Irena Rachůnková

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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FOUR-POINT BOUNDARY VALUE PROBLEMS FOR 2ND ORDER DIFFERENTIAL EQUATIONS

IRENA RACHŮNKOVÁ, Olomouc

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The three-point boundary value problems for differential equations of the second order were studied in [1], [2], [9] and [11]. The problem of existence of solutions of the equation

$$u'' = f(t, u)$$

satisfying the conditions

$$u(0) = u(a) = u(2a)\,, \quad a \in (-\infty, +\infty)$$

is solved in [1], [2]. Theorems on existence and uniqueness of solutions of the problem

$$u'' = f(t, u, u')\,,$$

$$u(a) = c_1\,, \quad u(b) = u(t_0) + c_2\,, \quad a, b, t_0\,, \quad c_1, c_2 \in (-\infty, +\infty)\,,$$

$a < t_0 < b$, are proved in [11] and, for the linear differential equation, in [9].

1. Our paper deals with the problem of existence and uniqueness of solutions of the equation

$$u'' = f(t, u, u') \tag{1.1}$$

defined on an interval $[a, b]$ and satisfying the conditions

$$u(c) - u(a) = A\,, \quad u(b) - u(d) = B\,, \tag{1.2}$$

where $A, B \in (-\infty, +\infty)$, $-\infty < a < c < d < b < +\infty$. Sufficient conditions for the existence of solutions of the problem (1.1), (1.2) were found in [12]. Other existence theorems for this problem are proved here. Moreover, the problem of uniqueness is solved.

We shall use the following notation:

$$R = (-\infty, +\infty)\,, \quad R_+ = [0, +\infty)\,, \quad D = [a, b] \times R^2\,, \quad D_+ = [a, b] \times R_+^2\,.$$

$$\tau = \begin{cases} \max\{c - a,\ b - c\} & \text{for } d - a > b - c \\ \max\{d - a,\ b - d\} & \text{for } d - a \leqq b - c\,, \end{cases}$$

$$g_0(t) = \alpha t^2 + \beta t + \gamma\,,$$

where

$$\alpha = (B/(b-d) - A/(c-a))(b-c+d-a)^{-1},$$

$$\beta = (A(b+d)/(c-a) - B(c+a)/(b-d))(b-c+d-a)^{-1}, \quad \gamma \in R,$$

$$r_0 = \max\{|g_0(t)| : a \leqq t \leqq b\}, \quad r_1 = \max\{|g_0'(t)| : a \leqq t \leqq b\}.$$

$AC^1(a, b)$ is the set of all real functions which are absolutely continuous together with their first derivatives on $[a, b]$.

$\mathrm{Car}_{\mathrm{loc}}(D)$ is the set of all real functions satisfying the local Carathéodory conditions on D, i.e. $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ iff

$f(\cdot, x, y): [a, b] \to R$ is measurable for every $(x, y) \in R^2$,

$f(t, \cdot, \cdot): R^2 \to R$ is continuous for almost every $t \in [a, b]$,

$\sup\{|f(\cdot, x, y)| : |x| + |y| \leqq \varrho\} \in L(a, b)$ for any $\varrho \in (0, +\infty)$.

Definition. A function $u \in AC^1(a, b)$ which fulfils (1.1) for almost every $t \in [a, b]$ will be called a *solution of the equation* (1.1). Each solution of (1.1) which satisfies the conditions (1.2) will be called a *solution of the problem* (1.1), (1.2).

In the whole paper we suppose that $f \in \mathrm{Car}_{\mathrm{loc}}(D)$ and $\lambda \in \{-1, 1\}$.

Theorem 1. *Let there exist $r \in (0, +\infty)$ such that one the set D the inequalities*

(1.3) $$\lambda(f(t, x, y) - 2\alpha) \operatorname{sgn} x \geqq 0 \quad \text{for} \quad |x| > r$$

and

(1.4) $$|f(t, x, y)| \leqq h_1(t)|x| + h_2(t)|y| + \omega(t, |x| + |y|)$$

hold, where $h_1, h_2 \in L^2(a, b)$ are non-negative functions satisfying

(1.5) $$(b-a)^{1/2}\left(\left(\int_a^b h_1^2(t)\,\mathrm{d}t\right)^{1/2} 2(b-a)/\pi + \left(\int_a^b h_2^2(t)\,\mathrm{d}t\right)^{1/2}\right) < 1$$

and $\omega \in \mathrm{Car}_{\mathrm{loc}}([a, b] \times R_+)$ is a non-negative function, non-decreasing with respect to its second variable and satisfying the condition

(1.6) $$\lim_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b \omega(t, \varrho)\,\mathrm{d}t = 0.$$

Then the problem (1.1), (1.2) is solvable.

