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ON THE LATTICE OF CONVEX SUBSETS OF A PARTIAL MONOUNARY ALGEBRA

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The lattice of all convex subsets of a partial monounary algebra will be denoted by $\mathrm{Co}(A, f)$.

In the paper [5] the author described all partial monounary algebras (A, g) having the property that $\mathrm{Co}(A, g) = \mathrm{Co}(A, f)$, where (A, f) was a given partial monounary algebra.

In the present paper necessary and sufficient conditions for a lattice L will be found under which there exists a partial monounary algebra such that L is isomorphic to $\mathrm{Co}(A, f)$.

An analogous question concerning the lattice of all convex subsets of a partially ordered set was investigated by G. Birkhoff and M. K. Bennett [4]; for related questions cf. also Bennett and Birkhoff [3], and Bennett [1], [2].

1. PRELIMINARIES

By a (partial) monounary algebra we understand a pair (A, f), where A is a non-empty set and $f: A \to A$ is a (partial) mapping. Let $\mathscr{U}$ be the class of all partial monounary algebras. To each $(A, f) \in \mathscr{U}$ there corresponds a directed graph $G(A, f) = (A, E)$ without loops and multiple edges which is defined as follows: an ordered pair (a, b) of distinct elements of A belongs to E iff $f(a) = b$.

A subset $B \subseteq A$ will be called *convex* (in (A, f)) if, whenever a, b_1, b_2 are distinct elements of A such that $b_1, b_2 \in B$ and there is a path (in $G(A, f)$) going from b_1 to b_2 and containing the element a, then a belongs to B as well.

The system $\mathrm{Co}(A, f)$ of all convex subsets of a partial monounary algebra (A, f) is partially ordered by inclusion, and it is a lattice.

Let Z be the set of all integers and N the set of all positive integers. Let $(A, f) \in \mathscr{U}$, $n \in N$, $x \in A$. Put $f^0(x) = x$. If $f^{n-1}(x)$ and $f(f^{n-1}(x))$ exist, then we put $f^n(x) = f(f^{n-1}(x))$. If $x, y \in A$, $f^n(x) = f^m(y)$ for some $n, m \in N \cup \{0\}$, then we write $x \equiv_f y$. The relation $\equiv_f$ is an equivalence relation on A. A partial monounary algebra (A, f) is said to be connected, if $A/\equiv_f$ is a one-element set. If $X \in A/\equiv_f$, then X is called a *connected component of* (A, f).

1.1. Notation. Let $\mathscr{V}_0$ be the class of all monounary algebras which are isomorphic to (Z, f), where $f(i) = i + 1$ for each $i \in Z$. Further, let $\mathscr{V}_1$ be the class of all connected monounary algebras possessing a one-element cycle, and let $\mathscr{V}_2$ be the class of all connected monounary algebras having a cycle C with card $C > 1$. The class of all connected monounary algebras (A, f) which possess no cycle and such that there are distinct elements x, y of A with $f(x) = f(y)$ will be denoted by the symbol $\mathscr{V}_3$.

In [5] the following result was proved (cf. [5], Thms. 5.3.2, 5.3.3, 5.4.2 and 5.5.2):

(R) Let (A, f) be a connected partial monounary algebra. Then there is $(A, g) \in \in \mathscr{V}_0 \cup \mathscr{V}_1 \cup \mathscr{V}_2 \cup \mathscr{V}_3$ such that $\mathrm{Co}(A, f) = \mathrm{Co}(A, g)$.

First we shall investigate conditions under which a lattice L is isomorphic to some $(A, f) \in \mathscr{V}_i$ $(i \in \{0, 1, 2, 3\})$.

If L is a lattice, then the set of all atoms of L will be denoted by $A(L)$; if no misunderstanding can occur, we shall write simply A instead of $A(L)$.

If $(A, f) \in \mathscr{U}$, then we shall denote the lattice-operations in $\mathrm{Co}(A, f)$ by the symbols $\vee^{\mathrm{Co}}$, $\wedge^{\mathrm{Co}}$. Further, we shall write $a \in \mathrm{Co}(A, f)$, instead of $\{a\} \in \mathrm{Co}(A, f)$.

From [5] (1.5, 1.6) we obtain

1.2. Lemma. *If $(A, f) \in \mathscr{U}$, then $\mathrm{Co}(A, f)$ is a complete atomic lattice.*

1.3. Definition. (cf. [4]). An atomic lattice L is said to have *Carathéodory rank* 2 if, whenever p is an atom of L and $B \subseteq A$, then the relation $p \leqq \bigvee_{b \in B} b$ yields $p \leqq b_1 \vee b_2$ for two suitably chosen $b_1, b_2 \in B$.

(ε) **Condition.** *L is a complete atomic lattice having Carathéodory rank* 2.

1.4. Lemma. *If $L \cong \mathrm{Co}(A, f)$ for some $(A, f) \in \mathscr{U}$, then L satisfies the condition* (ε).

Proof. This follows from 1.2 and from the definition of convexity for subsets of A.

1.5. Lemma. *Let L be a lattice satisfying the condition* (ε), $(A, f) \in \mathscr{U}$, $A = A(L)$. *Further suppose that the following conditions are valid:*

(i) *If $B \in \mathrm{Co}(A, f)$, then $B = \{p \in A : p \leqq \bigvee_{b \in B} b\}$.*

(ii) *If $u \in L$, then $\{p \in A : p \leqq u\} \in \mathrm{Co}(A, f)$.*

Then $\varphi: B \to \bigvee_{b \in B} b$ for $B \in \mathrm{Co}(A, f)$ is a bijective mapping of $\mathrm{Co}(A, f)$ onto L and $\varphi^{-1}(v) = \{p \in A : p \leqq v\}$ for each $v \in L$.

Proof. Assume that $B, C \in \mathrm{Co}(A, f)$, $\varphi(B) = \varphi(C)$. Then $\bigvee_{b \in B} b = \bigvee_{c \in C} c$; denote this lattice element by the symbol u. Hence (i) yields that $B = \{p \in A : p \leqq u\} = C$, therefore φ is injective. Now let $v \in L$. Then (ii) implies that $\{p \in A : p \leqq v\} \in \in \mathrm{Co}(A, f)$; put $D = \{p \in A : p \leqq v\}$. Hence

$$\varphi(D) = \bigvee_{b \in B} b = v$$

and φ is surjective. Further, $\varphi^{-1}(v) = D = \{p \in A : p \leqq v\}$.

1.6. Lemma. *If the assumption of* 1.5 *is valid and* $\varphi(B) = \bigvee_{b \in B} b$ *for each* $B \in \text{Co}(A, f)$, *then* φ *is a lattice isomorphism of* $\text{Co}(A, f)$ *onto* L.

Proof. According to 1.5, φ is a bijection. If $B, C \in \text{Co}(A, f)$, $B \subseteq C$, then obviously $\varphi(B) = \bigvee_{b \in B} b \leqq \bigvee_{c \in C} c = \varphi(C)$. If $u, v \in L$, $u \leqq v$, then $\varphi^{-1}(u) = \{p \in A: p \leqq u\} \subseteq \subseteq \{p \in A: p \leqq v\} = \varphi^{-1}(v)$ (in view of 1.5).

(α1) **Condition.** *Assume that* L *satisfies* (ε). *For each* $x, y \in A$, $x \neq y$ *there are uniquely determined* $n = n(x, y) \in N$ *and distinct atoms* $x = u_0(x, y), u_1(x, y), \ldots \ldots, u_n(x, y) = y$ *such that, whenever* $0 \leqq i < j \leqq n$, *then*

$$\{p \in A: p \leqq u_i(x, y) \vee u_j(x, y)\} = \{u_i(x, y), u_{i+1}(x, y), \ldots, u_j(x, y)\}\,.$$

1.7. Lemma. *If the condition* (α1) *is satisfied, then* $n(x, y) = n(y, x)$ *and* $u_k(x, y) = u_{n(x,y)-k}(y, x)$ *for each* $x, y \in A$, $0 \leqq k \leqq n(x, y)$.

Proof. The assertion is obvious.

