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LATTICE ORDERED GROUPS HAVING A LARGEST CONVERGENCE

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All lattice ordered groups dealt with in this paper are assumed to be abelian. For a lattice ordered group G let Conv G be the system of all convergences on G in the sense of [1] (such convergences on lattice ordered groups were studied also in [2], [3], [5] and [6]; the basic definitions are recalled in Section 1 below). Next, let $b(G)$ be the set of all bounded sequences in G and

$$\mathrm{Conv}_b\, G = \{\alpha \cap b(G) \colon \alpha \in \mathrm{Conv}\, G\} \,.$$

The systems Conv G and $\mathrm{Conv}_b\, G$ are partially ordered by inclusion.

We denote by T the class of all lattice ordered groups G such that Conv G possesses a largest element.

In [5] it was proved that if G is an archimedean and completely distributive lattice ordered group, then $G \in T$.

In the present paper the following results will be proved:

The existence of a largest element in Conv G depends merely on the lattice properties of G; i.e., if G_1 and G_2 are lattice ordered groups such that G_1 and G_2 are isomorphic as lattices and if $G_1 \in T$, then $G_2 \in T$. (Let us remark that under the above conditions G_1 and G_2 need not be isomorphic as lattice ordered groups.)

The partially ordered set $\mathrm{Conv}_b\, G$ has a largest element if and only if $G \in T$.

The class T is closed with respect to convex l-subgroups and with respect to joins of convex l-subgroups. (Thus T is a radical class in the sense of [4].) T fails to be a variety.

If H is a lattice ordered group, then the radical $T(H)$ of H corresponding to the radical class T is a closed l-subgroup of H.

The notion of homogeneous convergence will be introduced and some results concerning this notion will be established.

1. PRELIMINARIES

Throughout the paper, G denotes a lattice ordered group. Let N be the set of all positive integers. The direct product $\prod_{n\in N} G_n$, where $G_n = G$ for each $n \in N$, will be denoted by G^N. The elements of G^N will be written as $(g_n)_{n\in N}$, or simply (g_n). If $g \in G$ and $g_n = g$ for each $n \in N$, then we write $(g_n) = \mathrm{const}\, g$.

(g_n) is said to be a *sequence* in G. The notion of a subsequence has the usual meaning.

Let $(G^N)^+$ be the positive cone of G^N and let α be a convex subsemigroup of $(G^N)^+$ such that the following conditions are valid:

(I) If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.

(II) Let $(g_n) \in (G^N)^+$. If each subsequence of (g_n) has a subsequence belonging to α, then (g_n) belongs to α.

(III) Let $g \in G$. Then const g belongs to α if and only if $g = 0$.

Under these assumptions α is said to be a *convergence in* G. The system of all convergences in G will be denoted by Conv G; this system is partially ordered by inclusion.

For $(g_n) \in G^N$ and $g \in G$ we put $g_n \to_\alpha g$ if and only if $(|g - g_n|) \in \alpha$.

The pair (G, α) will be said to be a *convergence lattice ordered group* (or a convergence l-group). If no misunderstanding can occur, then we sometimes write G and $g_n \to g$ instead of (G, α) or $g_n \to_\alpha g$, respectively.

1.1. Proposition. (Cf. [3].) *The partially ordered set* Conv G *is a* $\wedge$*-semilattice. Each interval of* Conv G *is a complete Brouwerian lattice.*

1.2. Proposition. (Cf. [3].) *The following conditions are equivalent:*

(i) Conv G *has a greatest element.*

(ii) Conv G *is a lattice.*

(iii) Conv G *is a complete lattice.*

2. REGULAR SEQUENCES

A nonempty subset S of $(G^N)^+$ will be said to be *regular* if there exists $\alpha \in$ Conv G such that $S \subseteq \alpha$. A sequence (g_n) in G^+ is called *regular* if the one-element set $\{(g_n)\}$ is regular.

Let $A \subseteq (G^N)^+$. We denote by δA the system of all subsequences of sequences belonging to A. The convex closure (in$(G^N)^+$) of the set $A \cup \{\text{const } 0\}$ will be denoted by $[A]$. Next, let $\langle A \rangle$ be the subsemigroup of $(G^N)^+$ generated by the set A. The symbol A^* will denote the set of all sequences in G^+ each subsequence of which has a subsequence belonging to A.

2.1. Proposition. (Cf. [2].) *Let* $\emptyset \neq A \subseteq (G^N)^+$. *Then the following conditions are equivalent:*

(a) A *is regular.*

(b) *If* $g \in G$, const $g \in [\langle \delta A \rangle]$, *then* $g = 0$.

2.2. Proposition. (Cf. [2].) *Let* A *be a regular subset of* $(G^N)^+$. *Then*

(i) $[\langle \delta A \rangle]^* \in$ Conv G.

(ii) *If* $\alpha \in$ Conv G, $A \subseteq \alpha$, *then* $[\langle \delta A \rangle]^* \subseteq \alpha$.

If A is regular, then the convergence $[\langle \delta A \rangle]^*$ is said to be *generated by* A.