Theorem 2. *Let $a_1, a_2 \in (0, +\infty)$ satisfy*

(1.7) $$a_1(2/\pi)^2 \tau(b-a) + a_2(2/\pi)\tau < 1$$

and let there exist $h_1, h_2 \in L(a, b)$ such that

(1.8) $$0 < \lambda h_1(t) \leqq a_1, \quad |h_2(t)| \leqq a_2 \quad \text{for} \quad a < t < b$$

and on the set D the inequality

(1.9) $$|f(t, x, y) - h_1(t)x - h_2(t)y| \leqq \omega(t, |x| + |y|)$$

is fulfilled, where ω is the function from Theorem 1.

Then the problem (1.1), (1.2) is solvable.

Note. The inequality $0 < \lambda\, h_1(t)$ cannot be replaced by the inequality $0 \leqq \lambda\, h_1(t)$ because the problem $u'' = 1$, $u(0) = u(1/2) = u(1)$ has no solution.

Theorem 3. *Let there exist a non-negative function $h \in L(a, b)$ such that on the set D the inequality*

(1.10) $\quad f(t, x_1, y_1) - f(t, x_2, y_2) + h(t)\,|y_1 - y_2| > 0\,, \quad$ *where* $\quad x_1 > x_2\,,$

is satisfied.

Then the problem (1.1), (1.2) *does not have more than one solution.*

2. Lemmas. **Lemma 1** ([6], Theorem 256, p. 219). If $f \in AC(t_1, t_2)$, $f' \in L^2(t_1, t_2)$ *and* $f(t_0) = 0$, *where* $-\infty < t_1 < t_2 < +\infty$, $t_0 \in [t_1, t_2]$, *then*

$$\int_{t_1}^{t_2} f^2(t)\, dt \leqq (2(t_2 - t_1)/\pi)^2 \int_{t_1}^{t_2} f'^2(t)\, dt\,.$$

Lemma 2. *Let $a_1, a_2 \in (0, +\infty)$ satisfy* (1.7) *and let $h_1, h_2 \in L(a, b)$ satisfy* (1.8). *Then the problem*

$$v'' = h_1(t)\, v + h_2(t)\, v'\,, \tag{2.1}$$

$$v(c) - v(a) = 0\,, \quad v(b) - v(d) = 0 \tag{2.2}$$

has only the trivial solution.

Proof. Let v be a solution of the problem (2.1), (2.2). The equation (2.1) can be written in the form

$$\exp\left(\int_a^t h_2(s)\, ds\right) \left(\exp\left(-\int_a^t h_2(s)\, ds\right) v'(t)\right)' - h_1(t)\, v(t) = 0 \quad \text{for} \quad a \leqq t \leqq b\,. \tag{2.3}$$

By (2.2) there exist $t_1 \in (a, c)$, $t_2 \in (d, b)$ such that

$$v'(t_1) = v'(t_2) = 0\,.$$

Consequently, the function $\varphi(t) = \exp\left(-\int_a^t h_2(s)\, ds\right) v'(t)$ has two zeros on (a, b). Let $v(t) \neq 0$ for $a \leqq t \leqq b$. Then (1.8) and (2.3) imply that φ is strictly monotonous on $[a, b]$ and we get a contradiction. Therefore there exists $t_0 \in (a, b)$ such that $v(t_0) = 0$. By Lemma 1 we have

$$\left(\int_a^b v'^2(t)\, dt\right)^{1/2} \leqq (2\tau/\pi) \left(\int_a^b v''^2(t)\, dt\right)^{1/2}$$

and

$$\left(\int_a^b v^2(t)\, dt\right)^{1/2} \leqq (2/\pi)^2\, \tau(b - a) \left(\int_a^b v''^2(t)\, dt\right)^{1/2},$$

and by virtue of (2.1), the inequality

$$\left(\int_a^b v''^2(t)\, dt\right)^{1/2} \leqq \left(a_1 (2/\pi)^2\, \tau(b - a) + a_2\, 2\tau/\pi\right) \left(\int_a^b v''^2(t)\, dt\right)^{1/2}$$

is true. Since (1.7), we get $\left(\int_a^b v''^2(t)\, dt\right)^{1/2} = 0$ and thus $\left(\int_a^b v^2(t)\, dt\right)^{1/2} = 0$.