1.8. Lemma. *Let the condition* (α1) *be satisfied,* $x, y \in A$, $0 \leqq i < j \leqq n(x, y)$, $0 \leqq k \leqq j - i$. *Then*

(i) $n(u_i(x, y), u_j(x, y)) = j - i$,

(ii) $u_k(u_i(x, y), u_j(x, y)) = u_{i+k}(x, y)$.

Proof. Put $n = n(x, y)$, $a = u_i(x, y)$, $b = u_j(x, y)$. Then in view of (α1) we have

$$n(a, b) = n(u_i(x, y), u_j(x, y)) = \text{card}\,\{p \in A: p \leqq u_i(x, y) \vee u_j(x, y)\} - 1 =$$
$$= \text{card}\,\{u_i(x, y), u_{i+1}(x, y), \ldots, u_j(x, y)\} - 1 = j - i\,,$$

thus (i) is valid. Further put $v_k = u_{i+k}(x, y)$ for $0 \leqq k \leqq j - i$. We have $v_0 = a$, $v_{j-i} = b$. Let $0 \leqq m < l \leqq j - i$. According to ($\alpha$1) we get

$$\{p \in A: p \leqq u_{i+m}(x, y) \vee u_{i+l}(x, y)\} = \{u_{i+m}(x, y), u_{i+m+1}(x, y), \ldots, u_{i+l}(x, y)\}\,,$$

i.e.,

$$\{p \in A: p \leqq v_m \vee v_l\} = \{v_m, v_{m+1}, \ldots, v_l\}\,.$$

Therefore $a = v_0, v_1, \ldots, v_{j-i} = b$ are exactly the elements $a = v_0 = u_0(a, b)$, $v_1 = u_1(a, b), \ldots, b = v_{j-i} = u_{j-i}(a, b)$, since such elements are uniquely determined in view of (α1).

1.9. Lemma. *Let* $(A, f) \in \mathscr{V}_0 \cup \mathscr{V}_1 \cup \mathscr{V}_3$. *Then* $L = \text{Co}(A, f)$ *satisfies the condition* (α1). *Moreover, if* $x \in A$, $y = f^k(x) \neq f^{k-1}(x)$, $k \in N$, *then* $n(x, y) = k$, $u_i(x, y) = f^i(x)$ *for each* $0 \leqq i \leqq k$.

Proof. In view of 1.4, L satisfies the condition (ε). If $x, y \in A$, $x \notin \{f^k(y): k \in N \cup \cup \{0\}\}$, $y \notin \{f^k(x): k \in N \cup \{0\}\}$, then $x \vee^{\text{Co}} y = \{x, y\}$ and we can set $n(x, y) = 1$. Assume that $y = f^k(x) \neq f^{k-1}(x)$, $k \in N$. Then

$$x \vee^{\text{Co}} y = \{x, f(x), \ldots, f^k(x)\}$$

Put $u_i(x, y) = f^i(x)$ for each $i \in \{0, \ldots, k\}$, $n(x, y) = k$. We infer that $u_i(x, y) \neq u_j(x, y)$ and

$$(1) \qquad u_i(x, y) \vee^{\mathrm{Co}} u_j(x, y) = \{u_i(x, y), u_{i+1}(x, y), \ldots, u_j(x, y)\}$$

for each $0 \leqq i < j \leqq k$. If $m \in N$ and $x = v_0, v_1, \ldots, v_m = y$ are distinct elements of A such that

$$(2) \qquad v_i \vee^{\mathrm{Co}} v_j = \{v_i, v_{i+1}, \ldots, v_j\}$$

for each $0 \leqq i < j \leqq m$, then (in view of (1) and (2))

$$(3) \qquad \{u_0(x, y), \ldots, u_k(x, y)\} = u_0(x, y) \vee^{\mathrm{Co}} u_k(x, y) = x \vee^{\mathrm{Co}} y = \\ = v_0 \vee^{\mathrm{Co}} v_m = \{v_0, v_1, \ldots, v_m\}.$$

Thus $k = m$, $v_i = u_{\varphi(i)}(x, y)$ for some permutation φ. Since $u_0(x, y) \vee^{\mathrm{Co}} u_i(x, y)$ is covered by $u_0(x, y) \vee^{\mathrm{Co}} u_{i+1}(x, y)$ for each $i \in \{0, \ldots, k-1\}$, we get that $\varphi(i) = i$ for each $i \in \{0, \ldots, k\}$.

1.10. Lemma. *Let L satisfy the condition $(\alpha 1)$. Suppose that $a, b, c \in A$, $u_i(a, b) = u_j(a, c)$ for some $i, j \in N$, $i \leqq n(a, b)$, $j \leqq n(a, c)$. Then $i = j$ and $u_k(a, b) = u_k(a, c)$ for each $0 \leqq k \leqq i$.*

Proof. Let the assumption hold. Then $(\alpha 1)$ yields

$$\begin{aligned} i + 1 &= \operatorname{card}\{u_0(a, b), u_1(a, b), \ldots, u_i(a, b)\} = \\ &= \operatorname{card}\{p \in A : p \leqq u_0(a, b) \vee u_i(a, b)\} = \\ &= \operatorname{card}\{p \in A : p \leqq u_0(a, c) \vee u_j(a, c)\} = \\ &= \operatorname{card}\{u_0(a, c), u_1(a, c), \ldots, u_j(a, c)\} = j + 1, \end{aligned}$$

therefore $i = j$. Let $0 \leqq k \leqq i$. According to 1.8,

$$\begin{aligned} u_k(a, b) &= u_{k+0}(a, b) = u_k(u_0(a, b), u_i(a, b)) = \\ &= u_k(u_0(a, c), u_i(a, c)) = u_{k+0}(a, c) = u_k(a, c). \end{aligned}$$

1.11. Lemma. *Let L satisfy the condition $(\alpha 1)$. If $x, z, v_1, v_2 \in A$ are distinct and $z \leqq v_1 \vee v_2$, $x \leqq v_1 \vee v_2$, then either $x \nleqq z \vee v_2$ or $x \nleqq z \vee v_1$.*

Proof. Let x, z, v_1, v_2 be distinct elements of A such that $z \leqq v_1 \vee v_2$, $x \leqq v_1 \vee v_2$. Then $(\alpha 1)$ yields

$$\begin{aligned} z &= u_i(v_1, v_2) \quad \text{for some} \quad 0 < i < n(v_1, v_2), \\ x &= u_j(v_1, v_2) \quad \text{for some} \quad 0 < j < n(v_1, v_2). \end{aligned}$$

If $i < j$, then

$$\begin{aligned} \{p \in A : p \leqq z \vee v_1\} &= \{p \in A : p \leqq u_0(v_1, v_2) \vee u_i(v_1, v_2)\} = \\ &= \{u_0(v_1, v_2), \ldots, u_i(v_1, v_2)\} \end{aligned}$$

and this set does not contain $u_j(v_1, v_2) = x$, thus $x \nleqq z \vee v_1$. Analogously, if $i > j$, then

$$\{p \in A: p \leqq z \vee v_2\} = \{p \in A: p \leqq u_i(v_1, v_2) \vee u_{n(v_1,v_2)}(v_1, v_2)\} =$$
$$= \{u_i(v_1, v_2), \ldots, u_{n(v_1,v_2)}(v_1, v_2)\},$$

which implies that $x = u_j(v_1, v_2) \nleqq z \vee v_2$.

1.12. Lemma. *Let L satisfy the condition (α1), let $a, b, c \in A$ be such that $b \leqq a \vee c$. Then $n(a, b) + n(b, c) = n(a, c)$.*

Proof. Let the assumption be valid. Since $b \leqq a \vee c$, (α1) implies that there is $0 \leqq i \leqq n(a, c)$ with $b = u_i(a, c)$. We have

(1) $$u_i(a, c) = b = u_{n(a,b)}(a, b),$$

hence 1.10 yields

(2) $$i = n(a, b).$$

Further, by 1.7, $b = u_i(a, c) = u_{n(a,c)-i}(c, a)$, thus

(3) $$u_{n(a,c)-i}(c, a) = b = u_{n(c,b)}(c, b).$$

According to 1.10 we get

(4) $$n(a, c) - i = n(c, b).$$

In view of 1.7, (2) and (4) we have $n(a, c) = n(a, b) + n(c, b) = n(a, b) + n(b, c)$.