2.3. Lemma. *Let $a_1, a_2 \in G^+$. Then $a_1 + a_2 \leqq 2a_1 \vee 2a_2$.*

Proof. Denote $u = a_1 \wedge a_2$, $x = a_1 - u$, $y = a_2 - u$. Then $x \wedge y = 0$, hence $x \vee y = x + y$. Therefore

$$a_1 \vee a_2 = (u + x) \vee (u + y) = u + (x \vee y) = u + x + y\,,$$
$$u + (a_1 \vee a_2) = 2u + x + y = a_1 + a_2\,,$$
$$u + (a_1 \vee a_2) = (u + a_1) \vee (u + a_2) \leqq 2a_1 \vee 2a_2\,.$$

Thus $a_1 + a_2 \leqq 2a_1 \vee 2a_2$.

2.4. Lemma. *Let $a_1, a_2, \ldots, a_k \in G^+$, $k \geqq 2$. Then $a_1 + a_2 + \ldots + a_k \leqq$ $\leqq ma_1 \vee ma_2 \vee \ldots \vee ma_k$, where $m = 2^{k-1}$.*

Proof. This follows by induction from 2.3.

Let us remark that if x_i $(i \in I)$ are elements of G such that $\bigwedge_{i\in I} x_i = 0$ and if m is a positive integer, then $\bigwedge_{i\in I} mx_i = 0$.

2.5. Lemma. *Let $(g_n) \in (G^N)^+$. Then the following conditions are equivalent:*

(i) *(g_n) is regular.*

(ii) *If $(h_n^1), (h_n^2), \ldots, (h_n^k)$ are subsequences of (g_n), $h_n = h_n^1 \vee h_n^2 \vee \ldots \vee h_n^k$ $(n = 1, 2, \ldots)$, then $\bigwedge_{n\in N} h_n = 0$.*

Proof. Let (i) be valid. Assume that (ii) fails to hold. Hence there is $0 < g \in G$ such that $g \leqq h_n$ for each $n \in N$. There exists $\alpha \in \operatorname{Conv} G$ with $(g_n) \in \alpha$. Then $(h_n) \in \alpha$ and thus const $g \in \alpha$, which is a contradiction.

Conversely, assume that (ii) holds. Suppose that (g_n) fails to be regular. Put $A =$ $= \{(g_n)\}$. Thus in view of 2.1 there is $0 < g \in G$ such that $g \in [\langle\delta A\rangle]$. Hence there are subsequences $(h_n^1), (h_n^2), \ldots, (h_n^k)$ of (g_n) and positive integers $m_1, \ldots, m_k$ such that

$$g \leqq m_1 h_n^1 + m_2 h_n^2 + \ldots + m_k h_n^k$$

is valid for each $n \in N$. Put $m = \max\{m_1, m_2, \ldots, m_k\}$. Then

$$g \leqq m(h_n^1 + h_n^2 + \ldots + h_n^k) \quad \text{for} \quad n = 1, 2, \ldots\,.$$

Hence we cannot have $\bigwedge_{n\in N} (h_n^1 + h_n^2 + \ldots + h_n^k) = 0$. Thus there is $0 < g' \in G$

$$g' \leqq h_n^1 + h_n^2 + \ldots + h_n^k \quad \text{for} \quad n = 1, 2, \ldots\,.$$

According to 2.4 we obtain

$$g' \leqq mh_n^1 \vee mh_n^2 \vee \ldots \vee mh_n^k \quad \text{for} \quad n = 1, 2, \ldots$$

where $m = 2^{k-1}$. Because of $mh_n^1 \vee mh_n^2 \vee \ldots \vee mh_n^k = m(h_n^1 \vee h_n^2 \vee \ldots \vee h_n^k)$, we obtain that

$$g' \leqq mh_n \quad \text{for} \quad n = 1, 2, 3, \ldots\,.$$

Hence we cannot have $\bigwedge_{n\in N} h_n = 0$, which is a contradiction.

2.6. Corollary. *The system of all regular sequences of G is uniquely determined by the neutral element of G and by the partial order on G.*

Let us remark that if $(A; +, \leqq)$ and $(A; +_1, \leqq)$ are lattice ordered groups having the same neutral elements, then the groups $(A; +)$ and $(A; +_1)$ need not be isomorphic.

2.7. Lemma. *Let $(g_n), (h_n) \in (G^N)^+$. Then $(g_n + h_n)$ is regular if and only if $(g_n \vee h_n)$ is regular.*

Proof. Let $(g_n + h_n)$ be regular. Hence $(g_n + h_n) \in \alpha$ for some $\alpha \in \operatorname{Conv} G$. Since $g_n \vee h_n \leqq g_n + h_n$, we obtain $(g_n \vee h_n) \in \alpha$. Conversely, let $(g_n \vee h_n)$ be regular; thus $(g_n \vee h_n) \in \beta$ for some $\beta \in \operatorname{Conv} G$. Then $2(g_n \vee h_n) \in \beta$ and in view of 2.3, $(g_n + h_n) \in \beta$.

The following assertion is obvious.

2.8. Lemma. *Let $\alpha \in \operatorname{Conv} G$. Then the following conditions are equivalent:*

(i) *α is a largest element of* Conv G.