Lemma 3. *Let $g \in \mathrm{Car}_{\mathrm{loc}}(D)$, $h_1, h_2 \in L(a, b)$ and let the problem* (2.1), (2.2) *have only the trivial solution. If there exists $g^* \in L(a, b)$ such that*

$$|g(t, x, y)| \leqq g^*(t) \quad \text{on} \quad D\,,$$

then the problem

$$v'' = h_1(t)\, v + h_2(t)\, v' + g(t, v, v')\,, \quad (2.2)$$

is solvable

Proof. See [8], Theorem 2.4, p. 25.

Lemma 4. *Let* $a_1, a_2, b_1, b_2 \in L(a, b)$ *and for any* $h_1, h_2 \in L(a, b)$ *satisfying*

(2.4) $$a_i(t) \leqq h_i(t) \leqq b_i(t) \quad \text{for} \quad a \leqq t \leqq b\,, \quad i = 1, 2\,,$$

let the problem (2.1), (2.2) *have only the trivial solution.*

Then there exists $\gamma \in (0, +\infty)$ *such that for any* $h_1, h_2 \in L(a, b)$ *satisfying* (2.4) *the inequality*

(2.5) $$\left|\frac{\partial G(t, s)}{\partial t}\right| + |G(t, s)| \leqq \gamma\,, \quad a \leqq t\,, \quad s \leqq b$$

is fulfilled, where G is Green's function of the problem (2.1), (2.2).

Proof. See [8], Lemma 2.2, p. 12.

3. Lemmas for a priori estimates. Lemma 5. *Let* $r \in (0, +\infty)$, let $h_1, h_2 \in L^2(a, b)$ *be non-negative functions satisfying* (1.5) *and* $\omega \in \mathrm{Car}_{\mathrm{loc}}([a, b] \times R_+)$ *a non-negative function, non-decreasing with respect to its second variable and satisfying* (1.6). *Then there exists* $r^* \in (r, +\infty)$ *such that for any function* $v \in AC^1(a, b)$ *the conditions*

(3.1) $$v(a) = v(c)\,, \quad v(d) = v(b)\,,$$

(3.2) $$\lambda\, v''(t) \operatorname{sgn} v(t) > 0 \quad \text{for} \quad |v(t)| > r\,, \quad t \in [a, b]\,,$$

(3.3) $$|v''(t)| \leqq h_1(t)\, |v(t)| + h_2(t)\, |v'(t)| + \omega(t, |v| + |v'|)\,, \quad a \leqq t \leqq b$$

imply the estimate

(3.4) $$|v(t)| + |v'(t)| \leqq r^* \quad \text{for} \quad a \leqq t \leqq b\,.$$

Proof. The condition (3.1) implies the existence of $t_1, t_2 \in (a, b)$ such that $v'(t_1) = v'(t_2) = 0$. If $|v(t)| > r$ on (a, b), then by (3.2), v' has to be strictly monotonous on (a, b) and we get a contradiction. Therefore there exists $t_0 \in (a, b)$ such that $v(t_0) = c_0$, where $|c_0| \leqq r$. Put $y(t) = v(t) - c_0$ for $a \leqq t \leqq b$. By virtue of (3.3), $|y''(t)| \leqq h_1(t)\, |y(t)| + h_2(t)\, |y'(t)| + \omega(t, |v| + |v'|) + h_1(t)\, r$ for $a \leqq t \leqq b$. Integrating the last inequality from t to t_1 and applying the Hölder inequality, we get

$$|y'(t)| \leqq (\textstyle\int_a^b h_1^2(s)\, ds)^{1/2} (\int_a^b y^2(s)\, ds)^{1/2} + (\int_a^b h_2^2(s)\, ds)^{1/2} (\int_a^b y'^2(s)\, ds)^{1/2} +$$
$$+ \textstyle\int_a^b (\omega(s, |v| + |v'|) + h_1(s)\, r)\, ds\,, \quad a \leqq t \leqq b\,.$$

