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POLARS IN AUTOMETRIZED ALGEBRAS

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In the paper [3], K. L. N. Swamy introduced the notion of the autometrized algebra which is a generalization, for example, of l-groups and Brouwerian algebras. Ideals in autometrized algebras are studied by K. L. N. Swamy and N. P. Rao in [4], where the polars of ideals are also introduced. Prime ideals in autometrized algebras are studied by the author in [2]. In this paper properties of polars in connections with ideals and prime ideals are discussed.

A system $A = (A, +, \leqq, *)$ is called an *autometrized algebra* if

(1) $(A, +)$ is a commutative semigroup with zero element 0;

(2) $(A, \leqq)$ is an ordered set and

$$\forall a, b, c \in A\,; \quad a \leqq b \Rightarrow a + c \leqq b + c\,;$$

(3) $*: A \times A \to A$ is a mapping such that

$$\forall a, b \in A\,; \quad a * b \geqq 0 \quad \text{and} \quad a * b = 0 \Leftrightarrow a = b\,,$$
$$\forall a, b \in A\,; \quad a * b = b * a\,,$$
$$\forall a, b, c \in A\,; \quad a * c \leqq (a * b) + (b * c)\,.$$

If $(A, \leqq)$ is a lattice and

$$\forall a, b, c \in A\,; \quad a + (b \vee c) = (a + b) \vee (a + c)\,,$$
$$a + (b \wedge c) = (a + b) \wedge (a + c)\,,$$

then A is called a *lattice algebra* (an l-algebra).

If

$$\forall a \in A\,; \quad a \leqq a * 0\,,$$
$$\forall a, b, c, d \in A\,; \quad (a + c) * (b + d) \leqq (a * b) + (c * d)\,,$$
$$\forall a, b, c, d \in A\,; \quad (a * c) * (b * d) \leqq (a * b) + (c * d)\,,$$
$$\forall a, b \in A\,; \quad a \leqq b \Rightarrow \exists x \geqq 0\,; \quad a + x = b\,,$$

then we say that A is a *normal algebra*.

If

$$\forall a \in A\,; \quad a \geqq 0 \Rightarrow a * 0 = a\,,$$

then A is called a *semiregular algebra*.

An ordered semigroup A with zero element 0 is said to be an *interpolation semigroup* if

$$\forall a, b, c \in A ; \quad [0 \leqq a, b, c, \ a \leqq b + c \Rightarrow$$
$$\Rightarrow (\exists 0 \leqq b_1 \leqq b, \ 0 \leqq c_1 \leqq c; \ a = b_1 + c_1)] .$$

For instance, every commutative l-group and every Brouwerian algebra is an interpolation semigroup. (For l-groups see e.g. [1, p. 21], for Brouwerian algebras it follows from the distributivity.)

If $A = (A, +, \leqq, *)$ is an autometrized algebra, $\emptyset \neq I \subseteq A$, then I is said to be an *ideal in A* if

$$\forall a, b \in I ; \quad a + b \in I ,$$
$$\forall a \in I , \quad x \in A ; \quad x * 0 \leqq a * 0 \Rightarrow x \in I .$$

Let us suppose that $A = (A, +, \leqq, *)$ is an autometrized l-algebra, $a, b \in A$. We say that a and b are *orthogonal* (notation $a \perp b$) whenever

$$(a * 0) \wedge (b * 0) = 0 .$$

If $B \subseteq A$, then

$$B^{\perp} = \{x \in A; \ x \perp b \ \text{for all} \ b \in B\}$$

is called the *polar* of the set B.

We say that $C \subseteq A$ is a *polar in A* if there exists $B \subseteq A$ such that $C = B^{\perp}$. The set of all polars in an algebra A will be denoted by $\mathscr{P}(A)$.

