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NECESSARY AND SUFFICIENT CONDITIONS FOR BOUNDED OSCILLATIONS OF HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS OF EULER'S TYPE

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1. INTRODUCTION

Consider the linear delay differential equation of the form

$$x^{(n)}(t) + (-1)^{n+1}\, t^{-n} \sum_{i=1}^{m} p_i\, x(\sigma_i t) = 0\,, \quad t \geqq a > 0\,, \tag{1}$$

where $p_i > 0$, $0 < \sigma_i \leqq 1$, $i = 1, \ldots, m$, are constants.

It is well-known that if $x(t)$ is a nonoscillatory solution of (1) then according to a result of Kiguradze (see [3, Lemma 1]) there exist an even integer l, $0 \leqq l \leqq n$, and $t_1 \geqq a$ such that

$$x(t)\, x^{(i)}(t) > 0 \quad \text{on} \quad [t_1, \infty) \quad \text{for} \quad i = 0, 1, \ldots, l\,, \tag{2}$$

$$(-1)^{i-l}\, x(t)\, x^{(i)}(t) > 0 \quad \text{on} \quad [t_1, \infty) \quad \text{for} \quad i = l, l+1, \ldots, n\,. \tag{3}$$

Denote by N_l the set of all solutions of (1) which satisfy (2) and (3). Then the set N of all nonoscillatory solutions of (1) has the following decomposition:

$$N = N_0 \cup N_2 \cup \ldots \cup N_{n-1} \quad \text{if } n \text{ is odd}\,,$$

$$N = N_0 \cup N_2 \cup \ldots \cup N_n \quad \text{if } n \text{ is even}\,.$$

In the present paper we characterize the situation in which $N_0 = \emptyset$, that is, all bounded solutions of Eq. (1) are oscillatory. Next, we shall establish necessary and sufficient conditions under which $N_n = \emptyset$ for the equations of the advanced type

$$x^{(n)}(t) - t^{-n} \sum_{i=1}^{m} p_i\, x(\sigma_i t) = 0\,, \quad t \geqq a > 0 \tag{4}$$

where $p_i > 0$, $\sigma_i \geqq 1$, $i = 1, \ldots, m$, are constants. Kiguradze's decomposition of the set of all nonoscillatory solutions is the same as above if n is even, and becomes

$$N = N_1 \cup N_3 \cup \ldots \cup N_n \quad \text{if } n \text{ is odd}\,.$$

It is known that if the deviations of the arguments are absent in (1) (or in (4)) then the subclasses N_0 and N_n are always nonempty. However, this is not true in general when $\sigma_i < 1$ or $\sigma_i > 1$ for some i, $1 \leqq i \leqq m$ (see, for example, [5] and the references cited therein).

Recently, Ladas, Sficas and Stavroulakis [8] have studied the similar problem for the retarded differential equation of the form

$$x^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{m} p_i x(t - \tau_i) = 0\,, \quad t \geqq a\,, \tag{5}$$

where $p_i > 0$, $\tau_i \geqq 0$, $i = 1, \ldots, m$, are constants, and proved that every bounded solution of (5) is oscillatory if and only if the corresponding characteristic equation

$$\lambda^n + (-1)^{n+1} \sum_{i=1}^{m} p_i \, e^{-\lambda \tau_i} = 0 \tag{6}$$

has no real roots in $(-\infty, 0]$.

Surprisingly, however, in literature there are not many results on the oscillation of delay differential equations of the form (1) which would be sharp enough to provide a characterization of the oscillation of all bounded solutions. To the best of the author's knowledge, Nadareišvili [10] in the case $n = 1$ and Čanturia [1] in the case $n > 1$ are the only references which are related to the subject of this paper. For the case $n = 1$ see also the recent paper [2].

In what follows, the oscillatory character of the solution $x(t)$ of (1) defined on an interval $[t_x, \infty)$, $t_x \geqq a$, is considered in the usual sense, that is, $x(t)$ is said to be *oscillatory* if it has arbitrarily large zeros in $[t_x, \infty)$ and it s said to be nonoscillatory otherwise.

