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STRICT TOPOLOGY AND PERFECT MEASURES

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Let X be a completely regular Hausdorff space, E a normed space over K, the field of real or complex numbers, and $C_b(X, E)$ the space of all continuous E-valued functions on X. When $E = K = \mathbb{R}$, the set of real numbers, a topology β_p is defined, in [6], on $C_b(X, E)$ which gives its dual $M_p(X)$ the set of all Baire perfect measures on X. In this case we consider the general case when E is a normed space.

Notations of [3] will be used. X will always denote a completely regular Hausdorff space and E a normed space over K (scalars), the field of real or complex numbers. All linear spaces are taken over K. $C_b(X, E)$ will be denoted by $C_b(X)$ when $E = K$. $\tilde{X}$ and vX will be respectively the Stone-Čech compactification and real compactification of X. $M(X)$, $M_\sigma(X)$, $M(X, E')$ have the meanings as in ([3], p. 196). For a continuous function f from X into a topological space Y, $\tilde{f}$, $\bar{f}$ will respectively denote its unique extensions to $\tilde{X}$ and vX if extensions are possible. Notations of [7] for locally convex spaces will be used. A locally convex space F is strongly Mackey if every relatively countably compact subset of $(F', \sigma(F', F))$ is equicontinuous. A subset Z of X will be called a *zero set* if $Z = f^{-1}\{0\}$ for some $f \in C_b(X)$. The norm topology on $C_b(X, E)$ is defined by $\|f\| = \sup_{x \in X} |f(x)|$. For a $\mu \in (C_b(X), \|\cdot\|)'$, $\tilde{\mu}$: $C(\tilde{X}) \to K$, is defined by $\tilde{\mu}(f) = \mu(f \setminus X)$.

As in [6], $M_p(X)$ denotes the space of all scalar-valued Baire perfect measures on X. A subset G in a completely regular space Y will be called *distinguished* if there exists a continuous mapping φ from Y onto a separable metric space such that $G = \varphi^{-1}(\varphi(G))$. The class of all distinguished subsets of $\tilde{X}$ disjoint from X will be denoted by $\mathscr{D}(\tilde{X}) = D$. For a $D \in \mathscr{D}$, the topology γ_D, on $C_b(X, E)$ is defined to be the one generated by the seminorms $\|\cdot\|_g$, as g varies over $B_D(X)$, all bounded scalar-valued functions on $\tilde{X}$, vanishing at infinity and zero on D, $\|f\|_g = \sup_{x \in \tilde{X}} (\|f\|\tilde{}(x)\, |g(x)|)$.

As in ([9], Theorem 2.4) it can be verified that γ_D is the finest locally convex topology agreeing with the topology of uniform convergence on compact subsets of $\tilde{X} \setminus D$, on norm-bounded subsets of $C_b(X, E)$ (note for a compact $C \subset \tilde{X} \setminus D$, $f \in C_b(X, E)$

norm sup of f over C is in the sense sup $\|f\|^{\sim}(C)$). We define $\beta_p = \Lambda\{\gamma_D\colon D \in \mathscr{D}\}$. As observed in [6] for every $D \in \mathscr{D}$, $D \cap vX = \emptyset$. For a function $f \in C_b(X, E)$, $\|f\|\colon X \to R$ is defined by $\|f\|(x) = \|f(x)\|$. A locally convex topology on $C_b(X, E)$ will be called *locally solid* if it has a 0-nbd. base consisting of absolutely convex sets V, such that $f \in V$, $\|g\| \leqq f$ implies $g \in V$ (f, g in $C_b(X, E)$). For a duality $\langle F, G\rangle$, $A \subset F$. The polar of $A = A^0 = \{g \in G, |\langle a, g\rangle| \leqq 1$, for each $a \in A\}$. For any collection $\{A_\alpha\}_{\alpha \in I}$ of subsets of a locally convex space F, $\Gamma_\alpha A_\alpha$ will denote the absolutely convex hull of $\bigcup_{\alpha \in I} A_\alpha$. We define $M_p(X, E') = \{\mu \in M_\sigma(X, E'),\ \mu_x \in M_p(X)$ for every $x \in E\}$. It is known that $\mu \in M_p(X)$ implies $|\mu| \in M_p(X)$. First we prove that $\mu \in M_p(X, E')$ implies $|\mu| \in M_p(X)$.