1.13. Lemma. *Let L satisfy (α1), let y, z, r, v be distinct elements of A such that $z \leqq y \vee r$, $r \leqq y \vee v$. Then $z \leqq y \vee v$ and $r \leqq z \vee v$.*

Proof. Since $z \leqq y \vee r$, $r \leqq y \vee v$, we get

(1) $$z = u_i(y, r) \quad \text{for some} \quad 0 < i < n(y, r),$$

(2) $$r = u_j(y, v) \quad \text{for some} \quad 0 < j < n(y, v).$$

Then $u_j(y, v) = r = u_{n(y,r)}(y, r)$ and 1.10 implies

(3) $$j = n(y, r),$$

(4) $$u_k(y, v) = u_k(y, r) \quad \text{for each} \quad 0 \leqq k \leqq n(y, r).$$

According to (1) and (4) we have

(5) $$z = u_i(y, r) = u_i(y, v),$$

hence

(6) $$z \leqq y \vee v.$$

Further, in view of (α1), (1), (3) and (5),

$$r = u_j(y, v) \in \{u_i(y, v), \ldots, u_j(y, v), \ldots, u_{n(y,v)}(y, v)\} =$$
$$= \{p \in A: p \leqq u_i(y, v) \vee u_{n(y,v)}(y, v)\} = \{p \in A: p \leqq z \vee v\},$$

and therefore

(7) $$r \leqq z \vee v.$$

In this section we shall investigate conditions under which the lattice L is isomorphic to $\mathrm{Co}(A, f)$ for some $(A, f) \in \mathscr{V}_0$.

($\alpha2$) **Condition.** *L satisfies the condition ($\alpha1$) nad whenever $x, y \in A$, then there is $z \in A$ such that $x \neq z \neq y$, $x \leqq y \vee z$.*

(α) **Condition.** *The condition ($\alpha2$) is valid and whenever $x, y, z \in A$, then either $x \leqq y \vee z$ or $y \leqq x \vee z$ or $z \leqq x \vee y$.*

2.1. Lemma. *If $(A, f) \in \mathscr{V}_0$, then $L = \mathrm{Co}(A, f)$ satisfies the condition (α).*

Proof. Let us show that (Z, f) with $f(i) = i + 1$ (for each $i \in Z$) satisfies (α). According to 1.9, ($\alpha1$) is valid. Let x, y be integers, x less than y. Then

$$x \in y \vee^{\mathrm{Co}} (x - 1) = \{x - 1, x, x + 1, \ldots, y\}$$

and ($\alpha2$) holds. If $x, y, z \in Z$, x is less than y, y is less than z, then

$$y \in x \vee^{\mathrm{Co}} z = \{x, x + 1, \ldots, y, y + 1, \ldots, z\}.$$

Therefore $\mathrm{Co}(Z, f)$ satisfies the condition (α).

2.2. Lemma. *Let L satisfy the condition (α). There are distinct elements $a, a' \in A$ with $\{p \in A: p \leqq a \vee a'\} = \{a, a'\}$.*

Proof. In view of ($\alpha2$), card $A > 1$. There are $a, b \in A$, $a \neq b$. According to ($\alpha1$), $u_1(a, b) \neq u_0(a, b) = a$; put $a' = u_1(a, b)$. We get

$$\{p \in A: p \leqq a \vee a'\} = \{p \in A: p \leqq u_0(a, b) \vee u_1(a, b)\} = \\ = \{u_0(a, b), u_1(a, b)\} = \{a, a'\}.$$

2.3. Lemma. *Let L satisfy the condition (α) and suppose that x, z, v_1, v_2 are distinct elements of A such that $x \leqq z \vee v_1$, $x \leqq z \vee v_2$. Then either $v_1 \leqq x \vee v_2$ or $v_2 \leqq x \vee v_1$.*

Proof. Let the assumption be valid and let $v_1 \nleqq x \vee v_2$, $v_2 \nleqq x \vee v_1$. According to (α) we obtain

$$x \leqq v_1 \vee v_2 . \tag{1}$$

From (1) and 1.11 we obtain

$$z \nleqq v_1 \vee v_2 . \tag{2}$$

In view of (2) and (α) we have either $v_1 \leqq z \vee v_2$ or $v_2 \leqq z \vee v_1$. Both the cases are analogous; let us suppose that

$$v_1 \leqq z \vee v_2 \tag{3}$$

holds. Since $x \leqq z \vee v_1$, $x \leqq z \vee v_2$,

$$x = u_i(z, v_1) = u_j(z, v_2) \quad \text{for some} \quad 0 \leqq i \leqq n(z, v_1), \quad 0 \leqq j \leqq n(z, v_2). \tag{4}$$

According to 1.10,

(5) $$i = j\,, \quad x = u_i(z, v_1) = u_i(z, v_2)\,.$$

Further, 1.7 implies

(6) $$x = u_{n(z,v_2)-i}(v_2, z)\,.$$

Moreover, $x \leqq v_2 \vee v_1$, hence $x = u_k(v_2, v_1)$ for some $0 \leqq k \leqq n(v_2, v_1)$ and this, in view of (6), yields

(7) $$k = n(z, v_2) - i\,, \quad x = u_{n(z,v_2)-i}(v_2, z) = u_{n(z,v_2)-i}(v_2, v_1)\,.$$

Further, according to 1.7 we have

(8) $$u_{n(z,v_2)-i}(v_2, v_1) = u_{n(v_2,v_1)-n(z,v_2)+i}(v_1, v_2)\,.$$

From (5) and 1.7 we get

(9) $$x = u_i(z, v_1) = u_{n(z,v_1)-i}(v_1, z)\,,$$

therefore (8), (9) and 1.10 imply

(10) $$n(v_2, v_1) - n(z, v_2) + i = n(z, v_1) - i\,,$$
$$n(v_2, v_1) + 2i = n(z, v_1) + n(z, v_2)\,.$$

According to (3) and 1.12 we obtain

(11) $$n(v_2, v_1) + n(v_1, z) = n(v_2, z)\,,$$

thus (10) and (11) yield

(12) $$i = n(z, v_1)\,.$$

Therefore $x = u_i(z, v_1) = u_{n(z,v_1)}(z, v_1) = v_1$, which is a contradiction.

2.4. Lemma. *Let L satisfy the condition (α) and let $a, a' \in A$ be as in 2.2, $n \in N$.*

(i) *There is $v \in A$ such that $a' \leqq a \vee v$, $n(a', v) \geqq n$.*

(ii) *There is $x \in A$ such that $a \leqq a' \vee x$, $n(a, x) \geqq n$.*

Proof. We shall show only (i); the proof of (ii) is analogous. From $(\alpha 2)$ it follows that there is $v_1 \in A$ such that

(1) $$a \neq v_1 \neq a'\,, \quad a' \leqq a \vee v_1\,.$$

If $n(a', v_1) \geqq n$, then the assertion is valid. Let us prove that there is $v_2 \in A$ with

(2) $$a' \leqq a \vee v_2\,, \quad n(a', v_2) > n(a', v_1)\,.$$

Then we obtain by induction that (i) holds.

In view of $(\alpha 2)$ there is $v_2 \in A$ such that

(3) $$a \neq v_2 \neq v_1\,, \quad v_1 \leqq a \vee v_2\,.$$

If $a' = v_2$, then $v_1 \leqq a \vee a'$, i.e., $v_1 \in \{a, a'\}$ (in view of 2.2), which is a contradiction to (1). Hence the elements a, a', v_1, v_2 are distinct and (1), (3) and 1.13 (with

a, a', v_1, v_2 instead of y, z, r, v) imply

(4) $$a' \leqq a \vee v_2\,,$$

(5) $$v_1 \leqq a' \vee v_2\,.$$

Then 1.12 and (5) yield

(6) $$n(a', v_2) = n(a', v_1) + n(v_1, v_2) \geqq n(a', v_1) + 1 > n(a', v_1)\,.$$

Combining (4) and (6) we obtain that (2) holds.