(ii) *Let $(g_n) \in (G^N)^+$. Then $(g_n) \in \alpha$ if and only if (g_n) is regular.*

3. THE LARGEST ELEMENT OF Conv G

We denote by $R(G)$ the set of all regular sequences of G.

3.1. Lemma. *The following conditions are equivalent:*

(i) Conv G *has no largest element.*

(ii) *The set $R(G)$ fails to be regular.*

Proof. This is an immediate consequence of 2.8.

3.2. Corollary. *The following conditions are equivalent:*

(i) Conv G *has no largest element.*

(ii) *There is a positive integer $k \geqq 2$ and there are regular sequences $(h_n^1), (h_n^2), \ldots$ $\ldots, (h_n^k)$ and $0 < g' \in G$ such that $g' \leqq h_n^1 + h_n^2 + \ldots + h_n^k$ holds for each $n \in N$.*

3.3. Lemma. *Assume that the condition* (ii) *from* 3.2 *is valid. Let k be the least positive integer having the above mentioned property. Then $k = 2$.*

Proof. By way of contradiction, assume that $k > 2$. Let $(h_n^1), (h_n^2), \ldots, (h_n^k)$ and g' be as in the condition (ii) of 3.2. Then the set $A = \{(h_n^1), (h_n^2), \ldots, (h_n^{k-1})\}$ is regular. Put $h_n = h_n^1 + h_n^2 + \ldots + h_n^{k-1}$ for each $n \in N$. Hence there exists $\alpha \in \operatorname{Conv} G$ with $A \subseteq \alpha$ and thus $(h_n) \in \alpha$. This yields that (h_n) is regular. In view of the assumption, the set $\{(h_n), (h_n^k)\}$ is regular as well. Then $(h_n + h_n^k)$ is regular, which is a contradiction to (ii) of 3.2.

From 3.2, 3.3 and 2.7 we obtain:

3.4. Proposition. *The following conditions are equivalent:*

(i) Conv G *has a largest element.*

(ii) *If (h_n^1) and (h_n^2) are regular, then $(h_n^1 + h_n^2)$ is regular.*

(iii) *If (h_n^1) and (h_n^2) are regular, then $(h_n^1 \vee h_n^2)$ is regular.*

Let T be as above.

3.5. Theorem. *Let G_1 and G_2 be lattice ordered groups. Assume that G_1 and G_2 are isomorphic as lattices and that $G_1 \in T$. Then $G_2 \in T$.*

Proof. There exists an isomorphism φ of the lattice G_1 onto the lattice G_2 such that $\varphi(0) = 0$. Let $(g_n) \in (G_1^N)^+$. In view of 2.5 we have

$$(g_n) \in R(G_1) \Leftrightarrow (\varphi(g_n)) \in R(G_2) .$$

Now, from 3.4 (namely, from the equivalence (i) $\Leftrightarrow$ (iii)) we infer that $G_2 \in T$.

The following example shows that under the assumptions as in 3.5, the lattice ordered groups G_1 and G_2 need not be isomorphic.

3.6. Example. Let G_1 and Z be the additive groups of all rational numbers and all integers, respectively, with the natural linear order. Let G_2 be the lexicographic product $Z \circ G_1$. Then G_1 and G_2 are isomorphic as lattices, but they are not isomorphic as lattice ordered groups.

4. BOUNDED SEQUENCES

We denote by $b(G)$ the system of all bounded sequences of G. For $\alpha \in \operatorname{Conv} G$ we put $b(\alpha) = \alpha \cap b(G)$. Let $\operatorname{Conv}_b G = \{b(\alpha) : \alpha \in \operatorname{Conv} G\}$. The system $\operatorname{Conv}_b G$ is partially ordered by inclusion.

The following assertion is obvious.

4.1. Lemma. $\operatorname{Conv}_b G \subseteq \operatorname{Conv} G$, *and whenever* $\alpha \in \operatorname{Conv} G$, $\beta \in \operatorname{Conv}_b G$, $\alpha \leqq \beta$, *then* $\alpha \in \operatorname{Conv}_b G$.

From 4.1 and 1.1 we obtain

4.2. Corollary. *The partially ordered set* $\operatorname{Conv}_b G$ *is a* $\wedge$*-semilattice. Each interval of* $\operatorname{Conv}_b G$ *is a Brouwerian lattice.*

Also, the definition of $b(\alpha)$ immediately yields

4.3. Lemma. *If α_0 is the largest element of* $\operatorname{Conv} G$, *then $b(\alpha_0)$ is the largest element of* $\operatorname{Conv}_b G$.

4.4. Lemma. *Let $0 < v \in G$, $(x_n) \in (G^N)^+$. Put $\{(x_n)\} = A$. Assume that $x_n \in [0, v]$ for each $n \in N$ and that $(y_n) \in [\langle \delta A \rangle]$. Then there is a positive integer m such that $y_n \in [0, mv]$ for each $n \in N$.*

Proof. There is a positive integer m and there are $(h_n^1), \ldots, (h_n^m) \in \delta A$ such that

$$0 \leqq y_n \leqq h_n^1 + h_n^2 + \ldots + h_n^m \quad \text{for each} \quad n \in N .$$

Then $y_n \in [0, mv]$ for each $n \in N$.