Theorem 1. *For a* $\mu \in M_p(X, E')$, $|\mu| \in M_p(X)$.

Proof. From ([6], Theorem 2.1) if is sufficient to prove that $|\mu|^{\sim *}(D) = 0$ for every $D \in \mathscr{D}$. Take a Baire subset V_0 of $\tilde{X}$ such that $|\mu|^{\sim}(B) = 0$, for any Baire $B \subset V_0 \setminus D$. For any $x \in E$, $\|x\| \leqq 1$, $|\mu_x|^{\sim} \leqq |\mu|^{\sim}$, and so $|\mu_x|^{\sim}(B) = 0$. Since $|\mu_x|$ is perfect, this implies $|\mu_x|^{\sim}(V_0) = 0$. So we get $|\mu_x|(V_0 \cap X) = 0$. Take any finite partition $\{V_i\colon 1 \leqq i \leqq n\}$ of $V_0 \cap X$, and any collection $\{x_i\colon 1 \leqq i \leqq n\}$ in E with $|x_i| \leqq 1$, for every i. From what is proved above it follows that $|\sum_{i=1}^{n} \mu_{x_i}(V_i)| = 0$ and so $|\mu|(V_0 \cap X) = 0$. This means $|\mu|^{\sim}(V_0) = 0$ and so $|\mu|^{\sim *}(D) = 0$. Thus $|\mu|$ is perfect.

The following theorem is a simple consequence of the definition of β_p.

Theorem 2.

(i) $\beta_0 \leqq \beta_p \leqq \beta_1$.

(ii) β_p is the finest locally convex topology agreeing with itself on norm-bounded subsets of $C_b(X, E)$.

(iii) $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$ if it is dense in $(C_b(X, E), \beta_1)$ ([3], p. 206).

Proof follows easily from the definition of β_p.

Theorem 3. *Let Y be a completely regular Hausdorff space and $\varphi\colon X \to Y$ a continuous mapping. Then the canonical mapping $(C_b(Y, E), \beta_p) \to (C_b(X, E), \beta_p)$. . $(f \to f \circ \varphi)$ is continuous.*

Proof. For a distinguished set D_0 in $\tilde{Y}$, $D_0 \subset \tilde{Y} \setminus Y$, $D = \tilde{\varphi}^{-1}(D_0) \in \mathscr{D}$, where $\tilde{\varphi}\colon \tilde{X} \to \tilde{Y}$ is the unique continuous extension. This implies that the mapping

$$(C_b(Y, E), \gamma_{D_0}) \to (C_b(X, E), \beta_p)$$

is continuous. The result follows now.

For a $\mu \in (C_b(X, E), \|\cdot\|)'$, define $\lambda_\mu\colon C_b(X)^+ \to [0, \infty)$, $\lambda_\mu(f) = \sup\{|\mu(g)|\colon g \in C_b(X, E),\ \|g\| \leqq f\}$.

Lemma 4. *λ_μ is additive and positively homogeneous.*

Proof. Take $f \in C_b(X)^+$, $g \in C_b(X)^+$, such that $f + g \geqq \eta > 0$, on X, for some η.

Taking any $h \in C_b(X, E)$, with $\|h\| \leqq f + g$, we get that

$$\left\|\frac{hf}{f+g}\right\| \leqq f, \quad \left\|\frac{hg}{f+g}\right\| \leqq g.$$

From

$$\mu(h) = \mu\left(\frac{hf}{f+g}\right) + \mu\left(\frac{hg}{f+g}\right)$$

it easily follows that

$$\lambda_\mu(f + g) \leqq \lambda_\mu(f) + \lambda_\mu(g).$$

On the other hand, take h_1, h_2 in $C_b(X, E)$, $\|h_1\| \leqq f$, $\|h_2\| \leqq g$. This gives that $\|h_1 + h_2\| \leqq f + g$, which implies $\lambda_\mu(f + g) \geqq \lambda_\mu(f) + \lambda_\mu(g)$. Thus $\lambda_\mu(f) + \lambda_\mu(g) = = \lambda_\mu(f + g)$. In particular, $\lambda_\mu(\frac{1}{2}) + \lambda_\mu(\frac{1}{2}) = \lambda_\mu(1)$. Take any f_1, g_1 in $C_b(X)^+$. From above it follows that $\lambda_\mu(f_1) + \lambda_\mu(g_1) + \lambda_\mu(1)$, $\lambda_\mu(f_1) + \lambda_\mu(g_1 + 1) = = \lambda_\mu(f_1 + g_1 + 1) = \lambda_\mu(f_1 + g_1) + \lambda_\mu(1)$ and so $\lambda_\mu(f_1 + g_1) = \lambda_\mu(f_1) + \lambda_\mu(g_1)$. Also it is easily verified that $\lambda_\mu(pf) = p\,\lambda_\mu(f)$, for any $p \geqq 0$ and $f \in C_b(X)^+$.