2.5. Notation. Let L satisfy the condition (α) and let $a, a' \in A$ be as in 2.2, $x \in A$. Then 2.2 and (α) yield that some of the following conditions is valid:

(1.1) $$x = a\,,$$

(1.2) $$x \neq a \leqq x \vee a'\,,$$

(1.3) $$a' \leqq x \vee a\,.$$

Let $\varkappa$ be a mapping of A into Z which is defined as follows:

$$\varkappa(x) = \begin{cases} 0\,, & \text{if} \quad x = a\,, \\ -n(a, x)\,, & \text{if} \quad x \neq a \leqq x \vee a'\,, \\ n(a, x)\,, & \text{if} \quad a' \leqq x \vee a\,. \end{cases}$$

2.6. Lemma. *Let L satisfy the condition (α). The mapping $\varkappa: A \to Z$ defined in 2.5 is surjective.*

Proof. Let $k \in Z$, $k > 0$. From 2.4 (i) it follows that there is $v \in A$ with $a' \leqq \leqq a \vee v$, $n(a', v) \geqq k$. Put $x = u_{k-1}(a', v)$. Then $x \leqq a' \vee v$, $a' \leqq a \vee v$ and 1.13 (with v, x, a', a instead of y, z, r, v) yields that $a' \leqq a \vee x$ whenever $a' \neq x \neq v$. If $x = a'$ or $x = v$, then obviously $a' \leqq a \vee x$, hence

(1) $$a' \leqq a \vee x\,.$$

According to 2.5 we have $\varkappa(x) = n(a, x)$. Further, (1) and 1.12 imply

(2) $$\varkappa(x) = n(a, x) = n(a, a') + n(a', x) = 1 + n(a', x)$$

(by 2.2). Since $x = u_{k-1}(a', v) = u_{n(a',x)}(a', x)$, 1.10 yields

(3) $$n(a', x) = k - 1\,.$$

From (2) and (3) we obtain that $\varkappa(x) = k$.

Let $m \in Z$, $m < 0$. In view of 2.4 (i) there is $u \in A$ such that $a \leqq a' \vee u$, $n(a, u) \geqq \geqq -m$. If we put $x = u_{-m}(a, u)$, then analogously as above we get that $a \leqq a' \vee x$, $n(a, x) = -m$, thus $\varkappa(x) = m$.

2.7. Lemma. *Let L satisfy the condition (α). The mapping $\varkappa: A \to Z$ defined in 2.5 is injective.*

Proof. If $x \in A$, $\varkappa(x) = 0$, then $x = a$, since $n(a, x) \geqq 1$ for $x \neq a$. Let $k \in N$,

$x, y \in A$ with $\varkappa(x) = \varkappa(y) = k$. Then 2.5 yields

(1) $$k = n(a, x) = n(a, y),$$

(2) $$a' \leqq x \vee a, \quad a' \leqq y \vee a.$$

According to (2) and 2.3 we get that either $x \leqq y \vee a'$ or $y \leqq x \vee a'$; without loss of generality assume that $x \leqq y \vee a'$. Thus (α1) implies

(3) $$x = u_j(a', y) \quad \text{for some} \quad 0 \leqq j \leqq n(a', y).$$

Further, (2), 1.12 and 2.2 yield

$$k = n(a, x) = n(a, a') + n(a', x) = 1 + n(a', x),$$
$$k = n(a, y) = 1 + n(a', y),$$

hence

(4) $$n(a', x) = n(a', y) = k - 1.$$

Since $x = u_{n(a',x)}(a', x) = u_{k-1}(a', x)$, 1.10 (in view of (3)) implies $j = k - 1$, thus

(5) $$x = u_{k-1}(a', y).$$

Further, according to (4) and (5),

$$x = u_{n(a',y)}(a', y) = y.$$

Now let $m \in Z$, $m < 0$, $z, v \in A$ with $\varkappa(z) = \varkappa(v) = m$. Then

(6) $$-m = n(a, z) = n(a, v),$$

(7) $$a \leqq z \vee a', \quad a \leqq v \vee a', \quad z \neq a \neq v.$$

According to 2.3, either $z \leqq v \vee a$ or $v \leqq z \vee a$; we can suppose that $z \leqq v \vee a$, i.e., $z = u_i(a, v)$ for some $0 \leqq i \leqq n(a, v)$. Analogously as above,

$$z = u_{n(a,z)}(z) = u_i(a, v),$$

and 1.10 yields $i = n(a, z)$, thus $i = n(a, z) = n(a, v)$ (by (6)),

$$z = u_{n(a,v)}(a, v) = v.$$

2.8. Notation. Let the assumption of 2.5 be valid. For $x \in A$ put

$$f(x) = \varkappa^{-1}(\varkappa(x) + 1).$$

2.9. Lemma. *Let the assumption of* 2.5 *hold and let f be as in* 2.8. *Then f is a unary operation on A and $(A, f) \in \mathscr{V}_0$.*

Proof. According to 2.6 and 2.7, $\varkappa$ is a bijective mapping of A onto Z. Then $f(x)$ is defined for each $x \in A$ and it is obvious that $(A, f) \in \mathscr{V}_0$.

2.10. Lemma. *Let L satisfy the condition* (α) *and let f be as in* 2.8. *If $B \in \mathrm{Co}(A, f)$, then $B = \{p \in A: p \leqq \bigvee_{b \in B} b\}$.*

Proof. Let $B \in \mathrm{Co}(A, f)$. Then $B \subseteq \{p \in A: p \leqq \bigvee_{b \in B} b\}$. Suppose that $p \in A$,

$p \leqq \bigvee_{b \in B} b$. According to (ε), $p \leqq x \vee y$ for some $x, y \in B$, thus $(\alpha 1)$ yields

(1) $$p = u_i(x, y) \quad \text{for some} \quad 0 \leqq i \leqq n(x, y)\,.$$

Since $(A, f) \in \mathscr{V}_0$, without loss of generality we can assume that $y = f^k(x)$, $k \in N \cup \{0\}$. Then 1.9 implies

(2) $$n(x, y) = k\,,$$

(3) $$f^j(x) = u_j(x, y) \quad \text{for each} \quad 0 \leqq j \leqq k\,.$$

By (1)–(3) we obtain

(4) $$p = f^i(x)\,, \quad 0 \leqq i \leqq k\,, \quad y = f^k(x)\,,$$

therefore $p \in x \vee^{\mathrm{Co}} y \subseteq B$.

2.11. Lemma. *Let the assumption of* 2.10 *hold. If* $u \in L$, *then* $\{p \in A: p \leqq u\} \in \mathrm{Co}(A, f)$.

Proof. Put $B = \{p \in A: p \leqq u\}$. Assume that $x, y \in B$, $y = f^k(x)$, $k \in N$, $p = f^i(x)$, $0 < i < k$. According to 1.9 we obtain

(1) $$n(x, y) = k\,, \quad p = u_i(x, y)\,,$$

therefore $p \leqq x \vee y \leqq u$. Hence $p \in B$ and $B \in \mathrm{Co}(A, f)$.

2.12. Corollary. *Let* L *satisfy the condition* (α) *and let* f *be as in* 2.8. *For* $B \in \mathrm{Co}(A, f)$ *put* $\varphi(B) = \bigvee_{b \in B} b$. *Then* φ *is a lattice isomorphism of* $\mathrm{Co}(A, f)$ *onto* L, *and* $(A, f) \in \mathscr{V}_0$.

Proof. The assertion follows from 2.9, 2.10, 2.11 and 1.6.

2.13. Theorem. *Let* L *be a lattice. Then* $L \cong \mathrm{Co}(A, f)$ *for some* $(A, f) \in \mathscr{V}_0$ *if and only if* L *satisfies the condition* (α).

Proof. The assertion follows from 2.12 and 2.1.

3. THE CLASS $\mathscr{V}_1$

In this part we shall investigate conditions under which a lattice L is isomorphic to $\mathrm{Co}(A, f)$ for some $(A, f) \in \mathscr{V}_1$.

3.0. Notation. Let L be a lattice, $x_0 \in A$. The pair (L, x_0) will be said to *satisfy the condition* (β^0), if L satisfies $(\alpha 1)$ and if, whenever $p, a, b \in A$, $a \neq p \neq b$, $p \leqq a \vee b$, then either $a \leqq b \vee x_0$ or $b \leqq a \vee x_0$.