Theorem 1. *Any polar in a normal interpolation autometrized l-algebra A is an ideal in A.*

Proof. Let $B \subseteq A$, $x, y \in B^{\perp}$, $b \in B$. Since A is normal, we have

$$[(x + y) * 0] \wedge (b * 0) \leqq [(x * 0) + (y * 0)] \wedge (b * 0) .$$

But A is also an interpolation algebra, hence we obtain

$$0 \leqq [(x + y) * 0] \wedge (b * 0) \leqq [(x * 0) \wedge (b * 0)] + [(y * 0) \wedge (b * 0)] = 0 ,$$

therefore $x + y \in B^{\perp}$.

If $x \in B^{\perp}$, $a \in A$, $a * 0 \leqq x * 0$, then evidently $a \in B^{\perp}$.

For an autometrized algebra A the set of all its ideals will be denoted by $\mathscr{I}(A)$. If A is normal, then $\mathscr{I}(A)$ ordered by set inclusion is a complete algebraic lattice in which the infimum of any system of ideals is formed by the intersection of that system ([4, Theorem 1]). If $B \subseteq A$, then we denote the smallest ideal in A containing B by $I(B)$. For $a \in A$ we shall write $I(a)$ instead of $I(\{a\})$.

We have

$$I(B) = \{x \in A; \ x * 0 \leqq (b_1 * 0) + \ldots + (b_n * 0), \ b_1, \ldots, b_n \in B\} ,$$
$$I(a) = \{x \in A; \ x * 0 \leqq m(a * 0), \ \text{for some positive integer} \ m\} .$$

Theorem 2. *If A is a semiregular normal interpolation autometrized l-algebra,*

$B \subseteq A$, *then*

$$B^{\perp} = \{x \in A;\ I(x) \cap I(B) = \{0\}\}\ .$$

Proof. a) Let $x \in B^{\perp}$, $z \in I(x) \cap I(B)$. Then there exist an integer $m \geqq 0$ and elements $b_1, \ldots, b_n \in B$ such that

$$z * 0 \leqq m(x * 0)\ ,$$
$$z * 0 \leqq (b_1 * 0) + \ldots + (b_n * 0)\ .$$

Since A is an interpolation algebra, it follows that

$$0 \leqq z * 0 \leqq m(x * 0) \wedge [(b_1 * 0) + \ldots + (b_n * 0)] \leqq$$
$$\leqq [m(x * 0) \wedge (b_1 * 0)] + \ldots + [m(x * 0) \wedge (b_n * 0)] \leqq$$
$$\leqq m[(x * 0) \wedge (b_1 * 0)] + \ldots + m[(x * 0) \wedge (b_n * 0)] = 0 + \ldots + 0 = 0\ ,$$

hence $z * 0 = 0$, and this means that $z = 0$. Therefore $I(x) \cap I(B) = \{0\}$.

b) Let us suppose that $x \in A$ is such that $I(x) \cap I(B) = \{0\}$. Let $b \in B$. Let us denote $c = (x * 0) \wedge (b * 0)$. Then the semiregularity of the algebra A implies

$$c * 0 = c \leqq x * 0\ ,$$

hence $c \in I(x)$. Similarly

$$c * 0 = c \leqq b * 0\ ,$$

and thus $c \in I(B)$.

But then $c = 0$ by the assumption, therefore $x \in B^{\perp}$.

Corollary. *Any polar in a semiregular normal interpolation autometrized l-algebra A is the polar of an ideal in A.*

Proof. If $B^{\perp}$ is a polar in A, then Theorem 2 implies $B^{\perp} = I(B)^{\perp}$.

An ideal I in an autometrized algebra A is called a *prime ideal* in A if

$$\forall J, K \in \mathscr{I}(A)\ ;\quad J \cap K = I \Rightarrow J = I \quad \text{or} \quad K = I\ .$$

In addition, if A is a semiregular normal autometrized l-algebra, I a prime ideal in A, then

$$\forall a, b \in A\ ;\quad 0 = a \wedge b \Rightarrow a \in I \quad \text{or} \quad b \in I\ .$$

([2, Theorem 4].)

We denote the set of all prime ideals in A by $\mathscr{I}_p(A)$.

