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STRUCTURE SPACES OF LATTICE ORDERED GROUPS

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Structure spaces of (non-ordered) rings were studied in the book [2]. In the present paper we introduce structure spaces of lattice ordered groups (l-groups). If G is an l-group, then the structure space of G is the set $\mathscr{C}_p(G)$ of all its proper prime subgroups with a topology induced by a topological closure operator on $\exp \mathscr{C}_p(G)$. For any l-ideal I of G we study homeomorphisms between some subspaces of $\mathscr{C}_p(G)$ and the structure spaces of the l-groups G/I and I. Furthermore, analogous homeomorphisms for the spaces of closed prime subgroups are also found. It is proved that the space of closed prime subgroups of any l-group is homeomorphic to the corresponding space of some completely distributive l-group.

In the second part we solve analogous problems for spaces of prime ideals of l-groups. It is shown that for any representable l-group G there exists a completely distributive representable l-group with the homeomorphic space of prime ideals.

The paper uses notions and results from the books [1] and [3] (in the additive form).

1. Let $G = (G, +, \leqq, \wedge, \vee)$ be a *lattice ordered group* (an *l-group*), A a convex l-subgroup of G. For any $x, y \in G$, we put $x + A \leqq y + A$ if and only if there exists $a \in A$ such that $x \leqq y + a$. Then the relation "$\leqq$" is an order of the set $G/_l A$ of all left classes modulo A and $(G/_l A, \leqq)$ is a lattice. A convex l-subgroup A is called a *prime subgroup* if $G/_l A$ is a linearly ordered set. If $G \neq 0$ is an l-group, then we denote the set of all proper (i.e. different from G) prime subgroups of G by $\mathscr{C}_p(G)$.

Let $\mathbf{x} \subseteq \mathscr{C}_p(G)$. Then we will denote

$$\mathscr{D}\mathbf{x} = \bigcap(P;\, P \in \mathbf{x}),$$
$$\bar{\mathbf{x}} = \{Q \in \mathscr{C}_p(G);\ \mathscr{D}\mathbf{x} \subseteq Q\}.$$

Theorem 1.1. *If $G \neq 0$ is an l-group, then* $^-$: $\exp \mathscr{C}_p(G) \to \exp \mathscr{C}_p(G)$, *where* $^-$: $x \mapsto \bar{x}$, *is a topological closure operator on* $\exp \mathscr{C}_p(G)$.

Proof. 1. $\mathscr{D}\phi = \bigcap(P;\, P \in \phi) = G$, hence $\bar{\phi} = \{P;\ P \in \mathscr{C}_p(G),\ G \subseteq P\} = \phi$.

2. If $P \in \mathbf{x}$, then $\mathscr{D}\mathbf{x} \subseteq P$, thus $\mathbf{x} \subseteq \bar{\mathbf{x}}$.

3. If $P \in \bar{\bar{\mathbf{x}}}$, then $\mathscr{D}\bar{\mathbf{x}} = \bigcap(Q;\ Q \in \bar{\mathbf{x}}) \subseteq P$. But for any $Q \in \bar{\mathbf{x}}$ we have $\mathscr{D}\mathbf{x} \subseteq Q$, therefore $\mathscr{D}\mathbf{x} \subseteq P$, and so $P \in \bar{\mathbf{x}}$. This means $\bar{\bar{\mathbf{x}}} = \bar{\mathbf{x}}$.

4. Let $\mathbf{x}, \mathbf{y} \subseteq \mathscr{C}_p(G)$. We have $\overline{\mathbf{x}} \cup \overline{\mathbf{y}} \subseteq \overline{\mathbf{x} \cup \mathbf{y}}$. Let $P \in \overline{\mathbf{x} \cup \mathbf{y}}$, i.e. $\mathscr{D}(\mathbf{x} \cup \mathbf{y}) \subseteq P$. If $a \in P$, then

$$a \in \mathscr{D}(\mathbf{x} \cup \mathbf{y}) \Leftrightarrow a \in \bigcap(Q;\ Q \in \mathbf{x} \cup \mathbf{y}) \Leftrightarrow a \in \bigcap(R;\ R \in \mathbf{x})$$

