.3) *has at least one solution.*

**2.** In what follows we shall need some lemmas.

**Lemma 1.** *Suppose that* $a_0, a, r \in (0, \infty)$, $a_0 < a$, $\alpha \in R_+$, $k_1, k_2 \in N$, $h_0, h_1, h_2 \in L_{loc}(R_+)$ *are nonnegative functions* $\omega \in C(R_+)$ *is a positive function satisfying* (1.6), $h \in L(\langle 0, a_0 \rangle)$ *is a positive function satisfying* (1.7) *and* $\delta_0 \colon \langle 0, a \rangle \times R_+ \to R_+$ *is a function satisfying* (1.8).

*Then there exists* $r^* \in \langle r, \infty)$ *such that for any* $c \in (a, \infty)$ *and* $v \in AC^2(\langle 0, c \rangle)$ *the inequalities*

$$v''' \leqq -\delta_0(t, v(t)) \quad \textit{for} \quad 0 \leqq t \leqq a\,, \tag{2.1}$$

$$v''' \geqq -h(t)\,(1 + v''(t))^2 \quad \textit{for} \quad 0 \leqq t \leqq a_0\,, \tag{2.2}$$

$$v''' \leqq \big[h_0(t) + \sum_{i=1}^{2} h_i(t)\,|v^{(i-1)}(t)|^{k_i} + \alpha\, v''(t)\big]\, \omega(v''(t)) \quad \textit{for} \quad a < t \leqq c\,, \tag{2.3}$$

$$v'(0) = r\,, \quad v(t) \geqq 0\,, \quad v'(t) \leqq 0\,, \quad v''(t) \geqq 0 \quad \textit{for} \quad 0 \leqq t \leqq c \tag{2.4}$$

*imply the estimates*

$$v(t) \leqq r^*\,, \quad v'(t) \geqq -r^*\,, \quad v''(t) \leqq \Omega^{-1}(r^* + r^* \textstyle\int_0^t (\sum_{i=0}^{2} h_i(\tau))\, d\tau)\,, \tag{2.5}$$

$$\textit{where} \quad \Omega(x) = \int_0^x \frac{ds}{\omega(s)}\,.$$

Proof. The conditions (2.4) imply

$$-r \leqq v'(t) \leqq 0 \quad \text{for} \quad 0 \leqq t \leqq c\,. \tag{2.6}$$

Due to (1.8) there exists $r_0 \in \langle r, \infty)$ such that

$$\textstyle\int_0^a t\, \delta_0(t, r_0) > r\,. \tag{2.7}$$

Integrating (2.1) we obtain $v''(t) \geqq \int_0^a \delta_0(\tau, v(a))\, d\tau$ and $v'(a) - v'(0) \geqq$
$\geqq \int_0^a t\, \delta_0(t, v(a))\, dt$. Thus

$$r \geqq \textstyle\int_0^a t\, \delta_0(t, v(a))\, dt\,. \tag{2.8}$$

According to (1.8), (2.7) and (2.8) we have $v(a) < r_0$. Since $v(0) = v(a) + \int_0^a |v'(\tau)| \, d\tau$ and (2.4) hold, we have

(2.9) $$0 \leqq v(t) \leqq r_1 \quad \text{for} \quad 0 \leqq t \leqq c, \quad \text{where} \quad r_1 = (1 + r_0)(1 + a).$$

According to the Lagrange Theorem there exists $t_1 \in (0, a_0)$ such that $v''(t_1) = (v'(a_0) - v'(0))/a_0 \leqq r/a_0$, and by (2.1) $v''(t) \leqq r/a_0$ for $t_1 \leqq t \leqq a$. From (2.2) it follows that

(2.10) $$(1 + v''(t))' \geqq -h(t)(1 + v''(t))^2 \quad \text{for} \quad 0 \leqq t \leqq a_0.$$

Let us consider the differential equation

(2.11) $$\varrho'(t) = -h(t)\,\varrho^2(t) \quad \text{for} \quad 0 \leqq t \leqq a_0.$$

Integrating (2.11) from 0 to $t$ we get $\varrho(t) = (1/\varrho(0) + H(t))^{-1}$, where $H(t) = \int_0^t h(\tau) \, d\tau$. According to (1.7) there exists $\varepsilon \in (0, 1)$ and $a_1 \in (0, a_0)$ such that