As is customary, we shall say that a continuous real-valued function $u(t)$ defined on an interval $[t_u, \infty)$ eventually has some property if there is a $T \geqq t_u$ such that $u(t)$ has this property on $[T, \infty)$.

2. MAIN RESULTS

The problem of the characterization of bounded oscillations of Eq. (1) can be examined in a variety of ways. In this paper we use the method which has been developed by Ladas, Sficas and Stavroulakis in [7] and [8]. For an alternative approach we refer to Čanturia [1].

We begin with a simple lemma which will be needed in the proof of our main theorem.

Lemma. *Let $x(t)$ be a bounded nonoscillatory solution of the retarded differential inequality*

$$\operatorname{sgn} x(\sigma t) \{(-1)^n x^{(n)}(t) - p t^{-n} x(\sigma t)\} \geqq 0\,, \quad t \geqq a > 0\,, \tag{7}$$

where $p > 0$, $0 < \sigma < 1$, defined on the interval $[\sigma a, \infty)$. Then

$$|x(t)| \geqq A^{n+1} |x(\sigma t)|$$

for all large t, where $A = (p \ln (1/\sigma))/2$ *if* $n = 1$ *and*

$$A = p \ln 1/\sigma \frac{n}{(n+1)!} \prod_{k=1}^{n-1} (1 - \sigma^{(n-k)/(n+1)}) \quad \text{if} \quad n > 1 .$$

Proof. Without loss of generality, we may assume that $x(t)$ is positive and decreasing on $[T, \infty)$, $T \geqq a$. Then the inequality (7) becomes

$$(-1)^n x^{(n)}(t) - pt^{-n} x(\sigma t) \geqq 0 , \quad t \geqq \sigma^{-1} T. \tag{9}$$

Let $s \geqq \sigma^{-1}T$ be given. Integrating both sides of (9) from s to $\sigma^{-1/n+1}s$ and using the decreasing character of $x(t)$ on $[T, \infty)$ we obtain that

$$(-1)^n x^{(n-1)}(\sigma^{-1/n+1}s) - (-1)^n x^{(n-1)}(s) - px(\sigma^{n/n+1}s) \int_s \sigma^{(-1/n+1)} t^{-n} \, dt \geqq 0 .$$

If $n = 1$, then

$$x(s) \geqq \frac{p \ln 1/\sigma}{2} x(\sigma^{1/2}s) , \quad s \geqq \sigma^{-1}T. \tag{10}$$

Given $t \geqq \sigma^{-3/2}T$ we apply (10) to $s = \sigma^{1/2}t$ and so $s = t$ and get

$$x(\sigma^{1/2}t) \geqq [(p \ln (1/\sigma))/2] \, x(\sigma t) ,$$

$$x(t) \geqq [(p \ln (1/\sigma))/2] \, x(\sigma^{1/2}t) ,$$

respectively. Combining these inequalities we obtain the desired relation (8) in the case $n = 1$.

If $n > 1$, then repeating the above procedure of integration of (9) from s to $\sigma^{(-1/n+1)}s$ and taking into account that $(-1)^i x^{(i)}(\sigma^{-1/n+1}s) > 0$, $i = 0, 1, \ldots, n$, we find

$$x(s) \geqq p \ln (1/\sigma) \frac{n}{(n+1)!} \prod_{k=1}^{n-1} (1 - \sigma^{(n-k)/(n+1)}) \, x(\sigma^{(1/n+1)}s) . \tag{11}$$

Given $t \geqq \sigma^{-(2n+1)/(n+1)}T$ we apply (11) successively to $s = t$, $s = \sigma^{(1/n+1)}t, \ldots, s = \sigma^{(n/n+1)}t$, and conclude that

$$x(t) \geqq \left[p \ln (1/\sigma) \frac{n}{(n+1)!} \prod_{k=1}^{n-1} (1 - \sigma^{(n-k)/(n+1)}) \right]^{n+1} x(\sigma t)$$

which is the relation (8) in the case $n > 1$.

