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Czechoslovak Mathematical Journal, Vol. 40 (1990), No. 1, 159–163

Persistent URL: `http://dml.cz/dmlcz/102367`

Czechoslovak Mathematical Journal, 40 (115) 1990, Praha

EXISTENCE OF PRIME IDEALS AND ULTRAFILTERS IN PARTIALLY ORDERED SETS

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(Received August 10, 1988)

In the existing literature the algebraic notions of an ideal and a prime ideal [1, p. 122] and the corresponding dual notions of a filter and an ultrafilter [1, p. 142] are predominantly generalized to the case of lattices [2] and [3]. Here we introduce these notions in partially ordered sets. An ideal in a partially ordered set can be defined in various (not necessarily pairwise equivalent) ways. The same is the case for the definitions of a prime ideal, a filter and an ultrafilter.

In what follows we refer to a partially ordered set simply as a poset. Also, we introduce the following notations:

(2.1) $\quad [x, y]$ for $\sup\{x, y\}$

(2.2) $\quad (x, y)$ for $\inf\{x, y\}$

Based on the above notations, we introduce:

Definition 2.3. A nonempty proper subset I of a poset $(P, \leqq)$ is called an *ideal of P* iff

(2.4) $\quad x \in I$ and $y \leqq x$ imply $y \in I$ for every $x, y \in P$.

(2.5) $\quad x \in I$ and $y \in I$ then $[x, y] \in I$ for every $x, y \in P$.

Moreover, an ideal D of P is called a *prime ideal of P* iff

(2.6) $\quad$ if $(x, y) \in D$ then $x \in D$ or $y \in D$ for every $x, y \in P$.

Lemma 2.7. *Let $(P, \leqq)$ be a poset with a maximum* 1. *Then for every $a, b, c \in P$ it is the case that:*

(i) *If $b \leqq c$ and $[a, b] = 1$ then $[a, c] = 1$.*

(ii) *Let $[b, c]$ exist. Then $[a, [b, c]] = 1$ iff $[a, b, c] = 1$.*

Proof. (i) Let $b \leqq c$ and $[a, b] = 1$. Then 1 is the only upper bound of $\{a, b\}$. If x is an upper bound of $\{a, c\}$ then x is an upper bound of $\{a, b\}$. Thus $x = 1$ and consequently 1 is the only upper bound of $\{a, c\}$. Hence $[a, c] = 1$.

(ii) Assume $[b, c]$ exists and $[a, [b, c]] = 1$. Again 1 is the only upper bound of $\{a, [b, c]\}$. If x is an upper bound of $\{a, b, c\}$ then x is also an upper bound of $\{a, [b, c]\}$. Thus $x = 1$. Consequently 1 is the only upper bound of $\{a, b, c\}$, i.e.,

$[a, b, c] = 1$. Conversely, let $[b, c]$ exist and $[a, b, c] = 1$. Since $[b, c] \geqq c$, by (i) we have $[a, b, [b, c]] = 1$. But $[b, c] \geqq b$ and therefore $1 = [a, b, [b, c] = = [a, [b, c]]$. Thus (ii) is established. Q.E.D.

Let $(P, \leqq)$ be a poset with a maximum 1 such that:

(2.8) The supremum of every two elements of P exists.

And for every finite subset $\{x, a_1, \ldots, a_n\}$ of P, the following distributivity condition holds:

(2.9) If $[x, a_1] = \ldots = [x, a_n] = 1$ and $[x, (a_i, \ldots, a_n)]$ exists then $[x, (a_1, \ldots, a_n)] = 1$.

Moreover, as usual, a subset A of P is said to have the *finite supremum property* iff:

(2.10) 1 is not the supremum of any finite subset of A.

Theorem 2.11. *Let $(P, \leqq)$ be a poset with a maximum 1 satisfying (2.8) and (2.9). Let D_0 be a nonempty subset of P satisfying (2.10). Then there exists a subset D of P such that:*

(i) *$1 \notin D$ and $D_0 \subseteq D$.*

(ii) *$x \in D$ and $y \leqq x$ imply $y \in D$ for every $x, y \in P$.*

(iii) *$[d_1, \ldots, d_n] \in D$ for every finite subset $\{d_1, \ldots, d_n\}$ of D.*

(iv) *If $(a_1, \ldots, a_n) \in D$ then $a_i \in D$ for some $1 \leqq i \leqq n$.*

Proof. Let

$$H' = \{H : H \subseteq P \text{ and } D_0 \subseteq H \text{ and } H \text{ satisfies (2.10)}\}.$$

Clearly, $(H', \subseteq)$ is a nonempty partially ordered set since $D_0 \in H'$. By Zorn's Lemma, it can be readily verified that H' has a maximal element D. We show that D satisfies (i) to (iv).