Theorem 3. *If A is an autometrized algebra, $I \in \mathscr{I}(A)$, $a \in A$, $a \notin I$, then there exists a prime ideal in A containing I and not containing a.*

Proof. Let $a \in A$, $I \in \mathscr{I}(A)$, $a \notin I$. Let us denote

$$Z = \{J \in \mathscr{I}(A);\ I \subseteq J,\ a \notin J\}\ .$$

Let us consider an arbitrary linearly ordered system $(J_\alpha;\ \alpha \in \Gamma)$ of elements in Z and let

$$K = \bigcup_{\alpha \in \Gamma} J_\alpha\ .$$

If $a, b \in K$, then there exist $\alpha_1, \alpha_2 \in \Gamma$ such that $a \in J_{\alpha_1}$, $b \in J_{\alpha_2}$ and, for example, $J_{\alpha_1} \supseteq J_{\alpha_2}$. Hence $a, b \in J_{\alpha_1}$, and so $a + b \in J_{\alpha_1} \subseteq K$.

It is obvious that if $x \in A$, $a \in K$, $x * 0 \leqq a * 0$, then $x \in K$.

Thus K is an ideal in A and $a \notin K$, therefore $K \in Z$. This means that Z is an inductive set, therefore Z contains a maximal element.

Let us consider any maximal element L in Z. Let $M, N \in \mathscr{I}(A)$, $M \cap N = L$, and let $M \supset L$, $N \supset L$. Then $a \in M$, $a \in N$, hence $a \in M \cap N = L$, a contradiction.

Therefore $L \in \mathscr{I}_p(A)$.

Theorem 4. *For any element $a \neq 0$ in an autometrized algebra A there exists a prime ideal in A not containing the element a.*

Proof. Since $\{0\} \in \mathscr{I}(A)$, the assertion is an immediate consequence of Theorem 3.

Theorem 5. *If A is a semiregular autometrized l-algebra, $B \subseteq A$, then $B^\perp$ is equal to the intersection of all prime ideals in A not containing B.*

Proof. Let C be the intersection of all prime ideals in A not containing B.

Let $x \in B^\perp$, $I \in \mathscr{I}_p(A)$, $B \subseteq I$, $b \in B \setminus I$. Then $(x * 0) \wedge (b * 0) = 0$ and consequently $b * 0 \notin I$. (If $b * 0 \in I$, then also $b \in I$, because in the case of a semiregular algebra we have $b * 0 = (b * 0) * 0$.) Since $I \in \mathscr{I}_p(A)$, it follows $x * 0 \in I$, and so also $x \in I$. Therefore $B^\perp \subseteq C$.

Conversely, let $x \notin B^\perp$, i.e., let there exist $b \in B$ such that $(x * 0) \wedge (b * 0) > 0$. Let us consider $I \in \mathscr{I}_p(A)$ such that $(x * 0) \wedge (b * 0) \notin I$. Then $x * 0 \notin I$, $b * 0 \notin I$. The semiregularity of A yields $x \notin I$, $b \notin I$. Thus $x \notin I$, $B \nsubseteq I$, hence $x \notin C$. But this means $C \subseteq B^\perp$.

Corollary. *Any polar in a semiregular normal autometrized l-algebra A is an ideal in A.*

Now let A be a semiregular normal autometrized l-algebra. Then $\mathscr{I}(A)$ is a complete algebraic Brouwerian lattice and for $I \in \mathscr{I}(A)$ we have that $I^\perp$ is the pseudocomplement of I in $\mathscr{I}(A)$. Further, the mapping that to any $I \in \mathscr{I}(A)$ assigns $I^{\perp\perp}$ is a closure operator on $\mathscr{I}(A)$. ([4, Theorem 6, Lemma 7, Theorem 7].)

Theorem 6. a) *If $B_\alpha \subseteq A$, $\alpha \in \Gamma$, then*

$$\bigcap_{\alpha\in\Gamma} B_\alpha^\perp = \Big(\bigcup_{\alpha\in\Gamma} B_\alpha\Big)^\perp .$$

b) *If $B_\alpha \in \mathscr{I}(A)$, $\alpha \in \Gamma$, then*

$$\bigcap_{\alpha\in\Gamma} B_\alpha^\perp = \Big(\bigvee_{\alpha\in\Gamma} B_\alpha\Big)^\perp ,$$

for the supremum in $\mathscr{I}(A)$.