(β) **Condition.** *There exists* $x_0 \in L$ *such that the pair* (L, x_0) *satisfies the condition* (β^0).

3.1. Lemma. *If* $(A, f) \in \mathscr{V}_1$, $x_1 \in A$ *with* $f(x_1) = x_1$, $L = \mathrm{Co}(A, f)$, *then the pair* (L, x_1) *satisfies the condition* (β^0).

Proof. Put $x_1 = x_0$. According to 1.9, L satisfies $(\alpha 1)$. Let $a, b, p \in A$, $a \neq p \neq b$, $p \in a \vee^{Co} b$. Then either $b = f^k(a)$ for some $k \in N$, $k > 1$ or $a = f^k(b)$ for some $k \in N$, $k > 1$. We can suppose that $b = f^k(a)$. Hence

$$b \in \{a, f(a), \ldots, f^k(a), \ldots, x_0\} = a \vee^{Co} x_0 .$$

3.2. Notation. Let (L, x_0) satisfy the condition (β^0). Put

$$f(x) = \begin{cases} x_0 , & \text{if } x = x_0 \\ u_1(x, x_0) , & \text{if } x \in A - \{x_0\} . \end{cases}$$

3.3. Lemma. *Let L and f be as in 3.2. If $a, b \in A$, $a \neq b$, $b \leqq a \vee x_0$, $0 \leqq i < n(a, b)$, then $f(u_i(a, b)) = u_{i+1}(a, b)$.*

Proof. Let the assumption of the lemma be valid. Since $b \leqq a \vee x_0$, $(\alpha 1)$ yields that $b = u_j(a, x_0)$ for some $0 \leqq j \leqq n(a, x_0)$. In view of 1.8,

(1) $$i < n(a, b) = n(u_0(a, x_0), u_j(a, x_0)) = j .$$

Let $n = n(a, x_0)$. We obtain

(2) $$f(u_i(a, b)) = u_1(u_i(a, b), x_0) = u_1(u_i(u_0(a, x_0), u_j(a, x_0)), u_n(a, x_0)) .$$

According to 1.8 and (1) we have

$$u_i(u_0(a, x_0), u_j(a, x_0)) = u_{0+i}(a, x_0) ,$$

hence (2) implies

(3) $$f(u_i(a, b)) = u_1(u_i(a, x_0), u_n(a, x_0)) .$$

Since $i < n$, i.e., $i \leqq n - 1$, 1.8 yields

$$u_1(u_i(a, x_0), u_n(a, x_0)) = u_{i+1}(a, x_0) ,$$

thus, by (3),

(4) $$f(u_i(a, b)) = u_{i+1}(a, x_0) .$$

Further we obtain (in view of 1.8 and (1))

(5) $$u_{i+1}(a, b) = u_{i+1}(u_0(a, x_0), u_j(a, x_0)) = u_{i+1}(a, x_0) .$$

Therefore (4) and (5) yield

(6) $$f(u_i(a, b)) = u_{i+1}(a, b) .$$

3.4. Lemma. *Let L and f be as in 3.2. If $B \in \mathrm{Co}(A, f)$, then $B = \{p \in A : p \leqq \bigvee_{b \in B} b\}$.*

Proof. Let $B \in \mathrm{Co}(A, f)$. Obviously, $B \subseteq \{p \in A : p \leqq \bigvee_{b \in B} b\}$. Assume that $p \leqq \bigvee_{b \in B} b$, $p \in A$. According to (ε), $p \leqq a \vee b$ for some $a, b \in B$ and we can suppose that $a \neq p \neq b$. In view of (β) we obtain that either $a \leqq b \vee x_0$ or $b \leqq a \vee x_0$. Let $b \leqq a \vee x_0$ (the second case is similar). Put $n = n(a, b)$. Then 3.3 implies

(1) $$f(u_i(a, b)) = u_{i+1}(a, b) \quad \text{for each} \quad 0 \leqq i < n .$$

Thus

$$f(a) = f(u_0(a, b)) = u_1(a, b),$$
$$f^2(a) = f(u_1(a, b)) = u_2(a, b), \dots,$$
$$f^n(a) = u_n(a, b) = b.$$

Since $(\alpha 1)$ is valid and $p \leqq a \vee b$, we obtain

$$p \in \{u_0(a, b), u_1(a, b), \dots, u_n(a, b)\} = \{a, f(a), \dots, f^n(a) = b\} = a \vee^{\mathrm{Co}} b \subseteq B.$$

3.5. Lemma. *Let L and f be as in 3.2. If $u \in L$, then $\{p \in A: p \leqq u\} \in \mathrm{Co}(A, f)$.*

Proof. Let $u \in L$. Put $B = \{p \in A: p \leqq u\}$. Assume that $a, b \in B$, $b = f^n(a)$, $c = f^i(a)$, $x_0 = f^m(a) \neq f^{m-1}(a)$, where $0 < i < n \leqq m$. We shall prove that $c \in B$. Since $(A, f) \in \mathscr{V}_0$ ($f(x_0) = x_0$ in view of 3.2), 1.9 yields

(1) $$n(a, x_0) = m, n(a, b) = n,$$

(2) $$u_i(a, x_0) = f^i(a) = u_i(a, b).$$

Further, $(\alpha 1)$ implies

(3) $$\{p \in A: p \leqq a \vee b\} = \{u_0(a, b), u_1(a, b), \dots, u_n(a, b)\}.$$

We have $0 < i < n$, thus (2) and (3) imply

$$c \in \{p \in A: p \leqq a \vee b\} \subseteq \{p \in A: p \leqq u\} = B.$$

Therefore $B \in \mathrm{Co}(A, f)$.

3.6. Corollary. *Let L and f be as in 3.2. For $B \in \mathrm{Co}(A, f)$ put $\varphi(B) = \bigvee_{b \in B} b$. Then φ is a lattice isomorphism of $\mathrm{Co}(A, f)$ onto L and $(A, f) \in \mathscr{V}_1$.*

Proof. According to 3.2, $(A, f) \in \mathscr{V}_1$. Further, 3.4, 3.5 and 1.6 imply that φ is a lattice isomorphism of $\mathrm{Co}(A, f)$ onto L.

3.7. Theorem. *Let L be a lattice. Then $L \cong \mathrm{Co}(A, f)$ for some $(A, f) \in \mathscr{V}_1$ if and only if L satisfies the condition (β).*

Proof. The assertion follows from 3.1 and 3.6.

4. THE CLASS $\mathscr{V}_2$

In this section we shall characterize the lattices L satisfying the relation $L \cong \mathrm{Co}(A, f)$ for some $(A, f) \in \mathscr{V}_2$.

$(\gamma 1)$ **Condition.** *L satisfies the condition (ε) and there is a finite set $C \subseteq A$ with* card $C > 1$ *such that*

(i) $\bigvee_{c \in C} c$ *covers c_1 for each $c_1 \in C$;*

(ii) *for each $x \in A$ there is a uniquely determined $c(x) \in C$ with $x \vee c(x) \ngeqq \bigvee_{c \in C} c$;*

(iii) *if* $x \in A - C$, $y \in C$, $p \in A$, $p \leqq x \vee y$, $p \nleqq y \vee c(x)$, *then* $p \leqq x \vee c(x)$ *and* $c(x) \leqq x \vee y$;

(iv) *if* $x, y \in A - C$, $c(x) \neq c(y)$, *then* $\{p \in A: p \leqq x \vee y\} = \{x, y\}$.

4.1.1. Remark. The condition (i) in $(\gamma 1)$ is equivalent to the condition

(i') $c_1 \vee c_2 = \bigvee_{c \in C} c$ for each $c_1, c_2 \in C$, $c_1 \neq c_2$.