4.5. Lemma. *Let (x_n), v and A be as in 4.4. Next, let $(z_n) \in [\langle \delta A \rangle]^*$. Then there are positive integers m_0 and n_0 such that for each $n \geqq n_0$ we have $z_n \in [0, m_0 v]$.*

Proof. By way of contradiction, assume that the assertion of the lemma does not hold. Then there is a subsequence (t_n) of (z_n) such that for each $n \in N$ the relation

$$t_n \notin [0, nv]$$

is valid. Let m be a positive integer. Then no subsequence (q_n) of (t_n) has the property that $q_n \in [0, mv]$ for each $n \in N$. Hence, according to 4.4, no subsequence of (t_n) belongs to $[\langle \delta A \rangle]$. In this way we arrived at a contradiction.

4.6. Lemma. *Assume that* Conv G *has no largest element. Then there exist bounded regular sequences* (z_n^1) *and* (z_n^2) *in* G^+ *and an element* $h \in G$ *with* $h > 0$ *such that* $h = z_n^1 \vee z_n^2$ *is valid for each* $n \in N$.

Proof. In view of 3.4, there exist regular sequences (h_n^1) and (h_n^2) in G^+ such that (h_n) fails to be regular, where $h_n = h_n^1 \vee h_n^2$ for each $n \in N$.

Hence according to 2.5 there exist subsequences $(x_n^1), (x_n^2), \ldots, (x_n^k)$ of (h_n) and an element $0 < h \in G$ such that

$$0 < h \leqq x_n^1 \vee x_n^2 \vee \ldots \vee x_n^k \quad \text{for each} \quad n \in N\,.$$

Thus

$$h = (h \wedge x_n^1) \vee (h \wedge x_n^2) \vee \ldots \vee (h \wedge x_n^k) \quad \text{for each} \quad n \in N\,.$$

For each $j \in \{1, 2, \ldots, k\}$ and each $n \in N$ there is $n(j) \in N$ such that

$$x_n^j = h_{n(j)}^1 \vee h_{n(j)}^2\,.$$

Denote $y_n^{1j} = h_{n(j)}^1$, $y_n^{2j} = h_{n(j)}^2$. Hence we have

$$h = (h \wedge y_n^{11}) \vee \ldots \vee (h \wedge y_n^{1k}) \vee (h \wedge y_n^{21}) \vee \ldots \vee (h \wedge y_n^{2k}) =$$
$$= [h \wedge (y_n^{11} \vee \ldots \vee y_n^{1k})] \vee [h \wedge (y_n^{21} \vee \ldots \vee y_n^{2k})]\,.$$

The sequences (y_n^{1j}) $(j = 1, 2, \ldots, k)$ are subsequences of (h_n^1), hence they are regular. Similarly, the sequences (y_n^{2j}) $(j = 1, 2, \ldots, k)$ are regular. Thus both $(y_n^{11} \vee \ldots \vee y_n^{1k})$ and $(y_n^{21} \vee \ldots \vee y_n^{2k})$ are regular. Denote

$$z_n^1 = h \wedge (y_n^{11} \vee \ldots \vee y_n^{1k})\,, \quad z_n^2 = h \wedge (y_n^{21} \vee \ldots \vee y_n^{2k})\,.$$

Then (z_n^1) and (z_n^2) are regular and bounded; we have $h = z_n^1 \vee z_n^2$ for each $n \in N$.

As a corollary we obtain

4.7. Lemma. *Assume that* Conv G *has no largest element. Then* $\text{Conv}_b\, G$ *has no largest element.*

Summarizing, we have

4.8. Theorem. Conv G *has a largest element if and only if* $\text{Conv}_b\, G$ *has a largest element.*

5. CONVEX l-SUBGROUPS AND THEIR JOINS

Let H be a convex l-subgroup of G.

For each $\alpha \in \text{Conv}\, G$ we denote by $\varphi_H(\alpha)$ the set $\alpha \cap (H^N)^+$.

Next, for each $\beta \in \operatorname{Conv} H$ let $\psi_G(\beta)$ be the set of all $(z_n) \in (G^N)^+$ such that there is a positive integer m such that $(z_{m+n})_{n\in N} \in \beta$.

By the conditions (I), (II) and (III) of Section 1 we immediately obtain

5.1. Lemma. (i) *Let* $\alpha \in \operatorname{Conv} G$. *Then* $\varphi_H(\alpha) \in \operatorname{Conv} H$ *and* $\psi_G(\varphi_H(\alpha)) \subseteq \alpha$.
(ii) *Let* $\beta \in \operatorname{Conv} H$. *Then* $\psi_G(\beta) \in \operatorname{Conv} G$ *and* $\varphi_H(\psi_G(\beta)) = \beta$.
If $\alpha_1, \alpha_2 \in \operatorname{Conv} G$ and $\beta_1, \beta_2 \in \operatorname{Conv} H$, *then we clearly have*

$$\alpha_1 \leqq \alpha_2 \Rightarrow \varphi_H(\alpha_1) \leqq \varphi_H(\alpha_2),$$
$$\beta_1 \leqq \beta_2 \Leftrightarrow \psi_G(\beta_1) \leqq \psi_G(\beta_2).$$

Thus we get

5.2. Lemma. *If* α_1 *is the largest element of* $\operatorname{Conv} G$, *then* $\varphi_H(\alpha_1)$ *is the largest element of* $\operatorname{Conv} H$.