Theorem 5. *The space $(C_b(X, E), \beta_p)$ is locally solid, and has a 0-nbd. base consisting of solid absolutely convex sets.*

Proof. For any $D \in \mathscr{D}$, take $g_D: \tilde{X} \to R$, $g_D \equiv 0$ on D, g_D bounded and vanishing at infinity and put $V_D = \{f \in C_b(X, E), \sup_{x \in \tilde{X}} |\|f\|^\sim(x)\, g_D(x)| \leqq 1\}$ and $V = \bigcup_{D \in \mathscr{D}} V_D$. This gives $V^0 = \bigcap V_D^0$, polar being taken in $(C_b(X, E), \beta_p)'$. Take $\mu \in V^0$. Since $\beta_p \leqq \|\cdot\|$, $\mu \in (C_b(X, E), \|\cdot\|)'$. From Lemma 4 and the fact that each V_D is locally solid it follows that $\mu \in V^0$ if and only if $\lambda_\mu(\|g\|) \leqq 1$, for every $g \in V_D$, for each D. Using Lemma 4, it is easily verified that $W = \{f \in C_b(X, E), \lambda_\mu(\|f\|) \leqq 1$, for every $\mu \in V^0\}$ is convex, contains V, is contained in V^{00}, and is locally solid. This proves the result.

Corollary 6. *A net $f_\alpha \to 0$, in $(C_b(X, E), \beta_p)$, if and only if $\|f_\alpha\| \to 0$ in $(C_b(X), \beta_p)$.*

Proof. Assuming $f_\alpha \to 0$ in $(C_b(X, E), \beta_p)$ take an absolutely convex, solid β_p 0-nbd W in $(C_b(X), \beta_p)$. This means for every $D \in \mathscr{D}$, there exists a $g_D \in B_D(\tilde{X})$, such that $W \supset P = \Gamma_D\{g \in C_b(X): \sup_{x \in \tilde{X}} | |g|^\sim(x)\, g_D(x)| \leqq 1\}$. Take $W_0 = = \Gamma_D\{f \in C_b(X, E): \sup_{x \in X^\sim} | \|f\|^\sim(x)\, g_D(x)| \leqq 1\}$. Since W is locally solid, $f_\alpha \in W_0$ implies $\|f_\alpha\| \in W$. This proves $\|f_\alpha\| \to 0$. Conversely suppose $\|f_\alpha\| \to 0$ in $(C_b(X), \beta_p)$. Fix a $y \in E$ with $\|y\| = 1$. We first prove that $\|f_\alpha\|\, y \to 0$ in $(C_b(X, E), \beta_p)$. Take an absolutely convex solid 0-nbd. V_0 in $(C_b(X, E), \beta_p)$. With above notations $V_0 \supset W_0$, for some g_D's in $B_D(\tilde{X})$. Form P, as defined above, with these g_D's. It is easy to see now that $\|f_\alpha\| \in P$ implies $\|f_\alpha\| = \sum_{i=1}^{n} \lambda_i g_i$, with $\sum_{i=1}^{m} |\lambda_i| \leqq 1$, $\sup_{x \in X^\sim} | |g_i|^\sim(x)\, g_{D(i)}(x)| \leqq \leqq 1$, for $D(i) \in \mathscr{D}$, $g_i \in C_b(X)$, $1 \leqq i \leqq n$, for some n. Thus, $\|f_\alpha\|\, y = \sum_{i=1}^{m} \lambda_i g_i y$ and so $\|f_\alpha\| \, y \in W_0$. Since $\|f_\alpha\| \leqq \| \, \|f_\alpha\| y\|$ and V_0 is locally solid $f_\alpha \in V_0$. The result follows now.