Put $\varrho_0 = \max \{|y'(t)|: a \leqq t \leqq b\}$. Then $|y(t)| \leqq (b - a)\, \varrho_0$ and we get

$$\varrho_0 \leqq (\textstyle\int_a^b h_1^2(s)\, ds)^{1/2} (\int_a^b y^2(s)\, ds)^{1/2} + (\int_a^b h_2^2(s)\, ds)^{1/2} (\int_a^b y'^2(s)\, ds)^{1/2} +$$
$$+ \textstyle\int_a^b (\omega(s, r + \varrho_0(1 + b - a)) + h_1(s)\, r)\, ds\,.$$

By Lemma 1, we obtain

$$\left(\int_a^b y^2(s)\,ds\right)^{1/2} \leqq (2(b-a)/\pi)\left(\int_a^b y'^2(s)\,ds\right)^{1/2}$$

and since

$$\left(\int_a^b y'^2(s)\,ds\right)^{1/2} \leqq \varrho_0(b-a)^{1/2},$$

we have

$$\varrho_0 \leqq \left[\left(\int_a^b h_1^2(s)\,ds\right)^{1/2} 2(b-a)^{3/2}/\pi + \left(\int_a^b h_2^2(s)\,ds\right)^{1/2}(b-a)^{1/2}\right]\varrho_0 + \\ + \int_a^b \left(\omega(s, r + \varrho_0(1 + b - a)) + h_1(s)\,r\right) ds\,.$$

In view of (1.5) and (1.6) there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the inequality

$$\left[\left(\int_a^b h_1^2(s)\,ds\right)^{1/2} 2(b-a)^{3/2}/\pi + \left(\int_a^b h_2^2(s)\,ds\right)^{1/2}(b-a)^{1/2}\right]\varrho + \\ + \int_a^b \left(\omega(s, r + \varrho(1 + b - a)) + h_1(s)\,r\right) ds < \varrho$$

is satisfied. Consequently $\varrho_0 \leqq \varrho^*$. Putting

$$r^* = r + \varrho^*(b - a + 1),$$

we get the estimate (3.4).

Lemma 6. *Let $a_1, a_2 \in (0, +\infty)$ satisfy (1.7), let $h_1, h_2 \in L(a, b)$ satisfy (1.8) and let $\omega \in \mathrm{Car}_{\mathrm{loc}}([a, b] \times R_+)$ be a non-negative function, non-decreasing with respect to its second variable and satisfying (1.6). Then there exists $r^* \in (r, +\infty)$ such that for any function $v \in AC^1(a, b)$ the conditions (3.1) and*

$$(3.5) \qquad |v'' - h_1(t)\,v - h_2(t)\,v'| \leqq \omega(t, |v| + |v'|)\,, \quad a \leqq t \leqq b$$

imply the estimate (3.4).

Proof. Put $h_0(t) = v''(t) - h_1(t)\,v(t) - h_2(t)\,v'(t)$ for $a \leqq t \leqq b$ and consider the equation

$$(3.6) \qquad v'' = h_1(t)\,v + h_2(t)\,v' + h_0(t)\,.$$

Since h_1, h_2 satisfy the conditions of Lemma 2, the problem (2.1), (2.2) has only the trivial solution. Consequently, by Lemma 4, there exists $\gamma \in (0, +\infty)$ such that Green's function G for the problem (2.1), (2.2) fulfils the estimate (2.5). Therefore the solution

$$v(t) = \int_a^b G(t, s)\,h_0(s)\,ds$$

of the problem (3.6), (2.2) satisfies the estimate

$$|v(t)| + |v'(t)| \leqq \gamma \int_a^b \omega(s, |v| + |v'|)\,ds \quad \text{for} \quad a \leqq t \leqq b\,.$$

Putting $\max\{|v(t)| + |v'(t)| : a \leqq t \leqq b\} = \varrho_0$, we get

$$\varrho_0 \leqq \int_a^b \omega(t, \varrho_0)\,dt\,.$$

It follows from (1.6) that there exists $r^* > 0$ such that

$$\gamma \int_a^b \omega(t, \varrho)\,dt < \varrho \quad \text{for any} \quad \varrho > r^*\,.$$

Therefore $\varrho_0 \leqq r^*$ and Lemma 6 is proved.