5.3. Corollary. *The class* T *is closed with respect to convex* l*-subgroups.*

Let H_i $(i \in I)$ be convex l-subgroups of G and let $H = \bigvee_{i\in I} H_i$ be their join. It is well-known that for each $0 < h \in H$ there is a finite subset I_1 of I and there are elements $0 < h'_i \in H_i$ $(i \in I_1)$ such that $h = \sum h'_i$ $(i \in I_1)$. Hence according to 2.4 there are $0 < h_i \in H_i$ $(i \in I_1)$ such that $h = \bigvee h_i$ $(i \in I_1)$.

5.4. Lemma. *Assume that all* H_i $(i \in I)$ *belong to* T. *Then* H *belongs to* T *as well.*

Proof. By way of contradiction, assume that $\operatorname{Conv} H$ has no largest element. Then in view of 4.6 there exist regular sequences (z_n^1) and (z_n^2) in H and an element $0 < h \in H$ such that $h = z_n^1 \vee z_n^2$ is valid for each $n \in N$.

There are elements $i(1), i(2), \ldots, i(k)$ of I and $0 < t_1 \in H_{i(1)}, \ldots, 0 < t_k \in H_{i(k)}$ such that

$$h = t_1 \vee t_2 \vee \ldots \vee t_k.$$

Thus we have

$$(*) \qquad t_1 = t_1 \wedge h = t_1 \wedge (z_n^1 \vee z_n^2) = (t_1 \wedge z_n^1) \vee (t_1 \wedge z_n^2) \quad \text{for each} \quad n \in N.$$

The sequences $(t_1 \wedge z_n^1)$ and $(t_1 \wedge z_n^2)$ are regular in $H_{i(1)}$. In view of the assumption, $\operatorname{Conv} H_{i(1)}$ has a largest element and hence by 3.4 the sequence $((t_n \wedge z_n^1) \vee (t_n \wedge z_n^2)$ is regular in $H_{i(1)}$, which contradicts $(*)$.

Summarizing, from 5.2 and 5.4 we obtain

5.5. Theorem. *T is a radical class of lattice ordered groups.*

As usual, we denote by $T(G)$ the radical of G corresponding to the radical class T. Hence $T(G)$ is the largest convex l-subgroup of G belonging to T.

5.6. Theorem. *$T(G)$ is a closed* l*-subgroup of* G.

Proof. It suffices to verify that if h_i $(i \in I)$ are elements of $T(G)$ such that $0 < h_i$ for each $i \in I$ and the relation $\bigvee_{i\in I} h_i = h$ holds in G, then $h \in T(G)$.

By way of contradiction, assume that (under the above notation) the element h

does not belong to $T(G)$. Let H be the convex l-subgroup of G generated by h; thus

$$H = \bigcup_{n\in N} [-nh, nh] .$$

Hence in view of 5.5, H does not belong to T. Thus according to 4.6 there are regular sequences (x_n^1) and (x_n^2) in H and an element $0 < x$ in H such that

$$x = x_n^1 \vee x_n^2 \quad \text{for each} \quad n \in N .$$

There exists a positive integer n such that $x \leqq nh$. Now we can take nh instead of h, and thus without loss of generality we can assume that $x \leqq h$. Hence we have

$$h = x \wedge h = \bigvee_{i\in I} (x \wedge h_i) .$$

Thus there is $i \in I$ such that $x \wedge h_i > 0$; let such an element i be fixed.

From the regularity of (x_n^1) and (x_n^2) in H we infer that the sequences $(x_n^1 \wedge h_i)$ and $(x_n^2 \wedge h_i)$ are regular in $T(G)$; because $T(G) \in T$, we infer that the sequence $((x_n^1 \wedge h_i) \vee (x_n^2 \wedge h_i))$ is regular in $T(G)$. But

$$(x_n^1 \wedge h_i) \vee (x_n^2 \wedge h_i) = (x_n^1 \vee x_n^2) \wedge h_i = x \wedge h_i > 0$$

for each $n \in N$, which is a contradiction.

5.7. Example. Let I be the set of all reals x with $x \in [0, 1]$. For each $i \in I$ let G_i be the additive group of all reals with the natural linear order. Put $G = \prod_{i\in I} G_i$. Then G is completely distributive and archimedean, hence (cf. [5]) $G \in T$. There exists an l-subgroup H of G such that G does not belong to T (cf. [1]). Thus T fails to be closed with respect to l-subgroups. Hence T fails to be a variety.