Theorem 1. *All bounded solutions of Eq.* (1) *are oscillatory* (i.e., $N_0 = \emptyset$) *if and only if*

$$-\alpha(\alpha + 1) \ldots (\alpha + n - 1) + \sum_{i=1}^{m} p_i \sigma_i^{-\alpha} > 0 \tag{12}$$

for all $\alpha > 0$.

Proof. (The "only if" part.) Assume to the contrary that (12) does not hold.

Then there exists an $\alpha_0 > 0$ such that

$$-\alpha_0(\alpha_0 + 1) \dots (\alpha_0 + n - 1) + \sum_{i=1}^{m} p_i \sigma_i^{-\alpha_0} = 0$$

and so Eq. (1) has a nonoscillatory solution $x(t) = t^{-\alpha_0}$.

(The "if" part.) Let (12) hold and assume that there exists an eventually positive solution $x(t)$ of Eq. (1).

When all σ_i, $i = 1, \dots, m$, are equal to 1, then (12) obviously does not hold. Thus, we may assume without any loss of generality that $\sigma_m = \min\{\sigma_1, \dots, \sigma_m\} < 1$.

Define the set

$$A(x) = \{\alpha > 0 : (-1)^{n+1}\, t^n\, x^{(n)}(t) + \alpha\, x(t) < 0 \text{ eventually}\}\,.$$

From (1) we have

$$(-1)^{n+1}\, t^n\, x^{(n)}(t) + p_m\, x(t) < (-1)^{n+1}\, t^n\, x^{(n)}(t) + p_m\, x(\sigma_m t) \leqq 0$$

eventually, so that $p_m \in A(x)$. Consequently, the set $A(x)$ is non-empty.

Taking into account that $x(t)$ is decreasing and using Lemma, we find

$$0 = (-1)^{n+1}\, t^n\, x^{(n)}(t) + \sum_{i=1}^{m} p_i\, x(\sigma_i t) \leqq$$

$$\leqq (-1)^{n+1}\, t^n\, x^{(n)}(t) + \sum_{i=1}^{m} p_i\, x(\sigma_m t) \leqq$$

$$\leqq (-1)^{n+1}\, t^n\, x^{(n)}(t) + B_m^{n+1} \sum_{i=1}^{m} p_i\, x(t)$$

eventually, where $B_m = 2/(p_m \ln(1/\sigma_m))$ if $n = 1$ and

$$B_m = \left[p_m \ln(1/\sigma_m) \frac{n}{(n+1)!} \prod_{k=1}^{n-1} \left(1 - \sigma_m^{(n-k)/(n+1)}\right)\right]^{-1}$$

if $n > 1$. This proves that

$$B_m^{n+1} \sum_{i=1}^{m} p_i$$

is an upper bound of $A(x)$. Notice that this upper bound does not depend on $x(t)$.

Let $\alpha \in A(x)$ be given. Set $\alpha_0 = \alpha$, and for each $j = 1, 2, \dots$ let α_j be the (unique) solution of

$$\alpha_j(\alpha_j + 1) \dots (\alpha_j + n - 1) = \alpha_{j-1}\,.$$

Clearly, for every positive α_{j-1} such a solution α_j always exists since the continuous function $g(\beta) = \beta(\beta + 1) \dots (\beta + n - 1)$, $\beta \geqq 0$, satisfies $g(0) = 0$ and $\lim_{\beta \to \infty} g(\beta) = \infty$. Moreover, due to the increasing character of $g(\beta)$ for $\beta \geqq 0$, this solution is unique.

Define $x_0(t) = x(t)$ and

$$x_j(t) = \sum_{i=0}^{n-1} (-1)^i\, P_{n-i-1}(\alpha_j)\, t^i\, x_{j-1}^{(i)}(t)\,, \quad j = 1, 2, \dots$$

where $P_0(\alpha_j) = 1$ and $P_i(\alpha_j) = \prod_{k=1}^{i} (\alpha_j + n - k)$, $i = 1, 2, \ldots, n - 1$.