4.1.2. Remark. If $(\alpha 1)$ is valid, we shall always take a fixed set C with the property as in $(\gamma 1)$.

4.2. Notation. Let L satisfy the condition $(\gamma 1)$. If $c \in C$, we shall denote

$$X(c) = \{x \in A: c(x) = c\},$$
$$L(c) = \{\bigvee_{b \in B} b: B \subseteq X(c)\}.$$

From $(\gamma 1)$ (ii) we conclude

$$A = \bigcup_{c \in C} X(c), \quad X(c_1) \cap X(c_2) = \emptyset \quad \text{for} \quad c_1, c_2 \in C, \quad c_1 \neq c_2.$$

4.3. Lemma. *If* $(\gamma 1)$ *is valid, then* $\{p \in A: p \leqq \bigvee_{c \in C} c\} = C$.

Proof. It is obvious that $C \subseteq \{p \in A: p \leqq \bigvee_{c \in C} c\}$. Let $p \in A$, $p \leqq \bigvee_{c \in C} c$ and suppose that $p \notin C$. In view of $(\gamma 1)$ (ii) there is $c(p) \in C$ with

(1) $$p \vee c(p) \ngeqq \bigvee_{c \in C} c.$$

Further, card $C > 1$ and 4.1.1 yields that there is $a \in C$ with

(2) $$a \vee c(p) = \bigvee_{c \in C} c.$$

Then $p \leqq a \vee c(p)$, hence

(3) $$p \vee c(p) \leqq a \vee c(p).$$

Combining (1), (2) and (3) we obtain

(4) $$a \vee c(p) \nleqq p \vee c(p) \leqq a \vee c(p).$$

Then $p \vee c(p) \neq a \vee c(p)$ and $p \vee c(p) < a \vee c(p)$. Therefore

$$c(p) < p \vee c(p) < a \vee c(p) = \bigvee_{c \in C} c,$$

which is a contradiction to $(\gamma 1)$ (i) ($c(p)$ is not covered by $\bigvee_{c \in C} c$).

(γ) **Condition.** *The condition* $(\gamma 1)$ *is satisfied. If* $c \in C$, *then the pair* $(L(c), c)$ *satisfies the condition* (β^0).

4.4. Lemma. *If* $(A, f) \in \mathscr{V}_2$, *then* $L = \mathrm{Co}(A, f)$ *satisfies the condition* (γ).

Proof. The condition (ε) is satisfied by 1.4. Let C be the cycle of (A, f). If $x \in A$, then there exists a least non-negative integer k such that $f^k(x) \in C$. Put $c(x) = f^k(x)$. It is routine to verify that the conditions (i)–(iv) of $(\gamma 1)$ are valid. Let $c \in C$. Then

$$X(c) = \{x \in A: c(x) = c\} = \{c\} \cup \{x \in A - C: f^k(x) = c,\ f^{k-1}(x) \notin C,\ k \in N\},$$
$$L(c) = \{\bigvee^{\mathrm{Co}}_{b \in B} b: B \subseteq X(c)\}.$$

It is obvious that $L(c)$ is a sublattice of $\mathrm{Co}(A,f)$. Further, put

$$g(x) = \begin{cases} x\,, & \text{if} \quad x = c \\ f(x)\,, & \text{if} \quad x \in X(c) - \{c\}\,. \end{cases}$$

Then

$$L(c) = \mathrm{Co}(X(c), g)\,. \tag{1}$$

Since $(X(c), g) \in \mathscr{V}_1$, $g(c) = c$, 3.1 implies that the pair $(\mathrm{Co}(X(c), g), c)$ satisfies the condition (β^0).

4.5. Lemma. *Let L satisfy the condition (γ), $c \in C$. Then there is $(X(c), g_c) \in \mathscr{V}_1$ with $g_c(c) = c$, such that φ_c: $\mathrm{Co}(X(c), g_c) \to L(c)$, where $\varphi_c(B) = \bigvee_{b\in B} b$ for each $B \in \mathrm{Co}(X(c), g_c)$, is a lattice isomorphism.*

Proof. The assertion follows immediately from 3.6 and from the fact that the set of all atoms of $L(c)$ is $X(c)$.

4.6. Notation. Let L satisfy (γ) and suppose that for each $c \in C$, $(X(c), g_c) \in \mathscr{V}_1$ is as in 4.5. If $x \in A - C$, then $x \in X(c(x))$; put

$$f(x) = g_{c(x)}(x)\,.$$

Further, let f on C be such that C is a cycle of (A,f).

4.7. Lemma. *If L satisfies (γ) and f is as in 4.6, then $(A,f) \in \mathscr{V}_2$.*

Proof. The assertion follows from 4.6 and 4.5.

4.8. Lemma. *Let L and f be as in 4.6. If $B \in \mathrm{Co}(A,f)$, then $B = \{p \in A\colon p \leqq \bigvee_{b\in B} b\}$.*

Proof. Let $B \in \mathrm{Co}(A,f)$. Obviously, $B \subseteq \{p \in A\colon p \leqq \bigvee_{b\in B} b\}$. Assume that $p \in A - B$, $p \leqq \bigvee_{b\in B} b$. Since (ε) is valid, $p \leqq x \vee y$ for some $x, y \in B$. We can suppose that x, y, p are distinct. If $x, y \in C$, then $C \subseteq B$ and $p \leqq \bigvee_{c\in C} c$, thus 4.3 implies that $p \in C \subseteq B$. If $x, y \notin C$, $c(x) \neq c(y)$, then $(\gamma 1)$ (iv) yields that $\{t \in A\colon t \leqq x \vee y\} = \{x, y\}$, hence $p \in \{x, y\} \subseteq B$. Let $x \notin C$, $c(x) = c(y) = d$. Let $\vee^d$ and $\vee^{d\mathrm{Co}}$ be the lattice operations in $L(d)$ and in $\mathrm{Co}(X(d), g_d)$, respectively. Then we obtain (since $x, y \in X(d)$)

$$p \leqq x \vee^d y\,,$$

and by 4.5

$$p \leqq x \vee^{d\mathrm{Co}} y\,.$$

From 4.6 it follows that then $p \leqq x \vee^{\mathrm{Co}} y$. Since $x \vee^{\mathrm{Co}} y \subseteq B$, we obtain that $p \in B$. Now suppose that $y \in C, x \notin C$, $c(x) = a \neq y$. Then $x \vee a \ngeqq \bigvee_{c\in C} c$, $x \vee y \geqq \bigvee_{c\in C} c = a \vee y$ (according to $(\gamma 1)$ (ii) and 4.1.1) and $C \subseteq x \vee^{\mathrm{Co}} y \subseteq B$. Since $p \notin B$, $p \notin C$, thus $p \nleqq a \vee y$ in view of 4.3. According to $(\gamma 1)$ (iii) we have $p \leqq x \vee a$, hence $x, a \in X(a)$ and this case was already investigated.

4.9. Lemma. *Let L and f be as in 4.6. If $u \in L$, then $\{p \in A\colon p \leqq u\} \in \mathrm{Co}(A,f)$.*

Proof. Let $u \in L$. Put $B = \{p \in A: p \leqq u\}$. Assume that $x, y \in B$, $z \in A$, $n, m \in N$, $m < n$, $y = f^n(x) \neq f^k(x)$ for each $k \in N$, $k < n$, $z = f^m(x) \neq f^k(x)$ for each $k \in N$, $k < m$. If $x, y \in X(d)$ for some $d \in C$, then

$$z \in x \vee^{d\mathrm{Co}} y,$$

$$z \leqq x \vee^d y$$

(by 4.5) and thus (γ) yields $z \leqq x \vee y \leqq u$, i.e., $z \in B$. Now let $a = c(x) \neq c(y)$. From the assumption $y = f^n(x)$ it follows that then $y \in C$. Further, $c(x) = f^l(x)$ for some $l < n$. According to $(\gamma 1)$ (iii) we have $a \leqq x \vee y \leqq u$, therefore $a \in B$. If $m \leqq l$, then $x, a, z \in X(a)$, $z \in a \vee^{d\mathrm{Co}} x$ and 4.5 implies that $z \leqq a \vee^d x$, thus (γ) yields that $z \leqq a \vee x \leqq u$, i.e. $z \in B$. If $m > l$, then $z \in C$, $z \leqq a \vee y \leqq u$, i.e., $z \in B$ as well.

4.10. Lemma. *Let L and f be as in 4.6. For* $B \in \mathrm{Co}(A,f)$ *put* $\varphi(B) = \bigvee_{b \in B} b$. *Then* φ *is a lattice isomorphism of* $\mathrm{Co}(A,f)$ *onto L and* $(A,f) \in \mathscr{V}_2$.