Theorem 7. *If X is realcompact $\beta_p \leqq \beta_\infty$. In this case $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$.*

Proof. If $E = K = R$, this result is proved in ([6], Prop. 4.6). First we prove that $f_\alpha \to 0$ in $(C_b(X, E), \beta_\infty)$ if and only if $\|f_\alpha\| \to 0$ in $(C_b(X), \beta_\infty)$. Suppose $\|f_\alpha\| \to 0$ in $(C_b(X), \beta_\infty)$. Take $x \in E$ with $\|x\| = 1$. We claim that $\|f_\alpha\| \otimes x \to 0$ in $(C_b(X, E), \beta_\infty)$. Let A be an equicontinuous subset of $M_\infty(X, E')$. This means $|A| = \{|\mu|: \mu \in A\}$ is an equicontinuous subset of $M_\infty(X)$. (This is proved in [3], Theorem 3.7, p. 202). Now $|\mu(\|f_\alpha\| \otimes x)| \leqq |\mu| (\| \, \|f_\alpha\| \otimes x\|) = |\mu| (\|f_\alpha\|) \to 0.$ uniformly for $\mu \in A$. This proves the claim. Now $\|f_\alpha\| \leqq \| \, \|f_\alpha\| \otimes x\|$, and $\|f_\alpha\| \otimes x \to 0$. Since $(C_b(X, E), \beta_\infty)$ is locally solid ([3], Theorem 8.1), we get $f_\alpha \to 0$.

Now suppose $f_\alpha \to 0$ in β_∞. This means $\|f_\alpha\| \to 0$ in $(C_b(X), \beta_\infty)$ ([3], Theorem 8.1). Since $\beta_p \leqq \beta_\infty$, considered as topologies on $C_b(X)$ ([6], Theorem 4.1; this is proved for $K = \mathbb{R}$, but easily extends to the case when $K = \mathbb{C}$), we get $\|f_\alpha\| \to 0$ in β_p. By Corollary 6, this means $f_\alpha \to 0$ in $(C_b(X, E), \beta_p)$. Thus $\beta_p \leqq \beta_\infty$, as topologies on $C_b(X, E)$. Since $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\infty)$, it follows that $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$.

Theorem 8. *If $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$, then*

(i) *for any $\mu \in M_p(X, E')$, $L_1(\mu, X, E) \supset C_b(X, E)$;*

(ii) *$(C_b(X, E), \beta_p)' = M_p(X, E)$, $L \in (C_b(X, E), \beta_p)'$ being related to corresponding $\mu \in M_p(X, E')$ by $L(f) = \mu(f)$, $\# f \in C_b(X, E)$.*

Proof is very similar to ([3], Theorem 5.3) and is omitted.

In the next theorem we give a new characterization of the topology β_p, which avoids the use of distinguished sets.

Let

$$\mathscr{F} = \{(Y, \varphi): Y \text{ a separable metric space}, \; \varphi: X \to Y \text{ a continuous onto mapping}\}.$$

Every element $F = (Y, \varphi) \in \mathscr{F}$ gives rise to a linear mapping $T_F: C_b(Y, E) \to C_b(X, E)$, $h \to h \circ \varphi$.

Theorem 9. *β_p is the finest locally solid, locally convex topology V_p on $C_h(X, E)$ such that the mappings*

$$T_F: (C_h(Y, E), \beta_0) \to (C_h(X, E), V_p)$$

are continuous for every $F = (Y, \varphi) \in \mathscr{F}$.

Proof. When X is a separable metric space, $\beta_p = \beta_0$ (simple verification). Thus T_F is continuous when $V_p = \beta_p$. This means V_p exists and $V_p \geqq \beta_p$. To prove $V_p \leqq \|\cdot\|$ we take a sequence $\{f_n\} \subset C_b(X, E), f_n \to 0$ in $\|\cdot\|$. Fix $e \in E$, $\|e\| = 1$. The continuous mapping $\varphi: X \to R^N$, $\varphi(x) = \{\|f_n\| (x)\}$ maps X onto the separable metric space $Y = \varphi(X)$. The sequence $\{g_i\} \subset C_b(Y, E)$, $g_i(\{\|f_n\| (x)\}) = \|f_i\| (x) \otimes e$ uniformly converges to 0 and so converges to 0 in $(C_b(Y, E)\ \beta_0)$. Thus $\|f_i\| \otimes e$ converges to 0 in $(C_b(X, E), V_p)$. Since $(C_b(X, E), V_p)$ is locally solid, this means $f_i \to 0$ in $(C_b(X, E), V_p)$.