(2.12) $$\int_{a_1}^{a_0} (\varrho(t) - 1) \, dt > r, \quad \text{where} \quad \varrho(0) = 1/\varepsilon.$$

Let us suppose that $1 + v''(t) \geqq \varrho(t)$ for $a_1 \leqq t \leqq a_0$. Then by (2.12) we get

(2.13) $$\int_{a_1}^{a_0} v''(t) \, dt > r.$$

On the other hand, the equality $v'(a_0) - v'(a_1) = \int_{a_1}^{a_0} v''(t) \, dt$ implies by (2.6) that $\int_{a_1}^{a_0} v''(t) \, dt \leqq r$, which contradicts (2.13). Thus it is necessary that there exists $t_0 \in (a_1, a_0)$ such that

(2.14) $$1 + v''(t_0) \leqq \varrho(t_0).$$

Using the Chaplygin Lemma (see [5] or [11]) we get from (2.10), (2.11) and (2.14) that $1 + v''(t) \leqq \varrho(t) \leqq \varrho(0) = 1/\varepsilon$ for $0 \leqq t \leqq t_0$, and by virtue of (2.1) we have

(2.15) $$1 + v''(t) \leqq 1/\varepsilon \quad \text{for} \quad 0 \leqq t \leqq a.$$

Integrating (2.3) from $a$ to $t$ and putting $k = \max\{k_1, k_2\}$ we have

$$\Omega(v''(t)) \leqq \Omega(v''(a)) + r_1^k \int_a^t \sum_{i=0}^{2} h_i(\tau) \, d\tau + \alpha(v'(t) - v'(a)),$$

thus

(2.16) $$v''(t) = \Omega^{-1}\Big(r^* + r^* \int_0^t \Big(\sum_{i=0}^{2} h_i(\tau) \, d\tau\Big)\Big) \quad \text{for} \quad 0 \leqq t \leqq c,$$

where $r^* = \Omega(1/\varepsilon) + r_1^k + \alpha r$. (2.6), (2.9) and (2.16) yield (2.5).

**Lemma 2.** *Let $f \in \mathrm{Car}_{\mathrm{loc}}(R_+ \times R^3)$ be a function satisfying*

(2.17) $$f(t, 0, 0, 0) = 0, \quad f(t, x_1, x_2, 0) \leqq 0 \quad \text{on} \quad R_+ \times R^3.$$

*Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of functions $f_k \in \mathrm{Car}_{\mathrm{loc}}(R_+ \times R^3)$ satisfying (2.17) and the Lipschitz condition*

(2.18) $$|f_k(t, x_1, x_2, x_3) - f_k(t, y_1, y_2, y_3)| \leqq v_{k\varrho}(t) \sum_{i=1}^{3} |x_i - y_i|$$

*for $\varrho \in (0, \infty)$, $t \in R_+$, $|x_i| \leqq \varrho$, $|y_i| \leqq \varrho$ $(i = 1, 2, 3)$, $v_{k\varrho} \in L_{\mathrm{loc}}(R_+)$, $k \in N$, such* *that for any fixed* $t \in R_+$, $\{f_k\}_{k=1}^{\infty}$ *is uniformly convergent to* $f$ *on each compact subset of* $R^3$.

Proof. Let us consider functions $w_k: R \to R_+$, $w_k \in C_1(R)$, $k \in N$, such that $w_k(x) = 0$ for $|x| \geqq 1/k$, $\int_{-\infty}^{\infty} w_k(x)\,dx = 1$. Let us put

$$g_k(t, x_1, x_2, x_3) = \int_{-\infty}^{\infty} w_k(z_1 - x_1) \int_{-\infty}^{\infty} w_k(z_2 - x_2) \,.$$
$$. \int_{-\infty}^{\infty} w_k(z_3 - x_3) f(t, z_1, z_2, z_3)\,dz_3\,dz_2\,dz_1\,,$$
$$h_k(t, x_1, x_2) = \int_{-\infty}^{\infty} w_k(z_1 - x_1) \int_{-\infty}^{\infty} w_k(z_2 - x_2) f(t, z_1, z_2, 0)\,dz_2\,dz_1$$

and

$$f_k(t, x_1, x_2, x_3) = g_k(t, x_1, x_2, x_3) - g_k(t, x_1, x_2, 0) - \tag{2.19}$$
$$- |h_k(t, x_1, x_2) - h_k(t, 0, 0)|\,, \quad k \in N\,.$$