4. Proofs of Theorems.

Proof of Theorem 1. Let $\varepsilon_0 \in (0, +\infty)$ satisfy

$$(4.1) \qquad (b-a)^{1/2}\left[\left(\int_a^b (h_1(t)+\varepsilon_0)^2\,dt\right)^{1/2} 2(b-a)/\pi + \left(\int_a^b h_2^2(t)\,dt\right)^{1/2}\right] < 1$$

and let r^* be the constant constructed by means of Lemma 5 for the functions $h_1(t) + \varepsilon_0$, $h_2(t)$ and $\tilde{\omega}(t, s) = \omega(t, s + r_0 + r_1) + h_1(t) r_0 + h_2(t) r_1 + 2|\alpha|$ and for the constant $\tilde{r} = r + r_0$. Put

$$\chi(r^*, s) = \begin{cases} 1 & \text{for } 0 \leqq s \leqq r^* \\ 2 - s/r^* & \text{for } r^* < s < 2r^* \\ 0 & \text{for } s \geqq 2r^*, \end{cases}$$

$$g(t, x, y) = f(t, x + g_0(t),\ y + g_0'(t)) - 2\alpha,$$

$$\tilde{g}(t, x, y) = \chi(r^*, |x| + |y|)\, g(t, x, y)$$

and consider the equation

$$(4.2) \qquad v'' = \lambda \varepsilon v + \tilde{g}(t, v, v'), \quad \varepsilon \in (0, \varepsilon_0].$$

Since ε satisfies the assumptions of Lemma 2, the problem

$$v'' = \lambda \varepsilon v, \quad (2.2)$$

has only the trivial solution. Consequently, by Lemma 3, the problem (4.2), (2.2) has a solution v.

Clearly v satisfies (3.1). Now, let $v(t) > \tilde{r}$ for some $t \in [a, b]$. Then $v(t) + g_0(t) > r$ and

$$\lambda\, v''(t) = \lambda \chi(r^*, |v| + |v'|)\,(f(t, v + g_0(t), v' + g_0'(t)) - 2\alpha) + \varepsilon\, v(t) > 0.$$

Analogously, if $v(t) < -\tilde{r}$, then $v(t) + g_0(t) < -r$ and $\lambda\, v''(t) < 0$. Consequently, v satisfies (3.2) with the constant $\tilde{r}$. Further,

$$\begin{aligned} |v''(t)| &\leqq |f(t, v + g_0, v' + g_0') - 2\alpha| + \varepsilon|v| \leqq \\ &\leqq h_1(t)(|v| + r_0) + h_2(t)(|v'| + r_1) + \\ &+ 2|\alpha| + \varepsilon_0|v| + \omega(t, |v| + r_0 + |v'| + r_1) = \\ &= (h_1(t) + \varepsilon_0)|v| + h_2(t)|v'| + \tilde{\omega}(t, |v| + |v'|) \quad \text{for} \quad a \leqq t \leqq b. \end{aligned}$$

It follows from (4.1) that the functions $h_1 + \varepsilon_0$, h_2 satisfy (1.5). Since ω satisfies (1.6), there exists $\varrho^* > 0$ such that for any $\varrho > \varrho^*$ the conditions $r_0 + r_1 + \varrho \leqq 2\varrho$ and

$$\lim_{\varrho \to +\infty} \frac{1}{2\varrho} \int_a^b (\omega(t, 2\varrho) + h_1(t) r_0 + h_2(t) r_1 + 2|\alpha|)\,dt = 0$$

are fulfilled. Therefore $\tilde{\omega}$ satisfies (1.6) and, by Lemma 5, the estimate (3.4) is valid. Thus v is a solution of the equation

$$v'' = \lambda \varepsilon v + g(t, v, v')$$

and $u = v + g_0$ is a solution of the equation

$$u'' = \lambda\varepsilon(u - g_0(t)) + f(t, u, u') \tag{4.3}$$

and satisfies the conditions (1.2).