To prove $V_p = \beta_p$ we first consider the case when $E = K$. Take a sequence $\{f_n\} \subset \subset C_b(X)$, $f_n \downarrow 0$. The mapping

$$\varphi: X \to R^N, \quad x \to \{f_n(x)\}$$

is a continuous mapping from X onto a separable metric space $\varphi(X)$. Fix a $\mu \in \in (C_b(X), V_p)$. Then $\varphi * \mu \in M_t(\varphi(X))$ (note $\varphi * \mu(g) = \mu(g \circ \varphi)$). Since the sequence $\{g_n\} \subset C_b(\varphi(X))$, $g_n(\{f_i\}) = f_n$, monotonically decreases to 0, we get $\varphi * \mu(g_n) \to 0$, from which it follows that $\mu(f_n) \to 0$. Thus $(C_b(X), V_p) \subset M_\sigma(X)$. Next we will prove that $(C_b(X), V_p)' = M_p(X)$. Take a $\mu \in (C_b(X), V_p)'$. By the locally solid property of V_p, $|\mu| \in (C_b(X), V_p)' \subset M_\sigma$. Thus for any metric space Y and every $\varphi: X \to Y$, a continuous onto mapping, $\varphi * |\mu| \in M_t(Y)$. Thus $|\mu| \in M_p(X)$ ([6], Lemma 2.2, p. 469). By ([6], Theorem 2.1) $\mu \in M_p(X)$. (Though results proved in [6] are for $k = R$, they easily extend to when $K = C$). Since $\beta_p \leqq V_p$, we get $(C_b(X), V_p)' = = M_p(X)$. Now we are ready to prove that when $E = K$, $\beta_p = V_p$. Take $H \subset M_p(X)$, H V_p-equicontinuous. There exists an absolutely convex solid V_p 0-nbd W in $C_b(X)$, such that $W \subset H^0 = \{g \in C_b(X): |\mu(g)| \leqq 1,\ \forall \mu \in H\}$. If $g \in W$ and $\mu \in H$, then $|\mu|\,(|g|) = \sup\{|\mu(h)|: |h| \leqq |g|: h \in C_b(X)\}$. Since W is solid, we get $|\mu|\,(|g|) \leqq 1$. Thus $|H|$ is V_p-equicontinuous, and so for any $(Y, \varphi) \in \mathscr{F}$, $\varphi * |H|$ is β_0-equicontinuous in $M_t(Y)$. By ([6], Prop. 2.6, p. 471), H is β_p-equicontinuous. This proves V_p and β_p on $C_b(X)$.

Now we come to the general case $(C_b(X, E), V_p)$. Fix $e \in E$, $\|e\| = 1$. We shall prove that the mapping $\psi: (C_b(X), \beta_p) \to (C_b(X, E), V_p)$, $g \to g \otimes e$ is continuous. Taking any $\mathscr{F} = (Y, \varphi) \in F$, the mappings $T_F: (C_b(Y, E), \beta_0) \to (C_b(X, E), V_p)$, $g \to g \circ \varphi$, and $\psi_0: (C_b(Y), \beta_0) \to (C_b(Y, E), \beta_0)$, $g \to g \otimes e$ are continuous. Let $\psi_1: (C_b(Y), \beta_0) \to (C_b(X), \beta_p)$, $g \to g \circ \varphi$. For any locally solid, absolutely convex 0-nbd U in $(C_b(X, E), V_p)$, $\psi_0^{-1}(T_F^{-1}(U))$ is a 0-nbd in $(C_b(Y), \beta_0)$. Since $\psi_1^{-1}(\psi^{-1}(U)) = \psi_0^{-1}(T_F^{-1}(U))$ (simple verification), and $V_p = \beta_p$ on $C_b(X)$, we get ψ is continuous. Now take a net $f_\alpha \to 0$ in $(C_b(X, E), \beta_p)$. This gives $\|f_\alpha\| \to 0$ in $(C_b(X), \beta_p)_p$. Since ψ is continuous, $\|f_\alpha\| \otimes e \to 0$ in $(C_b(X, E), V_p)$. Since $(C_b(X, E), V_p)$ is locally solid and $\|f_\alpha\| \leqq \|(\|f_\alpha\| \otimes e)\|$, we get $f_\alpha \to 0$ in $(C_b(X, E), V_p)$. This proves the theorem.