Proof. The assertion follows from 4.7, 4.8, 4.9 and 1.6.

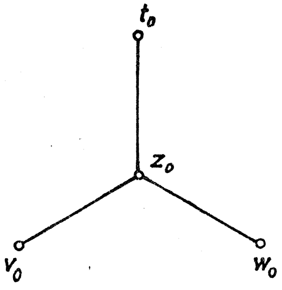
4.11. Theorem. *Let L be a lattice. Then* $L \cong \mathrm{Co}(A,f)$ *for some* $(A,f) \in \mathscr{V}_2$ *if and only if L satisfies the condition* (γ).

Proof. The assertion follows from 4.4 and 4.10.

5. THE CLASS $\mathscr{V}_3$

In this section we shall investigate the question when a lattice L is isomorphic to $\mathrm{Co}(A,f)$ for some $(A,f) \in \mathscr{V}_3$.

Let $(P_0, \preceq)$ be the poset drawn in the following figure (v_0, w_0, z_0, t_0 are distinct elements):



The lattice $\mathrm{Co}(P_0, \preceq)$ will be denoted by the symbol L_0.

$(\delta 1)$ **Condition.** *L satisfies the condition* $(\alpha 1)$ *and there are* $v, w, z, t \in A$ *such that the sublattice of L generated by* v, w, z, t *is isomorphic to* L_0 *under an isomorphism such that* $v \to v_0$, $w \to w_0$, $z \to z_0$, $t \to t_0$.

Remark. If L is lattice satisfying $(\delta 1)$, we shall always take fixed elements v, w, z, t as in $(\delta 1)$.

$(\delta 2)$ **Condition.** *Let* $(\delta 1)$ *hold. For each* $x \in A$ *there is* $y \in A$ *with* $\operatorname{card}\{p \in A: p \leqq x \vee y\} > 2$, $\operatorname{card}\{p \in A: p \leqq z \vee y\} > 2$, $t \leqq z \vee y$, $y \nleqq z \vee x$.

Notation. Let L be a lattice, $x \in A$. If $y \in A$ is such that the condition from $(\delta 2)$ is valid, then we shall say that y is convenient to x.

$(\delta 3)$ **Condition.** *Let* $(\delta 2)$ *hold. If* $x, y, y' \in A$, y *and* y' *are convenient to* x, *then either* $y \leqq x \vee y'$ *or* $y' \leqq x \vee y$. *If* $x, x', y \in A$, y *is convenient to* x, $x \leqq x' \vee y$, *then* y *is convenient to* x' *as well.*

5.1. Lemma. *Let* L *satisfy* $(\delta 3)$. *If* $y, y' \in A$ *are convenient to* $x \in A$, *then* $u_1(x, y) = u_1(x, y')$.

Proof. Let $y, y' \in A$ be convenient to $x \in A$. In view of $(\delta 3)$, either $y \leqq x \vee y'$ or $y' \leqq x \vee y$; we can suppose that $y' \leqq x \vee y$. Since $(\alpha 1)$ is valid, $y' = u_i(x, y)$ for some $i \in \{0, 1, \ldots, n(x, y)\}$. According to $(\delta 2)$ we obtain

$$2 < \operatorname{card}\{p \in A: p \leqq x \vee y'\} = \operatorname{card}\{p \in A: p \leqq u_0(x, y) \vee u_i(x, y)\} = i + 1\,,$$

i.e., $i > 1$. We have $0 \leqq 1 < i \leqq n(x, y)$, and 1.8 implies

$$u_1(x, y') = u_1(u_0(x, y), u_i(x, y)) = u_{1+0}(x, y) = u_1(x, y)\,.$$

5.2. Remark. In view of 5.1, if $(\delta 3)$ holds, then instead of $u_1(x, y)$ (for y convenient to x) we shall write $u_1(x, y) = u(x)$. Further put $u^0(x) = x$. By induction we define $u^k(x) = u(u^{k-1}(x))$ for each $k \in N$, $k > 1$.

5.3. Lemma. *Let* $(A, f) \in \mathscr{V}_3$, $L = \operatorname{Co}(A, f)$. *Then* L *satisfies the condition* $(\delta 1)$.

Proof. According to 1.9, L satisfies $(\alpha 1)$. Since $(A, f) \in \mathscr{V}_3$, there are distinct elements $v, w, z, t \in A$ such that

(1) $$f(v) = f(w) = z\,, \quad f(z) = t\,, \quad w \neq f(t) \neq v\,.$$

Now it can be easily shown that $(\delta 1)$ is valid, e.g., $v \vee^{\mathrm{Co}} t = \{v, z, t\}$, etc.

5.4. Lemma. *If* (A, f) *is a monounary algebra such that* $L = \operatorname{Co}(A, f)$ *satisfies* $(\delta 1)$, *then*

(1) $$f(v) = f(w) = z\,, \quad f(z) = t\,, \quad w \neq f(t) \neq v\,.$$

Proof. The assumption implies

(2) $$z \vee^{\mathrm{Co}} v = \{z, v\}\,, \quad z \vee^{\mathrm{Co}} w = \{z, w\}\,, \quad z \vee^{\mathrm{Co}} t = \{z, t\}\,,$$
$$t \vee^{\mathrm{Co}} v = \{t, z, v\}\,, \quad t \vee^{\mathrm{Co}} w = \{t, z, w\}\,, \quad v \vee^{\mathrm{Co}} w = \{v, w\}\,.$$

From the relation for $t \vee^{\mathrm{Co}} v$ we conclude that one of the following condition is satisfied:

(3.1) t, z, v form a 3-element cycle , (3.2) $$f(t) = z\,, \quad f(z) = v\,, \quad f(v) \neq t\,,$$

(3.3) $$f(v) = z\,, \quad f(z) = t\,, \quad f(t) \neq v\,.$$

From the relation for $z \vee^{\mathrm{Co}} v$ we get that (3.1) fails to hold. Since analogous conditions can be obtained if we use the relation for $t \vee^{\mathrm{Co}} w$, we infer that (1) is valid.

5.5. Lemma. *Let (A,f) be a monounary algebra and let $L = \mathrm{Co}(A,f)$ satisfy $(\delta 1)$. Suppose that $x \in A$ and that there is $y \in A$ such that y is convenient to x. Then $y \in \{f^k(x): k \in N,\ k > 1\}$.*

Proof. Assume that $y \notin \{f^k(x): k \in N,\ k > 1\}$. Since $\mathrm{card}\,(x \vee^{\mathrm{Co}} y) > 2$, we get that $x = f^i(y)$ for some $i \in N$, $i > 1$. According to 5.4 we have $f(z) = t$, and then the relation $t \in z \vee^{\mathrm{Co}} y$ implies that there is $j \in N$ with $f^j(t) = y$. Thus $f^{j+1}(z) = y$ and $y \in z \vee^{\mathrm{Co}} x$, which is a contradiction, since y was convenient to x.

5.6. Lemma. *Let $(A,f) \in \mathscr{V}_3$, $L = \mathrm{Co}(A,f)$. Then L satisfies $(\delta 3)$ and $u(x) = f(x)$ for each $x \in A$.*

Proof. According to 5.3, L satisfies $(\delta 1)$. We have (by 5.4)

(1) $$f(v) = f(w) = z\,, \quad f(z) = t\,, \quad v \neq f(t) \neq w\,.$$

Let $x \in A$. Since (A,f) is connected and possesses no cycle, there are $m, n \in N$, $y \in A$ with $m > 1$, $n > 1$, $y = f^m(x) = f^n(z)$. Then

$$\mathrm{card}\,(x \vee^{\mathrm{Co}} y) = m + 1 > 2\,, \quad \mathrm{card}\,(z \vee^{\mathrm{Co}} y) = n + 1 > 2\,,$$
$$t = f(z) \in z \vee^{\mathrm{Co}} y\,, \quad y \notin z \vee^{\mathrm{Co}} x\,,$$

i.e., y is convenient to x. Thus L satisfies $(\delta 2)$.