Due to Euler's nature of Eq. (1), every function $t^i x^{(i)}(t)$, $i = 1, \ldots, n - 1$, is also a solution of the same equation and, moreover, it belongs to N_0 provided that $x(t)$ belongs to N_0. Consequently, each $x_j(t)$, $j = 1, 2, \ldots$, as a linear combination of solutions from N_0 is itself an eventually positive solution from N_0 of Eq. (1).

The definition of $x_j(t)$ implies that

$$(13)\quad (-1)^{n+1} t^n x_j^{(n)}(t) + \alpha_j x_j(t) = t x'_{j+1}(t) + \alpha_{j+1} x_{j+1}(t), \quad j = 0, 1, \ldots.$$

Denote

$$F(\alpha) = -\alpha(\alpha + 1) \ldots (\alpha + n - 1) + \sum_{i=1}^{m} p_i \sigma_i^{-\alpha}.$$

Since

$$F(\alpha) > 0 \quad \text{for} \quad \alpha > 0, \quad F(0) = \sum_{i=1}^{m} p_i > 0 \quad \text{and} \quad \lim_{\alpha \to \infty} F(\alpha) = \infty,$$

we may conclude that

$$\mu = \min_{\alpha \geqq 0} F(\alpha)$$

exists and has a positive value.

We shall show that $\alpha_j \in A(x_j)$ implies $(\alpha_j + \mu) \in A(x_{j+1})$, $j = 0, 1, \ldots$, and, consequently, $\alpha \in A(x)$ implies $(\alpha + j\mu) \in A(x_j)$, $j = 1, 2, \ldots$ which contradicts the above conclusion that

$$B_m^{n+1} \sum_{i=1}^{m} p_i$$

is a common upper bound for all $A(x_{j+1})$, $j = 0, 1, 2, \ldots$.

In fact, since $x_{j+1}(t)$ is the solution of Eq. (1) we have

$$(-1)^{n+1} t^n x_{j+1}^{(n)}(t) + (\alpha_j + \mu) x_{j+1}(t) = -\sum_{i=1}^{m} p_i x_{j+1}(\sigma_i t) + (\alpha_j + \mu) x_{j+1}(t).$$

Define $u(t) = t^{\alpha_{j+1}} x_{j+1}(t)$. Then

$$\begin{aligned} u'(t) &= t^{\alpha_{j+1}-1}[t x'_{j+1}(t) + \alpha_{j+1} x_{j+1}(t)] = \\ &= t^{\alpha_{j+1}-1}[(-1)^{n+1} t^n x_j^{(n)}(t) + \alpha_j x_j(t)] < 0 \end{aligned}$$

eventually, so that $u(t)$ is an eventually decreasing function.

Thus

$$\begin{aligned} &(-1)^{n+1} t^n x_{j+1}^{(n)}(t) + (\alpha_j + \mu) x_{j+1}(t) = \\ &= -\sum_{i=1}^{m} p_i (\sigma_i t)^{-\alpha_{j+1}} u(\sigma_i t) + (\alpha_j + \mu) t^{-\alpha_{j+1}} u(t) < \\ &< t^{-\alpha_{j+1}} u(t) \left[-\sum_{i=1}^{m} p_i \sigma^{-\alpha_{j+1}} + \alpha_j + \mu\right] = \\ &= t^{-\alpha_{j+1}} u(t) \left[-\sum_{i=1}^{m} p_i \sigma^{-\alpha_{j+1}} + \alpha_{j+1}(\alpha_{j+1} + 1) \ldots (\alpha_{j+1} + n - 1) + \mu\right] \leqq \\ &\leqq t^{-\alpha_{j+1}} u(t) [-\mu + \mu] = 0 \end{aligned}$$

eventually, which proves that $(\alpha_j + \mu)$ belongs to $A(x_{j+1})$ and the proof is complete.

Due to its transcendental nature, (12) is not easily tractable. Therefore, we proceed further and derive some explicit sufficient conditions for oscillation of bounded solutions of Eq. (1) expressed in terms of p_i and σ_i only.