Due to (2.17), for any $\varrho \in (0, \infty)$ and $k \in N$ $f_k(t, 0, 0, 0) = 0$, $f_k(t, x_1, x_2, x_3) \leqq 0$ on $R_+ \times R^3$, and for $|x_i| \leqq \varrho$, $t \in R_+$ $(i = 1, 2, 3)$ we have

$$|g_k(t, x_1, x_2, x_3)| = \left|\int_{-\infty}^{\infty} w_k(z_1 - x_1) \int_{-\infty}^{\infty} w_k(z_2 - x_2) \,.\right.$$
$$. \int_{-\infty}^{\infty} w_k(z_3 - x_3) f(t, z_1, z_2, z_3)\,dz_3\,dz_2\,dz_1 \leqq$$
$$\leqq \int_{-1/k}^{1/k} w_k(p_1) / \int_{-1/k}^{1/k} w_k(p_2) \int_{-1/k}^{1/k} w_k(p_3)\, h_{k\varrho}(t)\,dp_3\,dp_2\,dp_1\,,$$

where $p_i = z_i - x_i$, $(i = 1, 2, 3)$ and

$$h_{k\varrho}(t) = \sup\{|f(t, p_1 + x_1, p_2 + x_2, p_3 + x_3)| : |p_i| \leqq 1/k, |x_i| \leqq \varrho$$
$$(i = 1, 2, 3)\} \in L_{\mathrm{loc}}(R_+)\,.$$

Thus for $|x_i| \leqq \varrho$, $|y_i| \leqq \varrho$ $(i = 1, 2, 3)$, $t \in R_+$ we have

$$|f_k(t, x_1, x_2, x_3) - f_k(t, y_1, y_2, y_3)| \leqq$$
$$\leqq h_{k\varrho}(t)\left\{\int_{-1/k}^{1/k} \int_{-1/k}^{1/k} \int_{-1/k}^{1/k} |(w_k(z_1 - x_1) - w_k(z_1 - y_1)| \,.\right.$$
$$. |w_k(z_2 - x_2) - w_k(z_2 - y_2)| \times$$
$$\times |w_k(z_3 - x_3) - w_k(z_3 - y_3)|\,dz_3\,dz_2\,dz_1 +$$
$$+ 2 \int_{-1/k}^{1/k} \int_{-1/k}^{1/k} |w_k(z_1 - x_1) - w_k(z_1 - y_1)| \,.$$
$$\left. . |w_k(z_2 - x_2) - w_k(z_2 - y_2)|\,dz_1\,dz_2\right\}\,.$$

Since $w_k \in C_1(R)$ we get from the above inequality

$$|f_k(t, x_1, x_2, x_3) - f_k(t, y_1, y_2, y_3)| \leqq v_{k\varrho}(t) \sum_{i=1}^{3} |x_i - y_i|\,,$$

where $v_{k\varrho}(t) \in L_{\mathrm{loc}}(R_+)$. Further, $\lim_{k\to\infty} g_k(t, x_1, x_2, x_3) = f(t, x_1, x_2, x_3)$ for any $t \in R_+$ uniformly on each compact subset of $R^3$ because

$$|g_k(t, x_1, x_2, x_3) - f(t, x_1, x_2, x_3)| \leqq$$
$$\leqq \int_{-\infty}^{\infty} w_k(z_1 - x_1) \int_{-\infty}^{\infty} w_k(z_2 - x_2) \int_{-\infty}^{\infty} w_k(z_3 - x_3) \,.$$
$$. |f(t, z_1, z_2, z_3) - f(t, x_1, x_2, x_3)|\,dz_3\,dz_2\,dz_1\,.$$

Similarly $\lim_{k\to\infty} h_k(t, x_1, x_2) = f(t, x_1, x_2, 0)$. Therefore by (2.19) the sequence $\{f_k\}_{k=1}^{\infty}$ is uniformly convergent to $f$ on each compact subset of $R^3$.