Proof. a) Let $x \in \bigcap_{\alpha\in\Gamma} B_\alpha^\perp$. Then $x \perp b$ for each $b \in \bigcup_{\alpha\in\Gamma} B_\alpha$, hence $\bigcap_{\alpha\in\Gamma} B_\alpha^\perp \subseteq \big(\bigcup_{\alpha\in\Gamma} B_\alpha\big)^\perp$. Conversely, if $y \in \big(\bigcup_{\alpha\in\Gamma} B_\alpha\big)^\perp$, then $y \perp b$ for all $b \in \bigcup_{\alpha\in\Gamma} B_\alpha$, hence $x \in \bigcap_{\alpha\in\Gamma} B_\alpha^\perp$, and so $\big(\bigcup_{\alpha\in\Gamma} B_\alpha\big)^\perp \subseteq \bigcap_{\alpha\in\Gamma} B_\alpha^\perp$.

b) The assertion now follows immediately from Corollary of Theorem 2.

Corollary. a) *If* $B \subseteq A$, *then*

$$B \subseteq B^{\perp\perp}, \quad B^{\perp} = B^{\perp\perp\perp}.$$

b) $B \subseteq A$ *is a polar in* A *if and only if* $B = B^{\perp\perp}$.

Now by [4, Theorem 7] we obtain that $\mathscr{P}(A)$ ordered by set inclusion is a Boolean algebra. If $B \in \mathscr{P}(A)$, then its complement is evidently formed by its polar $B^{\perp}$. In addition, by Theorem 6 $\mathscr{P}(A)$ is a Moore system, hence $\mathscr{P}(A)$ is a complete lattice.

Theorem 7. *If* B_α, $\alpha \in \Gamma$, *are any polars in* A, *then in the complete lattice* $\mathscr{P}(A)$ *we have*

$$\bigwedge_{\alpha\in\Gamma} B_\alpha = \bigcap_{\alpha\in\Gamma} B_\alpha, \quad \bigvee_{\alpha\in\Gamma} B_\alpha = \Big(\bigcap_{\alpha\in\Gamma} B_\alpha^{\perp}\Big)^{\perp}.$$

Proof. The first equality follows from Theorem 6.

Further, for any $C \in \mathscr{P}(A)$ we have $C \supseteq \bigcup_{\alpha\in\Gamma} B_\alpha$ if and only if $C^{\perp} \subseteq \big(\bigcup_{\alpha\in\Gamma} B_\alpha\big)^{\perp}$ and this is satisfied by Theorem 6 if and only if $C \supseteq \big(\bigcap_{\alpha\in\Gamma} B_\alpha^{\perp}\big)^{\perp}$. Hence the second equality follows.

Theorem 8. *The mapping that to any* $I \in \mathscr{I}(A)$ *assigns* $I^{\perp\perp}$ *is a surjective lattice homomorphism of* $\mathscr{I}(A)$ *onto* $\mathscr{P}(A)$.

Proof. The assertion follows immediately from [4, Theorem 7] and from Corollary of Theorem 6.

Let us denote $a^{\perp} = \{a\}^{\perp}$ for $a \in A$.

Theorem 9. *If* A *is an interpolation semiregular normal autometrized l-algebra,* $a, b \in A$, *then*

$$a^{\perp\perp} \cap b^{\perp\perp} = ((a * 0) \wedge (b * 0))^{\perp\perp},$$

$$a^{\perp\perp} \vee b^{\perp\perp} = ((a * 0) \vee (b * 0))^{\perp\perp}.$$

Proof. If A is an interpolation algebra, then by [2, Proposition 2] we have

$$I(a) \cap I(b) = I((a * 0) \wedge (b * 0)),$$

$$I(a) \bigvee_{\mathscr{I}(A)} I(b) = I((a * 0) \vee (b * 0)).$$

Therefore the assertion is a consequence of Theorem 8.

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