and

$$a \in \bigcap(S;\ S \in \mathbf{y}) \Leftrightarrow a \in \mathscr{D}\mathbf{x} \cap \mathscr{D}\mathbf{y}\,,$$

hence $\mathscr{D}(\mathbf{x} \cup \mathbf{y}) = \mathscr{D}\mathbf{x} \cap \mathscr{D}\mathbf{y}$. Moreover, $\mathscr{D}\mathbf{x}$ and $\mathscr{D}\mathbf{y}$ are convex l-subgroups of G, $P \in \mathscr{C}_p(G)$, hence we have (by [1, Théorème 2.4.1]) $\mathscr{D}\mathbf{x} \subseteq P$ or $\mathscr{D}\mathbf{y} \subseteq P$. Therefore $P \in \overline{\mathbf{x}}$ or $P \in \overline{\mathbf{y}}$, hence $\overline{\mathbf{x} \cup \mathbf{y}} \subseteq \overline{\mathbf{x}} \cup \overline{\mathbf{y}}$.

Definition. The set $\mathscr{C}_p(G)$ with the topology induced by the closure operator on $\exp \mathscr{C}_p(G)$ such that $\overline{\mathbf{x}} = \{P \in \mathscr{C}_p(G);\ \mathscr{D}\mathbf{x} \subseteq P\}$ for each $\mathbf{x} \subseteq \mathscr{C}_p(G)$, is called the *structure space* of an l-group G.

Note. It is clear that $\mathscr{C}_p(G)$ is a T_0-space but, in general, it is not a T_1-space.

Theorem 1.2. *Let I be an l-ideal of an l-group G, $\mathbf{x}_I = \{P \in \mathscr{C}_p(G);\ I \subseteq P\}$. Then the mapping $f\colon \mathbf{x}_I \to \mathscr{C}_p(G/I)$ such that $f(P) = P/I$ for any $P \in \mathbf{x}_I$, is a homeomorphism of the space $\mathbf{x}_I$ onto the structure space of the l-group G/I.*

Proof. Let I be an l-ideal of G, $\mathbf{x}_I = \{P \in \mathscr{C}_p(G); I \subseteq P\}$, $P \in \mathbf{x}_I$. Then the ordered sets $(G/I)/_l(P/I)$ and $G/_lP$ are isomorphic. Moreover, $G/_lP$ is, by the assumption, linearly ordered, hence P/I is a prime subgroup of G/I and $P/I \neq G/I$, therefore $P/I \in \mathscr{C}_p(G/I)$.

Conversely, let $R \in \mathscr{C}_p(G/I)$. Let us denote $P = \{x \in G;\ x + I \in R\}$. It is clear that P is a convex l-subgroup of G, $I \subseteq P \neq G$, and that $R = P/I$. Moreover, the ordered set $G/_lP$ is isomorphic to the ordered set $(G/I)/_lR$, thus $G/_lP$ is linearly ordered, i.e. $P \in \mathbf{x}_I$. Therefore $f\colon \mathbf{x}_I \to \mathscr{C}_p(G/I)$ such that $f(P) = P/I$ is a bijective mapping which evidently respects any set intersections.

Let $\mathbf{y} \subseteq \mathbf{x}_I$. Then we have $f(\overline{\mathbf{y}}) = \{f(P);\ P \supseteq \bigcap(Q;\ Q \in \mathbf{y})$. Since f respects intersections, we obtain

$$f(P) \supseteq f(\bigcap Q;\ Q \in \mathbf{y}) = \bigcap(f(Q;\ Q \in \mathbf{y})\,,$$

hence

$$f(\overline{\mathbf{y}}) = \{f(P); f(P) \supseteq \bigcap(f(Q);\ Q \in \mathbf{y})\,.$$

Moreover,

$$\mathscr{D}f(\mathbf{y}) = \bigcap(R;\ R \in f(\mathbf{y})) = \bigcap(f(Q);\ Q \in \mathbf{x}_I,\ f(Q) \in f(\mathbf{y}))\,,$$

thus

$$\overline{f(\mathbf{y})} = \{f(S) \in \mathscr{C}_p(G/I);\ \mathscr{D}f(\mathbf{y}) \subseteq f(S)\}\,.$$

Thus $f(\overline{\mathbf{y}}) = \overline{f(\mathbf{y})}$, therefore f is a homeomorphism of the space $\mathbf{x}_I$ (with the topology induced by the space $\mathscr{C}_p(G)$) onto the structure space of the l-group G/I.