**Lemma 3.** *Let* (1.4), (1.5) *be fulfilled. Suppose that*

(2.20) $\quad |f(t, x_1, x_2, x_3)| \leqq f^*(t)$

*takes place on the set* $R_+ \times D^3$, *where* $f^* \in L_{\mathrm{loc}}(R_+)$.

*Then for any* $c \in (0, \infty)$ *the boundary value problem*

(2.21) $\quad u''' = f(t, u, u', u'')$,

(2.22) $\quad \varphi(u(0), u'(0), u''(0)) = 0\,, \quad u(c) = u'(c) = 0$

*has at least one solution* $u \in AC^2(\langle 0, c\rangle)$ *satisfying on* $\langle 0, c\rangle$ *the inequalities*

(2.23) $\quad u(t) \geqq 0\,, \quad u'(t) \leqq 0\,, \quad u''(t) \geqq 0\,.$

Proof. First, let us prove Lemma 3 under the additional assumption that $f$ satisfies the Lipschitz condition

(2.24) $\quad |f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leqq v_\varrho(t) \sum_{i=1}^{3} |x_i - y_i|$

for $\varrho \in (0, \infty)$, $t \in R_+$, $|x_i| \leqq \varrho$, $|y_i| \leqq \varrho$ $(i = 1, 2, 3)$, $v_\varrho \in L_{\mathrm{loc}}(R_+)$. Let us put

(2.25) $$\sigma_i(s) = \begin{cases} 0 & \text{for} \quad (-1)^{i-1}\, s \leqq 0 \\ s & \text{for} \quad (-1)^{i-1}\, s > 0\,, \end{cases}$$

$$\tilde{f}(t, x_1, x_2, x_3) = f(t, \sigma_1(x_1), \sigma_2(x_2), \sigma_3(x_3))\,,$$

and consider the Cauchy problem

(2.26) $\quad u''' = \tilde{f}(t, u, u', u'')$,

$$u(c) = 0\,, \quad u'(c) = 0\,, \quad u''(c) = \alpha\,, \quad \alpha \in R\,.$$

According to (2.20) and (2.24), for any $\alpha \in R$ the problem (2.26) has a unique solution $u(t, \alpha) \in AC^2(\langle 0, c\rangle)$.

Let us put

$$h(t, x_1, x_2, x_3) = \begin{cases} \dfrac{\tilde{f}(t, x_1, x_2, x_3) - \tilde{f}(t, x_1, x_2, 0)}{x_3} & \text{for} \quad x_3 \neq 0 \\ 0 & \text{for} \quad x_3 = 0\,, \end{cases}$$

$$h_\alpha(t) = -h(t, u(t, \alpha), u'(t, \alpha), u''(t, \alpha))\,.$$

By virtue of (1.4) and (2.25), $u'''(t, \alpha) = -h_\alpha(t)\, u''(t, \alpha) + \tilde{f}(t, u(t, \alpha), u'(t, \alpha), 0) \leqq$ $\leqq -h_\alpha(t)\, u''(t, \alpha)$ for $0 \leqq t \leqq c$. Integrating the last inequality from $t$ to $c$ we get $u''(t, \alpha) \geqq \alpha \exp \int_t^c h_\alpha(\tau)\, d\tau$ for $0 \leqq t \leqq c$. Further $u'(t, \alpha) \leqq -\int_t^c \alpha\, H(\tau)\, d\tau$, where $H(\tau) = \exp \int_\tau^c h_\alpha(s)\, ds$ and $u(t, \alpha) \geqq \int_t^c \int_\tau^c \alpha\, H(s)\, ds\, d\tau$, and so

(2.27) $\quad u''(t, \alpha) \geqq 0\,, \quad u'(t, \alpha) \leqq 0\,, \quad u(t, \alpha) \geqq 0 \quad \text{for} \quad \alpha \in R_+\,, \quad t \in \langle 0, c\rangle\,.$