For this purpose, denote

$$M = \{1, 2, \dots, m\}, \quad J = \{j \in M: \sigma_j \neq 1\},$$
$$J_k = \{j \in J: k - 1 < n/(\ln(1/\sigma_j)) \leqq k\}$$

for $k = 1, 2, \dots, n - 1$ and

$$J_n = \{j \in J: n/(\ln(1/\sigma_j)) > n - 1\}.$$

Moreover, denote by K the set of all k, $1 \leqq k \leqq n$, such that $J_k \neq \emptyset$.

Theorem 2. *Assume that*

$$\sum_{k \in K} \frac{e^k(k-1)!}{n^k(n-1)!} \sum_{j \in J_k} p_j (\ln(1/\sigma_j))^k \sigma_j^{k(k-1)/2n} > 1. \tag{14}$$

Then all bounded solutions of Eq. (1) *are oscillatory.*

Proof. We shall show that (14) implies that the condition (12) of Theorem 1 is satisfied.

Let us consider the function

$$G(\alpha) = -1 + [\alpha(\alpha+1)\dots(\alpha+n-1)]^{-1} \sum_{j=1}^{m} p_j \sigma_j^{-\alpha}, \quad \alpha > 0.$$

Clearly,

$$G(\alpha) \geqq -1 + \sum_{k \in K} \sum_{j \in J_k} p_j [\alpha(\alpha+1)\dots(\alpha+n-1)\,\sigma_j^{\alpha}]^{-1} =$$
$$= -1 + \sum_{k \in K} \sum_{j \in J_k} p_j \prod_{i=1}^{n} [(\alpha + i - 1)\,\sigma_j^{\alpha/n}]^{-1}, \quad \alpha > 0.$$

Let $k \in K$ be fixed and denote $G_{ij}(\alpha) = (\alpha + i - 1)^{-1} \sigma_j^{-\alpha/n}$, $j \in J_k$, $i = 1, \dots, n$. Then we have

$$\min_{\alpha > 0} G_{ij}(\alpha) = \frac{e \ln(1/\sigma_j)}{n} \sigma_j^{(i-1)/n}$$

if $i \leqq k$ and

$$\min_{\alpha \geqq 0} G_{ij}(\alpha) = 1/(i-1)$$

if $i > k$. Thus,

$$G(\alpha) \geqq -1 + \sum_{k \in K} \sum_{j \in J_k} p_j \frac{e^k}{n^k} \frac{(k-1!)}{(n-1!)} (\ln(1/\sigma_j))^k \sigma_j^{k(k-1)/2n} > 0$$

for all $\alpha > 0$, which implies that

$$\alpha(\alpha+1)\dots(\alpha+n-1)\,G(\alpha) > 0 \quad \text{for all} \quad \alpha > 0$$

so that the condition (12) of Theorem 1 is satisfied.

Remark 1. In the case $J = J_1$ the condition (14) reduces to

$$\sum_{j=1}^{m} p_j \ln(1/\sigma_j) > n!/\mathrm{e}$$

and in the case $J = J_n$ it reduces to

$$\sum_{j=1}^{m} p_j (\ln(1/\sigma_j))^n \sigma_j^{(n-1)/2} > (n/\mathrm{e})^n .$$

Remark 2. An inspection of the proof of Theorem 2 shows that the functions $G_{ij}(\alpha)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $\alpha > 0$, can be estimated also as follows:

$$G_{ij}(\alpha) = \alpha^{-1}\sigma_j^{-\alpha/n} + (i-1)^{-1}\sigma_j^{-\alpha/n} \geqq \frac{\mathrm{e}\ln(1/\sigma_j)}{n} + 1/(i-1) .$$

This leads to the condition

$$\sum_{j=1}^{m} p_j(\ln(1/\sigma_j))^n \prod_{k=1}^{n}\left(1 + \frac{(k-1)\mathrm{e}}{n}\ln(1/\sigma_j)\right)^{-1} > (n/\mathrm{e})^n \tag{15}$$

(cf. Čanturia [1]). It is not difficult to see that (14) is weaker than (15).