Proposition 1.3. *If G is an l-group, I an l-ideal of G and Q a prime subgroup of I, then*

$$Q\colon I = \{z \in G;\ |z| \wedge |x| \in Q,\ \textit{for all}\ x \in I\}$$

is a prime subgroup of G and

$$Q = (Q : I) \cap I .$$

Proof. Let $z_1, z_2 \in Q : I$, $x \in I$. Then

$$0 \leqq |z_1 - z_2| \wedge |x| = (|z_1| + |z_2| + |z_1|) \wedge |x| \leqq$$
$$\leqq (|z_1| \wedge |x|) + (|z_2| \wedge |x|) + (|z_1| \wedge |x|) \in Q ,$$

hence $Q : I$ is a subgroup of G. It is evident that $Q : I$ is a convex l-subgroup of G.

Let $a, b \in G \setminus (Q : I)$, $a \wedge b = 0$. Since $a, b \notin Q : I$, there exist $x_a, x_b \in I$ such that $a \wedge |x_a| \notin Q$, $b \wedge |x_b| \notin Q$. Clearly $0 < a \wedge |x_a| \in I$, $0 < b \wedge |x_b| \in I$. Since Q is a prime subgroup of I, we have (by [1, Théorème 2.4.1], [3, Teorema III.3.1])

$$(a \wedge |x_a|) \wedge (b \wedge |x_b|) > 0 .$$

However, the assumption implies

$$0 = a \wedge b \geqq (a \wedge |x_a|) \wedge (b \wedge |x_b|) > 0 ,$$

a contradiction. Therefore $Q : I$ is a prime subgroup of G.

Let $c \in Q$, $x \in I$. Then $0 \leqq |c| \wedge |x| \leqq |c| \in Q$, hence $|c| \wedge |x| \in Q$. Thus $c \in Q : I$, and so $Q \subseteq (Q : I) \cap I$.

Conversely, let $d \in (Q : I) \cap I$. But then $d \in Q : I$, $d \in I$, hence necessarily $|d| = |d| \wedge |d| \in Q$. Since P is a convex l-subgroup, $d \in Q$, therefore $(Q : I) \cap I \subseteq Q$.

Theorem 1.4. *Let I be an l-ideal of an l-group G, $\mathbf{x}(I) = \{P \in \mathscr{C}_p(G); I \nsubseteq P\}$. Then the mapping $g: \mathbf{x}(I) \to \mathscr{C}_p(I)$ such that $g(P) = P \cap I$ for any $P \in \mathbf{x}(I)$, is a homeomorphism of the space $\mathbf{x}(I)$ onto the structure space of the l-group I.*

Proof. Let $P \in \mathbf{x}(I)$, $P_1 = P \cap I$. Then $P + I$ is a convex l-subgroup of G and the ordered sets $I/_l P$ and $(P + I)/_l P$ are isomorphic. Since P is a prime subgroup of G, it is a prime subgroup of $P + I$, too, and hence P_1 is a prime subgroup of I. Moreover, $P + I \neq P$, thus $P_1 \in \mathscr{C}_p(I)$.

Let $P, Q \in \mathbf{x}(I)$ be such that $g(P) = g(Q)$. Then $P \cap I \subseteq Q$, hence $P \subseteq Q$ or $I \subseteq Q$. By the assumption, $I \nsubseteq Q$, thus $P \subseteq Q$. Similarly $Q \subseteq P$, therefore g is an injection.

Hence, by Theorem 1.3, we get that g is a bijective mapping of $\mathbf{x}(I)$ onto $\mathscr{C}_p(I)$.