Let us put $\beta = \int_0^c f^*(t)\, dt$. Then (2.20) and (2.26) yield

$$\int_t^c u'''(\tau, \alpha)\, d\tau \leqq \int_t^c |\tilde{f}(\tau, u(\tau, \alpha), u'(\tau, \alpha), u''(\tau, \alpha)|\, d\tau \leqq \beta$$

for $0 \leqq t \leqq c$, thus $u''(c, \alpha) - u''(t, \alpha) \leqq \beta$ and so $u''(t, \alpha) \geqq \alpha - \beta$ for $0 \leqq t \leqq c$. Integrating the last inequality from $t$ to $c$ we get $u'(t, \alpha) \leqq -(c - t)(\alpha - \beta)$ for $0 \leqq t \leqq c$. Now put

$$\tilde{\varphi}(\alpha) = \varphi(u(0, \alpha), u'(0, \alpha), u''(0, \alpha)) \quad \text{for} \quad \alpha \in R_+, \quad \alpha^* = \beta + ((r + 1)/c).$$

Clearly $\tilde{\varphi}$ is a continuous function on $\langle 0, \alpha^* \rangle$ and

$$\tilde{\varphi}(0) = \varphi(u(0, 0), u'(0, 0), u''(0, 0)) = \varphi(0, 0, 0) < 0.$$

On the other hand, $\tilde{\varphi}(\alpha^*) > 0$. So there exists $\alpha_0 \in (0, \alpha^*)$ such that $\tilde{\varphi}(\alpha_0) = 0$. From (2.25), (2.27) it follows that $u(t) = u(t, \alpha_0)$ is a solution of the problem (2.21), (2.22) and satisfies (2.23).

If $f$ does not satisfy (2.24), we can use Lemma 2.

**3. Proof of Theorem.** Without loss of generality we may assume that $h_j$ $(j = 0, 1, 2)$ are nonnegative functions. Let

$$\Omega(x) = \int_0^x \frac{ds}{\omega(s)}$$

and let $r^*$ be the constant from Lemma 1. Let us choose $c_0 \in (r^*, \infty)$ and a function $\delta_0$: $\langle 0, a \rangle \times R_+ \to R_+$ satisfying the following conditions: $\delta_0(., x) \in L(\langle 0, a \rangle)$ for any $x \in R_+$, $\delta_0$ is nondecreasing in its second argument, $\delta(t, x) \geqq \delta_0(t, x)$ and $\delta_0(t, x) = \delta_0(t, c_0) > r/a$ for $t \in \langle 0, a \rangle$.

Now, let us put

$$\varrho(t) = \Omega^{-1}(r^* + r^* \textstyle\int_0^t \sum_{i=0}^{2} h_i(\tau)\, d\tau) + 2r^*,$$

$$\sigma_1(s) = \begin{cases} s & \text{for} \quad 0 \leqq s \leqq c_0, \\ c_0 & \text{for} \quad s > c_0 \end{cases}$$

$$\sigma_2(s) = \begin{cases} s & \text{for} \quad -r \leqq s \leqq 0, \\ -r & \text{for} \quad s < -r \end{cases}$$

$$\chi(t, s) = \begin{cases} s & \text{for} \quad 0 \leqq s \leqq \varrho(t), \\ \varrho(t) & \text{for} \quad \varrho(t) < s, \end{cases}$$

$$\chi_0(t, s) = \begin{cases} 1 & \text{for} \quad 0 \leqq s \leqq \varrho(t) \\ 2 - s/\varrho(t) & \text{for} \quad \varrho(t) < s \leqq 2\,\varrho(t), \\ 0 & \text{for} \quad s > 2\,\varrho(t) \end{cases}$$

$$\tilde{f}(t, x_1, x_2, x_3) = f(t, \sigma_1(x_1), \sigma_2(x_2), \chi(t, x_3)) \quad \text{for} \quad 0 \leqq t \leqq a,$$

$$\tilde{f}(t, x_1, x_2, x_3) = \chi_0(t, \sum_{i=1}^{3} |x_i|) f(t, x_1, x_2, x_3) \quad \text{for} \quad t > a.$$

It is clear that $\tilde{f}$ satisfies (1.4) and

(3.1) $$\tilde{f}(t, x_1, x_2, x_3) = f(t, x_1, x_2, x_3) \quad \text{for} \quad 0 \leqq x_1 \leqq c_0, \quad -r \leqq x_2 \leqq 0, \quad 0 \leqq x_3 \leqq \varrho(t), \quad t \in \langle 0, a \rangle,$$

$$\tilde{f}(t, x_1, x_2, x_3) = f(t, x_1, x_2, x_3) \quad \text{for} \quad \sum_{i=0}^{3} |x_i| \leqq \varrho(t), \quad t > a.$$