6. HOMOGENEOUS CONVERGENCES

A convergence $\alpha \in \operatorname{Conv} G$ will be called *homogeneous* if, whenever φ is an automorphism of the lattice ordered group G, then

$$(x_n) \in \alpha \Rightarrow (\varphi(x_n)) \in \alpha .$$

Next, α will be called *strongly homogeneous* if, whenever H and H' are convex l-subgroups of G and φ is an isomorphism of H onto H', then

$$(x_n) \in \alpha \cap (H^N)^+ \Rightarrow (\varphi(x_n)) \in \alpha .$$

The system of all homogeneous convergences or strongly homogeneous convergences on G will be denoted by $\operatorname{Conv}_h G$ or $\operatorname{Conv}_{sh} G$, respectively.

The following example shows that $\operatorname{Conv}_h G$ need not coincide with $\operatorname{Conv} G$.

6.1. Example. Let R be the additive group of all reals with the natural linear order. Let α be the o-convergence on R. Then $\alpha \in \operatorname{Conv} R$ (in fact, $\alpha \in \operatorname{Conv}_h R$). Let G be the direct product $R \times R$. We define $\beta \in (G^N)^+$ as follows. Let $(z_n) = ((x_n, y_n))$ be a sequence in G^+. We put $(z_n) \in \beta$ if and only if

(i) $x_n \to_\alpha 0$, and

(ii) there is $n_0 \in N$ such that $y_n = 0$ for each $n > n_0$. Then $\beta \in \operatorname{Conv} G$, and β fails to be homogeneous.

Clearly $\operatorname{Conv}_{sh} G \subseteq \operatorname{Conv}_h G$. The following example shows that Conv_{sh} need not coincide with $\operatorname{Conv}_h G$.

6.2. Example. Put $G = (R \circ R) \times R$, where $\circ$ denotes the operation of the lexicographic product. Thus the elements of G have the form (x, y, z), the operation $+$ being performed componentwise and $(x, y, z) \geqq 0$ if $(x, y) \geqq 0$ and $z \geqq 0$. (The relation $(x, y) \geqq 0$ means that either $x > 0$, or $x = 0$ and $y \geqq 0$.) Let $t_n = (x_n, y_n, z_n)$ be a sequence in G^+. We define $\alpha \subset (G^N)^+$ by putting $(t_n) \in \alpha$ if and only if

(a) (z_n) o-converges to 0 in R, and

(b) there is $n_0 \in N$ such that $x_n = y_n = 0$ for each $n > n_0$. Then $\alpha \in \operatorname{Conv}_h G$ and α does not belong to $\operatorname{Conv}_{sh} G$.

Let us consider a nonempty set $\{\alpha_i\}$ $(i \in I)$ of strongly homogeneous congruences on G. The following lemma is obvious.

6.3. Lemma. *$\bigwedge_{i \in I} \alpha_i$ is a strongly homogeneous convergence on G.*

6.4. Lemma. *Assume that $\bigvee_{i \in I} \alpha_i = \alpha$ holds in* Conv G. *Then $\alpha \in \operatorname{Conv}_{sh} G$.*

Proof. Let H and H' be convex l-subgroups of G and let φ be an isomorphism of H onto H'. Let $(h_n) \in \alpha \cap (H^N)^+$.

According to Lemma 2.3, [3] we have $\alpha = \langle \bigcup \alpha_i \rangle^*$. Hence we have to verify that $(\varphi(h_n)) \in \langle \bigcup \alpha_i \rangle^*$. Let $(\varphi(h_m))$ be a subsequence of $(\varphi(h_n))$. Then (h_m) is a subsequence of (h_n), hence there exists a subsequence (h_t) of (h_m) such that $(h_t) \in \langle \bigcup \alpha_i \rangle$. Thus there are $i(1), i(2), \ldots, i(k) \in I$ and $(h_t^1) \in \alpha_{i(1)}, \ldots, (h_t^k) \in \alpha_{i(k)}$ with

$$h_t = h_t^1 + \ldots + h_k^k \quad \text{for each} \quad t \in N .$$

Then $(h_t^1), \ldots, (h_t^k) \in (H^N)^+$ and so $(\varphi(h_t^1)) \in \alpha_{i(1)}, \ldots, (\varphi(h_t^k)) \in \alpha_{i(k)}$. Since

$$\varphi(h_t) = \varphi(h_t^1) + \ldots + \varphi(h_t^k) ,$$

we obtain that $(\varphi(h_t)) \in \langle \bigcup \alpha_i \rangle$, whence $(\varphi(h_n)) \in \langle \bigcup \alpha_i \rangle^*$, completing the proof.

For $\alpha_1, \alpha_2 \in \operatorname{Conv} G$ we denote, as usual,

$$[\alpha_1, \alpha_2] = \{\alpha \in \operatorname{Conv} G\colon \alpha_1 \leqq \alpha \leqq \alpha_2\} ;$$

next, for $\beta_1, \beta_2 \in \operatorname{Conv}_{sh} G$ we put

$$[\beta_1, \beta_2]_{sh} = \{\beta \in \operatorname{Conv}_{sh} G\colon \beta_1 \leqq \beta \leqq \beta_2\} .$$

Then Lemma 6.3, Lemma 6.4 and 1.1 yield

6.5. Proposition. $\operatorname{Conv}_{sh} G$ *is a $\bigwedge$-semilattice. Let $\beta_1, \beta_2 \in \operatorname{Conv}_{sh} G$, $\beta_1 \leqq \beta_2$. Then $[\beta_1, \beta_2]_{sh}$ is a closed sublattice of the lattice $[\beta_1, \beta_2]$. Hence $[\beta_1, \beta_2]_{sh}$ is a complete Brouwerian lattice.*

Let d be the least element of Conv G. Clearly $d \in \operatorname{Conv}_{sh} G$.