Clearly, $1 \notin D$ and $D_0 \subseteq D$ so that D satisfies (i). Let us observe that by the maximality of D we have:

(2.12) If $x \notin D$ then $[x, d_1, \ldots, d_n] = 1$ for some finite subset $\{d_1, \ldots, d_n\}$ of D.

Now, let $x \in D$ and $y \leqq x$ and let $y \notin D$. By (2.12) $[y, d_1, \ldots, d_n] = 1$ and by (i) of Lemma 2.7 we derive that $[x, d_1, \ldots, d_n] = 1$ which is a contradiction since D satisfies (2.10). Hence $y \in D$, i.e., (ii) is established.

Let $\{t_1, \ldots, t_n\}$ be a subset of D and assume that $t = [t_1, \ldots, t_m]$ is not an element of D. Then $[t, d_1, \ldots, d_n] = 1$, by (2.12). But then (ii) of Lemma 2.7 implies $[t_1, \ldots, t_m, d_1, \ldots, d_n] = 1$ which is a contradiction since D satisfies (2.10). Thus $t \in D$. Hence (iii) is established.

To show (iv), let us assume on the contrary that $(a_1, \ldots, a_n) \in D$ and $a_i \notin D$ for every i with $1 \leqq i \leqq n$. Then by (2.12) we have $[a_i, d_{i1}, \ldots, d_{im_i}] = 1$, for every $1 \leqq i \leqq n$. Now, let $x = [d_{11}, \ldots, d_{1m_1}, d_{21}, \ldots, d_{2m_2}, \ldots, d_{n1}, \ldots, d_{nm_n}]$. Then $x \in D$ by (iii) and if y_i is an upper bound of $\{x, a_i\}$ then y_i is an upper bound of $\{d_{i1}, \ldots, d_{im_i}\}$. But 1 is the only upper bound of $\{a_i, d_{i1}, \ldots, d_{im_i}\}$. Thus $y_i = 1$ and consequently $[x, a_i] = 1$. On the other hand, since $[x, (a_1, ..., a_n)]$ exists, then by (2.9) we have $[x, (a_1, \dots, a_n)] = 1$ which is a contradiction since D satisfies (2.10). Thus $a_i \in D$ for some $1 \leqq i \leqq n$. Hence (iv) is established. Q.E.D.

Remark 2.13. We note that Theorem 2.11 implies that the subset D of the poset P is a prime ideal of P. We also observe that the same is true if condition (2.9) is replaced by the weaker condition:

(2.9)′ If $[x, a_1] = [x, a_2] = 1$ and $[x, (a_1, a_2)]$ exists then $[x, (a_1, a_2)] = 1$.

The Theorem below which (in view of Remark 2.13) ensures the existence of a prime ideal of a poset follows readily from Theorem 2.11.

Theorem 2.14. *Let $(P, \leqq)$ be a poset with a maximum* 1 *satisfying* (2.8) *and* (2.9)′. *Let D_0 be a nonempty subset of P satisfying* (2.10). *Then there exists a prime ideal D of P such that $D_0 \subseteq D$.*

Remark 2.15. We observe that for every nonmaximum element x of a poset $(P, \leqq)$ satisfying (2.8), the subset $I(x)$ of P given by:

(2.16) $$I(x) = \{z : z \in P \text{ and } z \leqq x\}$$

is an ideal of $(P, \leqq)$.

As usual, $I(x)$ in (2.16) is called the *principal ideal of P generated by x*. Clearly, for every $x, y \in P$ with $x \neq y$ there exists an ideal of P containing, say, x but not y.

Next, we consider the case of the existence of a prime ideal of a poset without the maximum element. For this purpose we replace the distributivity condition (2.9)′ by:

(2.17) $$([x, a_1], [x, a_2]) \leqq [x, (a_1, a_2)]$$

with the understanding that (2.17) holds whenever the right side of $\leqq$ exists, and, this for every $x, a_1, a_2 \in P$.