Now, by (1.9) we obtain

$$(3.2) \qquad \tilde{f}(t, x_1, x_2, x_3) = f(t, \sigma_1(x_1), \sigma_2(x_2), \chi(t, x_3)) \leqq -\delta(t, \sigma_1(x_1)) \leqq$$
$$\leqq -\delta_0(t, \sigma_1(x_1)) = -\delta_0(t, x_1) \quad \text{on the set} \quad \langle 0, a\rangle \times D_r^3 ,$$

from (1.10) we get

$$(3.3) \qquad \tilde{f}(t, x_1, x_2, x_3) \geqq -h(t)(1 + \chi(t, x_3))^2 \geqq -h(t)(1 + x_3)^2$$
$$\text{on the set} \quad \langle 0, a_0\rangle \times D_r^3 ,$$

and from (1.11) we get

$$(3.4) \qquad f(t, x_1, x_2, x_3) = \chi_0(t, \sum_{i=1}^{3} |x_i|) f(t, x_1, x_2, x_3) \leqq$$
$$\leqq [h_0(t) + \sum_{i=1}^{2} h_i(t) |x_i|^{k_i} + \alpha x_3] \, \omega(x_3) \quad \text{on the set} \quad (a, \infty) \times D_r^3 .$$

Since $\tilde{f}$ satisfies the assumptions of Lemma 3, the boundary value problem

$$u''' = \tilde{f}(t, u, u', u''),$$
$$u(a + p) = u'(a + p) = 0, \quad \varphi(u(0), u'(0), u''(0)) = 0$$

has for any $p \in N$ at least one solution $u_p \in AC^2(\langle 0, a + p\rangle)$ satisfying on $\langle 0, a + p\rangle$ the inequalities $u_p(t) \geqq 0$, $u_p'(t) \leqq 0$, $u_p''(t) \geqq 0$. Moreover, (3.2), (3.3) and (3.4) imply $u_p'''(t) \leqq -\delta_0(t, u_p(t))$ for $0 \leqq t \leqq a$, $u_p'''(t) \geqq -h(t)(1 + u_p''(t))^2$ for $0 \leqq t \leqq a_0$ and

$$u_p'''(t) \leqq [h_0(t) + \sum_{i=1}^{2} h_i(t) |u_p^{(i-1)}(t)|^{k_i} + \alpha \, u_p''(t)] \, \omega(u_p''(t))$$
$$\text{for} \quad a < t \leqq a + p .$$

Thus $u_p$ satisfies the conditions of Lemma 1 on $\langle 0, a + p\rangle$ and so we get the estimates $u_p(t) \leqq r^*$, $u_p'(t) \geqq -r$, $u_p''(t) \leqq \varrho(t)$ for $t \in \langle 0, a + p\rangle$, therefore $u_p$ is also a solution of (1.1) on $\langle 0, a + p\rangle$. Denote

$$f_p(t, x_1, x_2, x_3) = \begin{cases} f(t, x_1, x_2, x_3) & \text{for} \quad 0 \leqq t \leqq a + p \\ 0 & \text{for} \quad t > a + p . \end{cases}$$

Then $|f_p(t, x_1, x_2, x_3)| \leqq |f(t, x_1, x_2, x_3)|$ for any $p \in N$ and $\lim_{p \to \infty} f_p(t, x_1, x_2, x_3) = f(t, x_1, x_2, x_3)$ on the set $R_+ \times D^3$. Moreover, $\sup \{ \sum_{i=1}^{3} |u_p^{(i-1)}(t)| : p \in N\} \leqq \varrho(t)$ for $t \in R_+$. Thus by the Arzela-Ascoli Lemma we can prove that the sequence $\{u_p\}_{p=1}^{\infty}$ contains a subsequence $\{u_{p_j}\}_{j=1}^{\infty}$ which is locally uniformly convergent together with $\{u_{p_j}'\}_{j=1}^{\infty}$ and $\{u_{p_j}''\}_{j=1}^{\infty}$ on $R_+$, and $u(t) = \lim_{j \to \infty} u_{p_j}(t)$ is a solution of (1.1), (1.2), (1.3) on $R_+$.

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