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ there exists a solution u_ε of the problem (4.3), (1.2) satisfying the estimate

$$|u_\varepsilon| + |u'_\varepsilon| \leqq r^* + r_0 + r_1 \quad \text{for} \quad a \leqq t \leqq b\,.$$

It follows that all functions of the set $\{u_\varepsilon\colon \varepsilon \in (0, \varepsilon_0]\}$ are uniformly bounded together with their derivatives and so also equi-continuous on $[a, b]$. Therefore, by the Arzelà-Ascoli lemma there exist a sequence $(\varepsilon_k)_{k=1}^{\infty}$, $\varepsilon_k \to 0$ for $k \to \infty$, and a sequence $(u_{\varepsilon_k})_{k=1}^{\infty}$ uniformly convering together with $(u'_{\varepsilon_k})_{k=1}^{\infty}$ on $[a, b]$, such that $u_0(t) = \lim_{k \to +\infty} u_{\varepsilon_k}(t)$ is a solution of the problem (1.1), (1.2).

Proof of Theorem 2. Let r^* be the constant constructed by means of Lemma 6 for the function

$$\tilde{\omega}(t, s) = \omega(t, s + r_0 + r_1) + |h_1(t)|\, r_0 + |h_2(t)|\, r_1 + 2|\alpha|\,.$$

Put

$$\chi(r^*, s) = \begin{cases} 1 & \text{for} \quad 0 \leqq s \leqq r^* \\ 2 - s/r^* & \text{for} \quad r^* < s < 2r^* \\ 0 & \text{for} \quad s \geqq 2r^*\,, \end{cases}$$

$$g(t, x, y) = f(t, x + g_0(t), y + g'_0(t)) - 2\alpha - h_1(t)\, x - h_2(t)\, y\,,$$

$$\tilde{g}(t, x, y) = \chi(r^*, |x| + |y|)\, g(t, x, y)$$

and consider the equation

$$v'' = h_1(t)\, v + h_2(t)\, v' + \tilde{g}(t, v, v')\,. \tag{4.4}$$

By Lemma 2, the problem (2.1), (2.2) has only the trivial solution. Consequently, by Lemma 3, the problem (4.4), (4.2) has a solution v. Now (1.9) implies

$$\begin{aligned} |v'' - h_1(t)\, v - h_2(t)\, v'| &\leqq |f(t, v + g_0, v' + g'_0) - h_1(t)\, v - h_2(t)\, v' - 2\alpha| \leqq \\ &\leqq |f(t, v + g_0, v' + g'_0) - h_1(t)\,(v + g_0(t)) - h_2(t)\,(v' + g'_0)| + \\ &+ |h_1(t)|\, r_0 + |h_2(t)|\, r_1 + 2|\alpha| \leqq \omega(t, |v + g_0| + |v' + g'_0|) + |h_1(t)|\, r_0 + \\ &+ |h_2(t)|\, r_1 + 2|\alpha| \leqq \tilde{\omega}(t, |v| + |v'|) \quad \text{for} \quad a \leqq t \leqq b\,. \end{aligned}$$

In the same way as in the proof of Theorem 1 we can show that $\tilde{\omega}$ satisfies (1.6). Consequently, by Lemma 6, the estimate (3.4) is valid and v is a solution of the equation

$$v'' = h_1(t)\, v + h_2(t)\, v' + g(t, v, v')\,.$$

Therefore $u = v + g_0$ is a solution of the problem (1.1), (1.2).

Proof of Theorem 3. Let us assume that the problem (1.1), (1.2) has two solutions u_1, u_2. Put $v = u_1 - u_2$ on $[a, b]$. Then

$$v(c) - v(a) = 0, \quad v(b) - v(d) = 0 \tag{4.5}$$

and thus there exist $t_1 \in (a, c)$, $t_2 \in (d, b)$ such that $v'(t_1) = v'(t_2) = 0$.