Theorem 10. *Let X be a P-space* ([2], p. 62). *If $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$, then $(C_b(X, E), \beta_p)$ is Mackey. If E is a Banach space $(C_b(X, E), \beta_p)$ is strongly Mackey.*

Proof. Putting $F = C_b(X, E)$ and $F' = M_p(X, E')$, let A be a norm-bounded relatively countably compact subset of $(F', \sigma(F', F))$. Since vX is topologically complete and P-space, $M_\infty(vX) = M_\tau(vX)$ ([10], p. 469). Further, since vX is also a P-space ([2], p. 125), $M_t(vX) = M_\tau(vX)$. Now using the fact that vX is realcompact, we get $M_p(vX) \subset M_\infty(vX)$. Combining these facts we get $M_t(vX, E') = M_p(vX, E')$. From this it easily follows that $(C_b(vX, E), \beta_p)' = (C_b(vX, E), \beta_C)'$. Since $(C_b(vX, E), \beta_0)$ is Mackey ([4]) and $\beta_0 \leqq \beta_p$, we get $\beta_0 = \beta_p$. By Theorem 3, $(C_b(vX, E), \beta_p) \to (C_b(X, E), \beta_p)$ $(f \to f|_X)$ is continuous. This means a $\mu \in M_p(X, E')$ gives a

$$\hat{\mu} \in M_p(vX, E'),$$

$$\hat{\mu}(f) = \mu(f|_X), \quad f \in C_b(vX, E).$$

Also it is a simple verification that $|\hat{\mu}|^\sim = |\mu|^\sim$. Thus $\hat{A} = \{\hat{\mu} : \mu \in A\}$ is norm-bounded and $\sigma(M_p(vX, E'), C(vX, E))$ relatively countable subset of $M_p(vX, E')$. Since $M_p(vX, E') = M_t(vX, E')$, $\hat{A}$ is a β_t-equicontinuous subset of $(C_b(vX, E), \beta_t)'$. There exists an increasing sequence of compact subsets K_n of vX, such that

$$|\mu|^\sim (\tilde{X} \setminus K_n) \leqq \frac{p}{(p+1)(n+1)\, 2^{n+1}},$$

for each $\mu \in H$, where $p = \sup\{|\mu|(X) : \mu \in H\}$ [4]. Take any $D \in \mathcal{D}$, $D \subset \tilde{X} \setminus X$. This means $D \subset \tilde{X} \setminus vX$. Define $g_D : \tilde{X} \to R$,

$$g_D = \sum_{i=1}^{\infty} \frac{4(p+1)}{n} \chi_{(K_n \setminus K_{n-1})} \quad (K_0 = \emptyset).$$

g_D vanishes at infinity.

Take $f \in C_b(X, E)$, $\|f\|^\sim (x)\, g_D(x) \leqq 1$, for every $x \in \tilde{X}$. This gives $\|f\|^\sim \leqq n/4(p+1)$ on $K_n \setminus K_{n-1}$. For a $\mu \in H$,

$$|\mu(f)| \leqq |\mu|(\|f\|) = |\mu|^\sim(\|f\|^\sim) = \sum_{i=1}^{\infty} \int_{K_n \setminus K_{n-1}} \|f\|^\sim \,\mathrm{d}|\mu|^\sim \leqq$$

$$\leqq \sum \frac{n}{4(p+1)} \frac{1}{n\, 2^n} p \leqq 1.$$

This proves A^0 is a 0-nbd. in $(C_b(X, E), \beta_p)$. This proves the theorem.

Theorem 11. *Let X be a paracompact locally compact D_0-space and E is a normed space. Then $(C_b(X, E), \beta_p)$ is Mackey. If E is a Banach space, then $(C_b(X, E), \beta_p)$ is strongly Mackey.*

Proof. A paracompact locally compact space is topologically complete ([1], Theorem 11.2, p. 92). Since X is a D_0-space, this implies X is realcompact ([6], Theorem 4.1). Thus $M_p(X) = M_t(X)$, which implies $M_p(X, E') = M_t(X, E')$. Since the result is known to be true in the case of topology β_0 ([5]), and $\beta_p \geqq \beta_0$, the result now follows.

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