6.6. Proposition. *The following conditions are equivalent:*

(i) $\mathrm{Conv}_{sh}\, G$ *is upper-directed.*

(ii) $\mathrm{Conv}_{sh}\, G$ *is a lattice.*

(iii) $\mathrm{Conv}_{sh}\, G$ *possesses a largest element.*

(iv) $\mathrm{Conv}_{sh}\, G$ *is a complete lattice.*

The proof is analogous to that concerning Conv G (cf. [3]); it will be omitted.

The following assertion is an immediate consequence of 2.1.

6.7. Lemma. *Let H be a convex l-subgroup of G and let $X \subseteq (H^N)^+$. Then the following conditions are equivalent*:

(i) *X is regular with respect to G.*

(ii) *X is regular with respect to H.*

6.8. Proposition. *Assume that* Conv G *has a largest element β. Then $\beta \in \mathrm{Conv}_{sh}\, G$.*

Proof. Let H, H' and φ be as in the proof of 6.4. Let $(h_n) \in (H^N)^+$ such that $(h_n) \in \beta$. Hence the sequence (h_n) is regular with respect to G. In view of 6.7, (h_n) is regular with respect to H. Thus $(\varphi(h_n))$ is regular with respect to H'. By applying 6.7 again we infer that $(\varphi(h_n))$ is regular with respect to G. Hence $(\varphi(h_n)) \in \beta$, completing the proof.

Consider the following conditions:

(a) Conv G has a largest element.

(b) $\mathrm{Conv}_{sh}\, G$ has a largest element.

In view of 6.8, (a) $\Rightarrow$ (b). The question whether for each lattice ordered group G the implication $(b) \Rightarrow (a)$ holds remains open.

From 6.5 and 6.8 we obtain

6.9. Corollary. *If* Conv G *has a largest element, then* $\mathrm{Conv}_{sh}\, G$ *is a closed sublattice of the lattice* Conv G.

6.10. Proposition. *Assume that* Conv G *has a largest element β. For each $\alpha \in$ Conv G let $h(\alpha)$ be the intersection of all $\alpha_i \in \mathrm{Conv}_{sh}\, G$ such that $\alpha \leqq \alpha_i$. Then h is a closure operation on the lattice* Conv G.

Proof. Let $\alpha \in$ Conv G. Let S be the set of all $\alpha_i \in \mathrm{Conv}_{sh}\, G$ with $\alpha \leqq \alpha_i$. In view of 6.8 we have $\beta \in S$, hence $S \neq \emptyset$. Thus $\alpha \leqq h(\alpha)$. According to 6.3, $h(\alpha)$ belongs to S and thus $h(\alpha)$ is the least element of S. Hence $h(h(\alpha)) = h(\alpha)$.

6.11. Remark. In all the assertions 6.3 – 6.10, $\mathrm{Conv}_{sh}\, G$ can be replaced by $\mathrm{Conv}_h\, G$. The proofs are either the same or analogous to those given above.

7. AN EXAMPLE

In this section an example will be given which shows that the partially ordered set $\mathrm{Conv}_h\, G$ need not have a largest element.

Let Q be the set of all rational numbers and let $a \in R$ be a positive irrational number. For each $n \in N$ we denote

$$B_{nk} = \left\{t \in Q: \frac{k-1}{2^n} a < t < \frac{k}{2^n} a\right\} \quad (k = 1, 2, \ldots, 2^n).$$

Let S be the system of all pairs (n, k) with $n \in N$ and $k \in \{1, 2, \ldots, 2^n\}$. The system S is lexicographically linearly ordered; i.e., we put $(n_1, k_1) < (n_2, k_2)$ if either $n_1 < n_2$, or $n_1 = n_2$ and $k_1 < k_2$. Let P be the set of all positive primes with the natural linear order. There exists a uniquely determined isomorphism f of S onto P. The image of (n, k) under f will be denoted by $f(n, k)$.

For $P_1 \subseteq P$ let $H(P_1)$ be the subgroup of the additive group Q generated by the set

$$\left\{\frac{1}{p^n}: p \in P_1, n \in N\right\}.$$

Let G be the set of all real functions x defined on the set $Q_1 = \{t \in Q: 0 < t < a\}$ which satisfy the following conditions:

(i) for each $t \in Q_1$ we have $x(t) \in H(P_1)$, where $P_1 = \{f(n, k): t \in B_{nk}\}$;

(ii) there is $n \in N$ such that, whenever $k \in \{1, 2, \ldots, 2^n\}$ and $t_1, t_2 \in B_{nk}$, then $x(t_1) = x(t_2)$.