Theorem 3. *Let k, $1 \leqq k \leqq n$, be the largest integer such that*

$$mn/\sum_{j=1}^{m}\ln(1/\sigma_j) > k - 1$$

and let

$$m^{1-k}\left(\prod_{j=1}^{m} p_j\right)^{1/m}\left(\prod_{j=1}^{m}\sigma_j\right)^{k(k-1)/2mn}\left(\sum_{j=1}^{m}\ln(1/\sigma_j)\right)^k > \frac{n^k(n-1)!}{\mathrm{e}^k(k-1)!} . \tag{16}$$

Then all bounded solutions of Eq. (1) *are oscillatory.*

Proof. Let $G(\alpha)$ be defined as in the proof of Theorem 2. The inequality for the aritmetic and geometric means yields that

$$G(\alpha) \geqq -1 + [\alpha(\alpha+1)\ldots(\alpha+n-1)]^{-1} m\left(\prod_{j=1}^{m} p_j\sigma_j^{-\alpha}\right)^{1/m} =$$

$$= -1 + m\left(\prod_{j=1}^{m} p_j\right)^{1/m}\prod_{i=1}^{n}\left[(\alpha+i-1)\left(\prod_{j=1}^{m}\sigma_j\right)^{\alpha/mn}\right]^{-1} .$$

Denote $H_i(\alpha) = (\alpha + i - 1)^{-1}\left(\prod_{j=1}^{m}\sigma_j\right)^{-\alpha/mn}$, $i = 1, \ldots, n$. Then we have

$$\min_{\alpha>0} H_i(\alpha) = (\mathrm{e}/mn)\left(\prod_{j=1}^{m}\sigma_j\right)^{(i-1)/mn}\sum_{j=1}^{m}\ln 1/\sigma_j$$

if $i \leqq k$ and

$$\min_{\alpha\geqq 0} H_i(\alpha) = 1/(i-1)$$

if $i > k$. Thus,

$$G(\alpha) \geqq -1 + m\left(\prod_{j=1}^{m} p_j\right)^{1/m}\left(\prod_{j=1}^{m}\sigma_j\right)^{k(k-1)/2mn}\left(\sum_{j=1}^{m}\ln 1/\sigma_j\right)^k \frac{\mathrm{e}^k(k-1)!}{(mn)^k(n-1)!} > 0$$

for every $\alpha > 0$ and the proof is complete.

Remark 3. The conditions (14) and (16) are independent. We illustrate this fact by the following simple examples.

For the equation

$$x'(t) + \frac{1}{9t} x(e^{-1}t) + \frac{1}{t} x(e^{-1/9}t) = 0, \quad t \geqq a > 0, \tag{17}$$

the condition (16) is satisfied but (14) does not hold.

On the other hand, for the equation

$$x'(t) + \frac{1}{8et} x(e^{-1}t) + \frac{1}{2et} x(e^{-2}t) = 0, \quad t \geqq a > 0, \tag{18}$$

(14) holds but (16) is not satisfied.

By similar arguments we can establish parallel results about the advanced equation

$$x^{(n)}(t) - t^{-n} \sum_{i=1}^{m} p_i\, x(\sigma_i t) = 0, \quad t \geqq a > 0, \tag{19}$$

where $p_i > 0$, $\sigma_i \geqq 1$, $i = 1, \ldots, m$, are constants.

Theorem 1′. *$N_n = \emptyset$ for Eq. (19) if and only if*

$$\alpha(\alpha - 1) \ldots (\alpha - n + 1) - \sum_{i=1}^{m} p_i \sigma_i^{\alpha} < 0 \tag{20}$$

for all $\alpha > n - 1$.

Theorem 2′. *Assume that*

$$\sum_{k \in K} \frac{e^{n-k+1}(n-k)!}{n^{n-k+1}(n-1)!} \sum_{j \in J_k'} p_j (\ln \sigma_j)^{n-k+1}\, \sigma_j^{(k-1)(n-1)/n + (n-k+1)(n+k-2)/2n} > 1, \tag{21}$$

where $J_k' = \{j \in J : n - k < n/(\ln \sigma_j) \leqq n - k + 1\}$ for $k = 2, \ldots, n$ and $J_1' = \{j \in J : n/(\ln \sigma_j) > n - 1\}$.