Let us show that g is a homeomorphism. Let $\mathbf{y} \subseteq \mathbf{x}(I)$. Let us put $\mathbf{y}_1 = \{g(R); R \in \mathbf{y}\}$. Let $P \in \bar{\mathbf{y}} \cap \mathbf{x}(I)$. Then $P \supseteq \mathscr{D}y$, hence $P \cap I \supseteq \mathscr{D}y \cap I = \bigcap(R \cap I; R \in \mathbf{y})$. This means that for the closure $\bar{\mathbf{y}}_1$ of the set $\mathbf{y}_1$ in $\mathscr{C}_p(I)$ we have $g(P) = P \cap I \in \bar{\mathbf{y}}_1$.

Conversely, let $P \in \mathbf{x}(I)$ be such that $g(P) \in \bar{\mathbf{y}}_1$. Then $P \cap I \supseteq \mathscr{D}\mathbf{y} \cap I$, thus $\mathscr{D}\mathbf{y} \cap I \subseteq P$. Since $I \nsubseteq P$, we have $\mathscr{D}\mathbf{y} \subseteq P$, and this means that $P \in \bar{\mathbf{y}}$.

If A is a subset of an l-group G, then A is called *closed* if it satisfies the following condition:

If $a_\alpha \in A$, $\alpha \in \Gamma$, and if there exists $b = \bigvee(a_\alpha; \alpha \in \Gamma)$ in G, then $b \in A$.

Let $G \neq 0$ be an l-group. Let us denote the set of all proper closed prime subgroups of G by $\mathscr{C}_{pc}(G)$. If $\mathbf{y} \subseteq \mathscr{C}_{pc}(G)$, then $\mathscr{D}\mathbf{y}$ is a closed convex l-subgroup of G

and (by [1, Proposition 6.1.10], [3, Lemma IX.2.4]) $\bar{\mathbf{y}} \subseteq \mathscr{C}_{pc}(G)$. Hence, if we consider $\mathscr{C}_{pc}(G)$ as a subspace of the structure space $\mathscr{C}_p(G)$, then the closure of $\mathbf{y}$ in $\mathscr{C}_{pc}(G)$ is the same as in $\mathscr{C}_p(G)$.

Theorem 1.5. *Let I be a closed l-ideal of an l-group G, $\mathbf{x}_I = \{P \in \mathscr{C}_p(G); I \subseteq P\}$, $\mathbf{x}_{cI} = \mathbf{x}_I \cap \mathscr{C}_{pc}(G)$, and let f_c be the restriction of the mapping $f: \mathbf{x} \to \mathscr{C}_p(G/I)$ from Theorem 1.2 on the set $\mathbf{x}_{cI}$. Then f_c is a homeomorphism of the space $\mathbf{x}_{cI}$ onto $\mathscr{C}_p(G/I)$.*

Proof. a) Let $P \in \mathbf{x}_{cI}$. Then $P/I \in \mathscr{C}_p(G/I)$. Let us suppose that $z_\alpha \in P$, $\alpha \in \Gamma$, and that $\bigvee(z_\alpha + I;\ \alpha \in \Gamma)$ exists. Since I is a closed l-ideal of G, the natural homomorphism $\psi: G \to G/I$ preserves all joins (by [1, Proposition 6.1.5], [3, Lemma IX.2.1]), hence

$$\bigvee(z_\alpha + I;\ \alpha \in \Gamma) = \bigvee(\psi(z_\alpha);\ \alpha \in \Gamma) =$$
$$= \psi(\bigvee z_\alpha;\ \alpha \in \Gamma) = \bigvee(z_\alpha;\ \alpha \in \Gamma) + I\,.$$

Since P is closed, we now get that $\bigvee(z_\alpha;\ \alpha \in \Gamma) \in P$, and so $\bigvee(z_\alpha;\ \alpha \in \Gamma) + I \in P/I$. Similarly we obtain that P/I is closed with respect to any meets. Therefore P/I is a closed prime subgroup of G/I, i.e. $P/I \in \mathscr{C}_{pc}(G/I)$.