We observe that (2.17) does not hold in every poset. For instance, it fails in the poset $(\{e, a, b, c, m\}, \leqq)$ with $e \leqq a$, $e \leqq b$, $e \leqq c$, $a \leqq m$, $b \leqq m$, $c \leqq m$.

Theorem 2.18. *Let $(P, \leqq)$ be a poset in which every two elements have a supremum and which satisfies* (2.17). *Let $x, y \in P$ with $y \nleqq x$. Then there exists a prime ideal D of P such that $x \in D$ and $y \notin D$.*

Proof. From (2.16) it follows that $I(x)$ is an ideal of P and that y is not the supremum of any finite subset of $I(x)$. This is because x is an upper bound of any subset of $I(x)$ and $y \nleqq x$.

Let H' be the set of all the ideals H of P such that $I(x) \subseteq H$ and y is not the supremum of any finite subset of H. It is obvious that $(H', \subseteq)$ is a nonempty poset. By Zorn's Lemma it can be readily verified that H' has a maximal element D.

We claim that D is a prime ideal of P. Let us assume on the contrary, i.e., there exist $a_1, a_2 \in P$ such that $(a_1, a_2) \in D$ but $a_1 \notin D$ and $a_2 \notin D$. Now, let us consider:

(2.19) $$D_i = D \cup \{z : z \in P \text{ and } z \leqq [a_i, d] \text{ and } d \in D\} \quad \text{with} \quad i = 1, 2\,.$$

One of the following two cases must occur:

Case 1. $D_i = P$ for some $i \in \{1, 2\}$. For this case, from (2.19) we derive:

(2.20) $$y \leqq [a_i, d_i] \quad \text{for some} \quad d_i \in D\,.$$

Case 2. D_i is a proper subset of P. For this case we show that D_i is an ideal of P which contains D properly. Let $t_1, t_2 \in D_i$. Thus, from (2.19) it follows that $t_1 \leqq$ $\leqq [a_i, d_3]$ and $t_2 \leqq [a_i, d_4]$ for some $d_3, d_4 \in D$. Based on the hypothesis of the Theorem, we let $d = [d_3, d_4]$. Since D is an ideal of P, we have $d \in D$. Also, it can be readily verified that $t_1 \leqq [a_i, d]$ and $t_2 \leqq [a_i, d]$. Thus, $[t_1, t_2] \leqq [a_i, d]$ which by (2.19) implies that $[t_1, t_2] \in D_i$. Hence, D_i satisfies (2.5). Now, let $t \in D_i$ and $r \leqq t$ with $r \in P$. But then, again from (2.19) it follows that $r \in D_i$. Hence, D_i also satisfies (2.4). Consequently, D_i is an ideal of P. However, the maximality of D implies that $y \leqq [a_i, d_i]$ for $i = \{1, 2\}$.

Thus, (2.20) holds in both of the abovementioned cases. Let $d = [d_1, d_2]$ which exists by the hypothesis of the Theorem. Clearly, y is a lower bound of $\{[a_1, d], [a_2. d]\}$. Since $[d, (a_1, a_2)]$ exists by the hypothesis of the Theorem and since $[d, (a_1, a_2)] \in D$, by (2.17) we have $y \leqq ([a_1, d], [a_2, d]) \leqq [d, (a_1, a_2)] \in D$. Since D is an ideal of P, by (2.4) we have $y \in D$. But this contradicts that $D \in H'$. Hence our assumption is false and D is a prime ideal of P. Q.E.D.

The existence of prime ideals in structures related to order (e.g., semilattices, lattices, Boolean rings, etc.) has been considered under assumptions generally stronger than those stated in Theorem 2.18. In this connection reference is made to [4], [5], [6].

Remark 2.21. We observe that the existence of prime ideals in posets is proved in Theorems 2.14 and 2.18 under the assumption that every two elements of the poset have a supremum. Next, we consider cases where this assumption is not satisfied by the posets. As shown below, for such cases we prove the existence of subsets of posets which will act almost like prime ideals.