Then $N_n = \emptyset$ for Eq. (19).

Theorem 3′. *Let k, $1 \leqq k \leqq n$, be the least integer such that*

$$mn / \sum_{j=1}^{m} \ln \sigma_j > n - k$$

and let

$$m^{k-n} \Big(\prod_{j=1}^{m} p_j\Big)^{1/m} \Big(\prod_{j=1}^{m} \sigma_j\Big)^{(k-1)(n-1)/mn + (n+k-2)(n-k+1)/2mn} \Big(\sum_{j=1}^{m} \ln \sigma_j\Big)^{n-k+1} > \frac{n^{n-k+1}(n-1)!}{e^{n-k+1}(n-k)!}. \tag{22}$$

Then $N_n = \emptyset$ for Eq. (19).

Finally, we note that all results presented in this paper remain valid if we replace Eqs. (1) and (19) by the retarded differential inequality

$$\operatorname{sgn} x(t) \Big\{(-1)^n x^{(n)}(t) - t^{-n} \sum_{i=1}^{m} p_i\, x(\sigma_i t)\Big\} \geqq 0, \tag{23}$$

where $p_i > 0$, $0 < \sigma_i \leqq 1$ are constants, and the advanced inequality

$$\operatorname{sgn} x(t) \{x^{(n)}(t) - t^{-n} \sum_{i=1}^{m} p_i x(\sigma_i t)\} \geqq 0 , \tag{24}$$

where $p_i > 0$, $\sigma_i \geqq 1$, $i = 1, \ldots, m$, are constants, respectively.

We shall provide a brief outline of the proof of Theorem 1 for the inequality (23).

The necessity part is obvious. In order to prove the sufficiency part, let us assume that the inequality (23) has an eventually positive solution $x(t)$. Then, according to the comparison result of Philos [11, Corollary 1], the corresponding differential equation (1) has an eventually positive solution $u(t)$, which is a contradiction with the assertion of Theorem 1 applied to (1).

Similarly we can prove that the sufficiency part of Theorem 1 (and, consequently, Theorems 2 and 3) remains true also for the nonlinear delay differential inequality

$$\operatorname{sgn} x(t) \{(-1)^n x^{(n)}(t) - f(t, x(g_1(t)), \ldots, x(g_m(t)))\} \geqq 0 , \tag{25}$$

where the following conditions are satisfied: there exist constants $p_i > 0, 0 < \sigma_i \leqq 1$, $i = 1, \ldots, m$, and $T \geqq a > 0$ such that

(i)
$$f(t, u_1, \ldots, u_m) \operatorname{sgn} u_1 \geqq t^{-n} \sum_{i=1}^{m} p_i |u_i|$$
$$\text{for} \quad t \geqq T, \quad u_1 u_i > 0 , \quad i = 1, \ldots, m , \quad \lim_{t \to \infty} g_i(t) = \infty$$
$$\text{and} \quad g_i(t) \leqq \sigma_i t \quad \text{for} \quad t \geqq T, \quad i = 1, \ldots, m .$$

The parallel results about the nonlinear advanced differential inequality

$$\operatorname{sgn} x(t) \{x^{(n)}(t) - f(t, x(g_1(t)), \ldots, x(g_m(t)))\} \geqq 0 \tag{26}$$

can be also proved without much difficulty if we assume that there exist constants $p_i > 0$, $\sigma_i \geqq 1$, $i = 1, \ldots, m$, and $T \geqq a > 0$ such that

(ii)
$$f(t, u_1, \ldots, u_m) \operatorname{sgn} u_1 \geqq t^{-n} \sum_{i=1}^{m} p_i |u_i|$$
$$\text{for} \quad t \geqq T, \quad u_1 u_i > 0 , \quad i = 1, \ldots, m , \quad \text{and} \quad g_i(t) \geqq \sigma_i t$$
$$\text{for} \quad t \geqq T, \quad i = 1, \ldots, m .$$

The details of this extension are left to the reader.

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