b) Let $P \in \mathscr{C}_p(G)$, $P/I \in \mathscr{C}_{pc}(G/I)$. Let us consider $u_\beta \in P$, $\beta \in \Delta$, and suppose that $\bigvee(u_\beta;\ \beta \in \Delta)$ exists. Then

$$\psi(\bigvee(u_\beta;\ \beta \in \Delta)) = \bigvee(\psi(u_\beta);\ \beta \in \Delta) = \bigvee(u_\beta + I;\ \beta \in \Delta)\,.$$

Since P/I is closed, we get that $\psi(\bigvee(u_\beta;\ \beta \in \Delta)) \in P/I$, hence $\bigvee(u_\beta;\ \beta \in \Delta) \in P$. Similarly for any meets. But this means that $P \in \mathscr{C}_{pc}(G)$, therefore $P \in \mathbf{x}_{cI}$.

Hence f_c is a bijection of $\mathbf{x}_{cI}$ onto $\mathscr{C}_{pc}(G/I)$. Moreover, the closures of the subsets of $\mathbf{x}_{cI}$ in $\mathscr{C}_{pc}(G)$ and in $\mathscr{C}_p(G)$ are the same, and also the closures of the subsets of $\mathscr{C}_{pc}(G/I)$ in $\mathscr{C}_{pc}(G/I)$ and in $\mathscr{C}_p(G/I)$ coincide. Thus f_c is a homeomorphism.

The *distributive radical* $D(G)$ of an l-group G is the intersection of all closed prime subgroups of G.

Theorem 1.6. *If $G \neq 0$ is an l-group, then its space $\mathscr{C}_{pc}(G)$ is homeomorphic to the space $\mathscr{C}_{pc}(G')$ for some completely distributive l-group G'.*

Proof. By ([1, 6.2.2]) the distributive radical $D(G)$ is a closed l-ideal of G which is contained in all prime subgroups of G. Hence the set $\mathbf{x}_c$ from Theorem 1.5 is equal to $\mathscr{C}_{pc}(G)$, and thus f_c is a homeomorphism of $\mathscr{C}_{pc}(G)$ onto $\mathscr{C}_{pc}(G/D(G))$. But the factor l-group $G/D(G)$ is (by [3, Teorema IX.2.2]) completely distributive, and this implies the assertion.

2. Let $G \neq 0$ be an l-group. Let us denote the set of all proper prime ideals (i.e. normal prime subgroups) of G by $\mathscr{L}_p(G)$. If $\mathbf{z} \subseteq \mathscr{L}_p(G)$, we put

$$\mathscr{D}_1\mathbf{z} = \bigcap(P;\ P \in \mathbf{z})\,,$$
$$\bar{\mathbf{z}} = \{Q \in \mathscr{L}_p(G);\ \mathscr{D}_1\mathbf{z} \subseteq Q\}\,.$$

Theorem 2.1. *If $G \neq 0$ is an l-group, then* $^{-}$: $\exp \mathscr{L}_p(G) \to \exp \mathscr{L}_p(G)$, *where* $^{-}$: $z \mapsto \bar{z}$, *is a topological closure operator on* $\exp \mathscr{L}_p(G)$.

Proof. It is analogous to the proof of Theorem 1.1.

Definition. The set $\mathscr{L}_p(G)$ with the topology induced by the closure operator on $\exp \mathscr{L}_p(G)$ from Theorem 2.1 is called the *space of prime ideals* of an *l*-group G.

Theorem 2.2. *Let I be an l-ideal of an l-group G, $\mathbf{z}_I = \{P \in \mathscr{L}_p(G);\ I \subseteq P\}$. Then the mapping $f_1: \mathbf{z}_I \to \mathscr{L}_p(G/I)$ such that $f_1(P) = P/I$ for any $P \in \mathbf{z}_I$, is a homeomorphism of the space $\mathbf{z}_I$ onto the space of prime ideals of the l-group G/I.*

Proof. The assertion follows from Theorem 1.2.