Definition 2.22. A nonempty proper subset D of a poset $(P, \leqq)$ is called a *pseudo ideal* of P iff

(2.23) $$x \in D \quad \text{and} \quad y \leqq x \quad \text{imply} \quad y \in D \quad \text{for every} \quad y \in P$$

and

(2.24) $$\text{if} \quad x, y \in D \quad \text{and} \quad [x, y] \quad \text{exists then} \quad [x, y] \in D\,.$$

Moreover, a pseudo ideal D of P is called a *pseudo prime ideal* of P iff

(2.25) $$(a, b) \in D \quad \text{implies} \quad a \in D \quad \text{or} \quad b \in D\,.$$

Let P be a poset with the maximum 1 satisfying the distributivity condition:

(2.26) $$[x_1, a_{11}, \ldots, a_{1n_1}] = [x_2, a_{21}, \ldots, a_{2n_2}] = 1$$

implies

$$[(x_1, x_2), (x_1, a_{21}), \ldots, (a_{1n_1}, x_2), \ldots, (a_{1n_1}, a_{2n_2})] = 1$$

for every $x_1, x_2, a_{21}, \ldots, a_{1n_1}, x_2, \ldots, a_{1n_1}, a_{2n_2} \in P$.

Theorem 2.27. *Let $(P, \leqq)$ be a poset with the maximum 1 satisfying (2.26). Let D_0 be a nonempty subset of P satisfying (2.10). Then there exists a pseudo prime ideal D of P such that $D_0 \subseteq D$.*

Proof. Let H' be the set of all the subsets H of P such that $D_0 \subseteq H$ and H satisfies (2.10).

Clearly, $(H', \subseteq)$ is a nonempty poset and by Zorn's Lemma H' has a maximal element D. We observe that D satisfies (2.12).

We show that D is a pseudo ideal of P. To show that D satisfies (2.23), we assume to the contrary that $x \in D$ and $y \leqq x$ but $y \notin D$ for some $y \in P$. Then by (2.12) we have $[y, d_1, \ldots, d_n] = 1$ for some $d_1, \ldots, d_n$ elements of D. Using (i) of Lemma 2.7, we obtain that $[x, d_1, \ldots, d_n] = 1$ which contradicts that $D \in H'$ and that D satisfies (2.10). Hence, $y \in D$. To show that D satisfies (2.24), we assume to the contrary that for some $t_1, t_2 \in D$ it is the case that $t = [t_1, t_2]$ exists but $t \notin D$. Again, from (2.12) it then follows that $[t, d_1, \ldots, d_k] = 1$ for some $d_1, \ldots, d_k \in D$. Also, by (ii) of Lemma 2.7, we obtain that $[t_1, t_2, d_1, \ldots, d_k] = 1$ which again contradicts that D satisfies (2.10). Hence, $t = [t_1, t_2] \in D$. Thus, D is a pseudo ideal of P.

Next we show that D is a pseudo prime ideal of P. We assume to the contrary that $(a_1, a_2) \in D$ for some $a_1, a_2 \in P$ but $a_1 \notin D$ and $a_2 \notin D$. Then by (2.12) we have $[a_i, d_{i1}, \ldots, d_{in_i}] = 1$, for $i = 1, 2$ and some $d_{i1}, \ldots, d_{in_i} \in D$. But then from (2.26) it follows that

$$[(a_1, a_2), (a_1, d_{21}), \ldots, (d_{1n_1}, a_2), \ldots, (d_{1n_1}, d_{2n_2})] = 1 \tag{2.28}$$

Clearly, for every term such as (a_i, d_{kj}) which appears in (2.28) we have $(a_i, d_{kj}) \leqq \leqq d_{kj} \in D$ and therefore, $(a_i, d_{kj}) \in D$ by (2.23). Also, by our assumption $(a_1, a_2) \in D$. Consequently, the entire left side of the equality sign in (2.28) is an element of D. But this contradicts that D satisfies (2.10). Thus, our assumption is false and the pseudo ideal D satisfies (2.25) and therefore D is a pseudo prime ideal of P. Q.E.D.

Bibliography

[1] *Hungerford, T. W.:* Algebra. Springer-Verlag, New York, 1974.

[2] *Birkhoff, G.:* Lattice Theory. AMS Colloquium Publications, Vol. *25*. Amer. Math. Soc., Providence, 1979.

[3] *Graetzer, G.:* General Lattice Theory. Academic Press, New York, 1978.

[4] *Bables, R.:* A Representation Theory for Prime and Implicative Semilattices. Transactions of AMS *136* (1969), 261—267.

[5] *Schein, B. M.:* On the Definition of Distributive Semilattices. Algebra Universalis *2* (1972), 1–2.

[6] *Markowski, G.:* The Representation of Posets and Lattices by Sets. Algebra Universalis *11* (1980), 173—192.

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