Let G_0 be the lattice ordered group of all real functions defined on the set Q_1 (the operations $+$, $\wedge$ and $\vee$ being defined componentwise).

The following assertion is obvious.

7.1. Lemma. *G is an l-subgroup of G_0.*

7.2. Lemma. *Let $0 < x \in G$, $t_0 \in Q_1$, $x(t_0) > 0$. Then there are $n \in N$, $k \in \{1, 2, \ldots, 2^n\}$ and $x_1 \in G$ such that*

(i_1) $0 < x_1 \leqq x$;

(ii_2) if $t_1, t_2 \in B_{nk}$, then $x(t_1) = x(t_2)$; if $t \in Q_1 \setminus B_{nk}$, then $x_1(t) = 0$.

Proof. The assertion is a consequence of the condition (ii) above.

For $Z \subseteq G$ we put

$$Z^\perp = \{y \in G: |y| \wedge |z| = 0 \text{ for each } z \in Z\}.$$

For $z \in G$ let Sup z be the support of z. In view of the definition of G we have

7.3. Lemma. *Let x and x_1 be as in* 7.2. *Then* Sup $y \subseteq$ Sup x_1 *whenever* $y \in \{x_1\}^{\perp\perp}$.

Lemmas 7.2 and 7.3 yield

7.4. Lemma. *Let φ be an automorphism of the lattice ordered group G. Let $0 < x \in G$. Then* Sup x = Sup $\varphi(x)$.

Let us denote by X the system of all sequences (x_n) in G^+ which satisfy the following condition:

if $n \in N$, $k \in \{1, 2, \ldots, 2^n\}$ and k is even, then $x_n(t) = 0$ for each $t \in B_{nk}$.

Next, let Y be defined analogously with the distinction that "even" is replaced by "odd".

7.5. Lemma. *Let φ be as in* 7.4. *If $(x_n) \in X$, then $(\varphi(x_n)) \in X$.*

Proof. This is a consequence of 7.4.

An analogous assertion holds for Y.

7.6. Lemma. *The sets X and Y are regular.*

The proof can be performed by a straight-forward application of 2.1.

As a corollary we obtain

7.7. Lemma. *There exist $\alpha, \beta \in$* Conv *G such that α is generated by X and β is generated by Y.*

7.8. Lemma. *If $(x_n) \in \alpha$, $(y_n) \in \beta$ and if φ is an automorphism of the lattice ordered group G, then $(\varphi(x_n)) \in \alpha$ and $(\varphi(y_n) \in \beta$. Hence both α and β are homogeneous.*

Proof. This is a consequence of 7.5 and of the corresponding result for Y.

7.9. Lemma. *The set $\{\alpha, \beta\}$ is not upper bounded in* Conv *G.*

Proof. Let us denote by z the element of G such that $z(t) = 1$ for each $t \in Q_1$. Next, we define a sequence (x_n) in G^+ as follows. Let $n \in N$, $t \in Q_1$. We put $x_n(t) = 1$ if $t \in B_{nk}$, where k is an even number, and $x_n(t) = 0$ otherwise. Let $y_n = z - x_n$ for each $n \in N$. Then $(x_n) \in X$ and $(y_n) \in Y$, hence $(x_n) \in \alpha$ and $(y_n) \in \alpha$. Since $x_n + y_n = z > 0$ for each $n \in N$, in view of 2.1 the set $\{\alpha, \beta\}$ is not upper-bounded in Conv G.

7.10. Proposition. *The partially ordered set* Conv_h *G has no greatest element.*

Proof. This is a consequence of 7.8 and 7.9.

7.11. Remark. If G is a lattice ordered group such that Conv G possesses a greatest element, then the results of Section 6 imply that for each $\alpha \in$ Conv G there exists a homogeneous convergence $h(\alpha)$ which has the following properties:

(i) $\alpha \leqq h(\alpha)$,

(ii) if $\beta \in \mathrm{Conv}_h G$ and $\alpha \leqq \beta$, then $h(\alpha) \leqq \beta$.

More generally (without assuming the existence of a greatest element in Conv G), a homogeneous convergence $h(\alpha)$ satisfying (i) and (ii) will be said to be a *homogeneous closure of α.*

7.12. Example. By modifying the above example we shall construct an example which shows that the homogeneous closure need not exist in general.

Let G be as above and let H be the set of all $g \in G$ having the property that $g(t)$ is an integer for each $t \in Q_1$. Then H is an l-subgroup of G. Let (x_n) and (y_n) be as above. Then (x_n) is a regular sequence in H, hence there is $\alpha_1 \in$ Conv H such that $(x_n) \in \alpha_1$. Assume that there exists $\beta_1 \in \mathrm{Conv}_h H$ with $\alpha_1 \leqq \beta_1$. It is not difficult to verify that there is an automorphism φ of the lattice ordered group H such that

$\varphi(x_n) = y_n$ for each $n \in N$. Hence $\{(x_n), (y_n)\} \subseteq \beta_1$. This is a contradiction, because we have shown above that the set $\{(x_n), (y_n)\}$ fails to be regular in G, and the same investigation shows that this set is not regular in H.

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