Proposition 2.3. *If G is an l-group, I an l-ideal of G and Q a prime ideal of I, then*

$$Q : I = \{z \in G;\ |z| \wedge |x| \in Q \text{ for all } x \in I\}$$

is a prime ideal of G and

$$Q = (Q : I) \cap I\,.$$

Proof. By Theorem 1.3, it is sufficient to prove that if Q is a prime ideal of I, then the prime subgroup $Q : I$ is normal. Let $z \in Q : I$, $x \in I$, $a \in G$. Then

$$|-a + z + a| \wedge |x| = |-a + z + a| \wedge |-a + a + x| =$$
$$= |-a + z + a| \wedge |-a + u_x + a|\,,$$

where $u_x \in I$. Moreover,

$$|-a + z + a| \wedge |-a + u_x + a| = (-a + |z| + a) \wedge$$
$$\wedge (-a + |u_x| + a) = -a + (|z| \wedge |u_x|) + a\,,$$

and since $|z| + |u_x| \in Q$, we have $-a + (|z| \wedge |u_x|) + a \in Q$, i.e. $|-a + z + a| \wedge \wedge |x| \in Q$.

Therefore $Q : I$ is a prime ideal of G.

Theorem 2.4. *Let I be an l-ideal of an l-group G, $\mathbf{z}(I) = \{P \in \mathscr{L}_p(G);\ I \nsubseteq P\}$. Then the mapping $g_1: \mathbf{z}(I) \to \mathscr{L}_p(I)$ such that $g_1(P) = P \cap I$ for any $P \in \mathbf{z}(I)$ is a homeomorphism of the space $\mathbf{z}(I)$ onto the space of prime ideals of the l-group I.*

Proof. By Theorem 1.4 and Proposition 2.3 it is evident that g_1 is a bijection from $\mathbf{z}(I)$ onto $\mathscr{L}_p(I)$.

The fact that g_1 is a homeomorphism can be proved in a similar way as for the mapping g in the proof of Theorem 1.4.

Let $G \neq 0$ be an *l*-group. Let us denote the set of all proper closed prime ideals of G by $\mathscr{L}_{pc}(G)$. If $\mathbf{v} \subseteq \mathscr{L}_{pc}(G)$, then $\mathscr{D}_1\mathbf{v}$ is a closed *l*-ideal of G and $\bar{\mathbf{v}} \subseteq \mathscr{L}_{pc}(G)$. Thus the closure of $\mathbf{v}$ in the subspace $\mathscr{L}_{pc}(G)$ coincides with its closure in the space $\mathscr{L}_p(G)$.

Theorem 2.5. *Let I be a closed l-ideal of an l-group G, $\mathbf{z}_I = \{P \in \mathscr{L}_p(G);\ I \subseteq P\}$, $\mathbf{z}_{cI} = \mathbf{z}_I \cap \mathscr{L}_{pc}(G)$, and let f_{1c} be the restriction of the mapping $f_1: \mathbf{z}_I \to \mathscr{L}_p(G/I)$*

from Theorem 2.2 *on the set* $\mathbf{z}_{cI}$. *Then* f_{1c} *is a homeomorphism of the space* $\mathbf{z}_{cI}$ *onto the space* $\mathscr{L}_{pc}(G/I)$.

Proof. The assertion follows immediately from Theorems 1.5 and 2.2.

Theorem 2.6. *If* $G \neq 0$ *is a representable l-group, then its space of prime ideals* $\mathscr{L}_{pc}(G)$ *is homeomorphic to the space of prime ideals* $\mathscr{L}_{pc}(G')$ *for some completely distributive representable l-group* G'.

Proof. If G is a representable l-group, then each of its minimal prime subgroups is normal. ([4, Satz 7.4], [1, Théorème 4.2.5]). Moreover, the closure of any l-ideal of G is an l-ideal of G, too. Hence the distributive radical $D(G)$ is in this case equal to the intersection of the closures of all minimal prime ideals of G, and therefore f_{1c} is a homeomorphism of $\mathscr{L}_{pc}(G)$ onto $\mathscr{L}_{pc}(G/D(G))$.

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