First, let us suppose that

$$v(t_0) \neq 0 \quad \text{for some} \quad t_0 \in (t_1, t_2).$$

Without loss of generality we may consider that $v(t_0) > 0$. Then there exist $t_*, t^* \in \in [t_1, t_2]$ such that

$$v(t) > 0 \quad \text{for} \quad t \in (t_*, t^*) \quad \text{and} \quad v'(t_*) \geqq 0, \quad v'(t^*) \leqq 0. \tag{4.6}$$

From (1.10) we get

$$v''(t) + \tilde{h}(t)\, v'(t) > 0 \quad \text{for} \quad t_* \leqq t \leqq t^*, \quad \text{where} \quad \tilde{h}(t) = h(t) \operatorname{sgn} v'(t),$$

and thus the inequality

$$(\exp(\textstyle\int_a^t \tilde{h}(s)\, ds)\, v'(t))' > 0 \quad \text{for} \quad t_* \leqq t \leqq t^* \tag{4.7}$$

is satisfied. Integrating (4.7) from t_* to t^*, we obtain, by (4.6), that

$$0 \geqq \exp(\textstyle\int_a^{t^*} \tilde{h}(s)\, ds)\, v'(t^*) - \exp(\textstyle\int_a^{t_*} \tilde{h}(s)\, ds)\, v'(t_*)) > 0. \tag{4.8}$$

The contradiction (4.8) implies $v(t) = 0$ for $t_1 \leqq t \leqq t_2$.

From this, according to (4.5), we get

$$v(a) = v(c) = v(d) = v(b) = 0. \tag{4.9}$$

Now, let us suppose that

$$v(t_0) > 0 \quad \text{for some} \quad t_0 \in (a, t_1) \ [t_0 \in (t_2, b)].$$

On the basis of (4.9) we can find $t_*, t^* \in [a, t_1]$ $[\in [t_2, b]]$ such that the conditions (4.6) are fulfilled. Therefore we obtain the contradiction (4.8) in the same way as in the first part of this proof. Thus $v(t) = 0$ for $a \leqq t \leqq b$.

References

[1] *J. Andres:* On a possible modification of Levinson's operator, Proceed. 11th ICNO held in Budapest, 1987.

[2] *J. Andres:* On some modification of the Levinson operator and its application to a three-point boundary value problem, Acta UPO 91, Math. *29* (1990).

[3] *P. B. Bailey, L. F. Shampine, P. E. Waltman:* Nonlinear two-point boundary value problems, Acad. Press., New York, 1968.

[4] *R. Conti:* Equazioni differenziali ordinarie quasilineari con condizioni lineari, Ann. Mat. Pura ed Appl., *57* (1962), 49–61.

[5] *V. V. Gudkov, J. A. Klokov, A. J. Lepin, V. D. Ponomarjev:* Dvuchtočečnyje krajevyje zadači dlja obyknověnnych differencialnych uravněnij, Zinatne, Riga, 1973.

[6] *G. H. Hardy, J. E. Littlewood, G. Pólya:* Inequalities (in Russian), IL Moscow, 1948.

[7] *P. Hartman:* Ordinary differential equations (in Russian), Mir, Moscow, 1970.

[8] *I. T. Kirguradze:* Nekotoryje singuljarnyje krajevyje zadači dlja obyknověnnych differencialnych uravněnij, ITU, Tbilisi, 1975.
[9] *I. T. Kiguradze, A. G. Lomtatidze:* On certain boundary value problems for second-order linear ordinary differential equations, J. Math. Anal. and Appl., *101* (1984), 325—347.
[10] *I. T. Kiguradze:* K teorii nelinějnych dvuchtočečnych krajevych zadač, Summer school on ordinary diff. eq. Difford 74, Czechoslovakia 1974.
[11] *A. G. Lomtatidze:* Ob odnoj singuljarnoj trechtočečnoj krajevoj zadače, Trudy IPM, Tbilisi *17* (1986), 122—134.
[12] *I. Rachůnková:* A four-point problem for differential equations of the second order, Arch. Math. Brno, Vol. *25*, No 4 (1989).
[13] *N. I. Vasiljev, J. A. Klokov:* Osnovy teorii krajevych zadač obyknověnnych differencialnych uravněnij, Zinatne, Riga, 1978.

Author's address: 771 46 Olomouc, Gottwaldova 15, Czechoslovakia (Přírodovědecká fakulta PU).