If $y' \in A$ is also convenient to x, 5.5 implies that $y' = f^k(x)$ for some $k \in N$, $k > 1$. Then either $k \leqq m$ and $y' \in x \vee^{\mathrm{Co}} y$, or $m < k$ and $y \in x \vee^{\mathrm{Co}} y'$.

Let $x' \in A$, $x \in x' \vee^{\mathrm{Co}} y'$, where $y' \in A$ is convenient to x. Then $x = f^l(x')$, $y' = f^j(x)$ for some $l \in N \cup \{0\}$, $j \in N$, $j > 1$ (by 5.5), and it is obvious that y' is convenient to x' as well. Hence L satisfies the condition $(\delta 3)$.

In view of 5.1, $u_1(x, y')$ does not depend on the choice of y' (convenient to x), hence

$$u(x) = u_1(x, y) = u_1(x, f^m(x)) = f(x)$$

according to 1.9.

$(\delta 4)$ **Condition.** *Let L satisfy $(\delta 3)$. If $x, y, x' \in A$, y is convenient to x and $\mathrm{card}\,\{p \in A: p \leqq x \vee x'\} > 2$, then either $x \leqq x' \vee y$, or $x' \leqq x \vee y$, or $y \leqq x \vee x'$.*

(δ) **Condition.** *Let $(\delta 4)$ hold. If $x, a, y \in A$, y is convenient to x and either $x \neq a \leqq x \vee y$ or $y \leqq x \vee a$, then $u_i(x, a) = u^i(x)$ for each $i \in \{0, 1, \ldots, n(x, a)\}$.*

5.7. Lemma. *Let $(A,f) \in \mathscr{V}_3$, $L = \mathrm{Co}(A,f)$. Then L satisfies the condition $(\delta 4)$.*

Proof. According to 5.6, L satisfies $(\delta 3)$. Let $x, y, x' \in A$, let y be convenient to x. In view of 5.5 we have that $y = f^k(x)$ for some $k \in N$, $k > 1$. Suppose that

(1) $$x \notin x' \vee^{Co} y\,, \quad x' \notin x \vee^{Co} y\,, \quad y \notin x \vee^{Co} x'$$

holds. Then $x' \notin \{f^i(x): i \in N \cup \{0\}\}$, $x \notin \{f^i(x'): i \in N\}$, which implies

$$\operatorname{card}(x \vee^{Co} x') = \operatorname{card}\{x, x'\} = 2\,.$$

5.8. Lemma. *Let $(A, f) \in \mathscr{V}_3$, $L = \mathrm{Co}(A, f)$. Then L satisfies the condition (δ).*

Proof. In view of 5.7, L satisfies $(\delta 4)$. Let $x, y, a \in A$, $x \neq a$, let y be convenient to x and suppose that either $a \in x \vee^{Co} y$ or $y \in x \vee^{Co} a$. By virtue of 5.5 we obtain that $y \in \{f^k(x): k \in N, k > 1\}$ and then $a \in \{f^k(x): k \in N\}$. Let $a = f^k(x)$, $k \in N$.

Then 1.9 implies that $n(x, a) = k$, $u_i(x, a) = f^i(x)$ for each $i \in \{0, 1, \dots, k\}$, therefore we get (in view of 5.6) that $u_i(x, a) = f^i(x) = u^i(x)$ for each $i \in \{0, 1, \dots \dots, n(x, a)\}$.

5.9. Notation. Let L be a lattice satisfying the condition (δ). Put $f(x) = u(x)$ for each $x \in A$.

5.10. Lemma. *Let L, (A, f) be as in 5.9. If $x, y \in A$, y is convenient to x, then $y \in \{f^k(x): k \in N\} \cap \{f^k(z): k \in N\}$.*

Proof. Assume that $x, y \in A$, y is convenient to x. Then $y \neq x$. Since $y \leqq x \vee y$, (δ) yields

(1) $$u_i(x, y) = u^i(x) = f^i(x) \quad \text{for each} \quad i \in \{0, 1, \dots, n(x, y)\}\,.$$

Further, $y = u_{n(x,y)}(x, y)$, hence (1) implies that $y \in \{f^k(x): k \in N\}$. The assertion that $y \in \{f^k(z): k \in N\}$ can be proved analogously, since y is convenient also to z.

5.11. Corollary. *If L, (A, f) are as in 5.9, then (A, f) is a connected monounary algebra.*

5.12. Lemma. *If (A, f) is a connected monounary algebra possessing a cycle C with $\operatorname{card} C > 2$, then $L = \mathrm{Co}(A, f)$ does not satisfy $(\alpha 1)$.*

Proof. Let $x, y \in C$, $x \neq y$. Then $x \vee^{Co} y = C$. Assume that $(\alpha 1)$ is valid. Then there are distinct elements $x = u_0(x, y), u_1(x, y), \dots, u_{n(x,y)}(x, y) = y$ with

$$C = x \vee^{Co} y = \{u_0(x, y), u_1(x, y), \dots, u_{n(x,y)}(x, y)\}\,.$$

Thus $u_0(x, y) \neq u_1(x, y) \neq u_{n(x,y)}(x, y)$ and

(1) $$u_{n(x,y)} = y \in C = u_0(x, y) \vee^{Co} u_1(x, y)\,,$$

a contradiction to $(\alpha 1)$.

5.13. Lemma. *If (A, f) is a connected monounary algebra such that $(A, f) \notin \mathscr{V}_3$, then $L = \mathrm{Co}(A, f)$ does not satisfy the condition $(\delta 2)$.*

Proof. Let (A, f) be a connected monounary algebra. According to (R) in Section 1, there is $(A, g) \in \mathscr{V}_0 \cup \mathscr{V}_1 \cup \mathscr{V}_2 \cup \mathscr{V}_3$ such that $\mathrm{Co}(A, f) = \mathrm{Co}(A, g)$.

Therefore we can suppose that $(A,f)\in\mathscr{V}_0\cup\mathscr{V}_1\cup\mathscr{V}_2\cup\mathscr{V}_3$. If $(A,f)\in\mathscr{V}_0$, then L does not satisfy $(\delta 1)$ in view of 5.4. If (A,f) contains a cycle C with card $C>2$, then L does not satisfy $(\alpha 1)$ by 5.12, hence L does not satisfy $(\delta 2)$. Assume that (A,f) satisfies $(\delta 2)$ and that there is a cycle C with card $C\leqq 2$.

Let $x\in C$. There is $y\in A$ convenient to x. From 5.5 it follows that $y\in\{f^k(x): k\in N, k>1\}$, thus $y\in C$. Then

$$\operatorname{card}(x\vee^{\mathrm{Co}} y)\leqq \operatorname{card} C\leqq 2\,,$$

which is a contradiction to the fact that y is convenient to x.

5.14. Corollary. *Let* L, (A,f) *be as in* 5.9. *Then* $(A,f)\in\mathscr{V}_3$.

Proof. The assertion follows from 5.11 and 5.13.

5.15. Lemma. *Let* L, (A,f) *be as in* 5.9. *Then*

(i) $B=\{p\in A: p\leqq\bigvee_{b\in B} b\}$ *for each* $B\in\mathrm{Co}(A,f)$.

Proof. Let $B\in\mathrm{Co}(A,f)$, $p\in A$, $p\leqq\bigvee_{b\in B} b$. Since L satisfies the condition (ε), there are $x, x'\in B$ with $p\leqq x\vee x'$. We will show that $p\in B$; thus assume that $p\notin\{x,x'\}$. According to $(\alpha 1)$, $p=u_k(x,x')$ for some $k\in\{0,\dots,n(x,x')\}$. Let y be convenient to x (such an element does exist in view of $(\delta 2)$). Then $(\delta 4)$ yields that some of the following conditions is valid:

(1.1) $$x\leqq x'\vee y\,,$$

(1.2) $$x'\leqq x\vee y\,,$$

(1.3) $$y\leqq x\vee x'\,.$$

Denote $n=n(x,x')$. If (1.2) or (1.3) holds, then (δ) implies

(2) $$u_i(x,x')=u^i(x)=f^i(x)\quad\text{for each}\quad i\in\{0,\dots,n\}\,,$$

hence

(3) $$x'=u_n(x,x')=f^n(x)\,,$$

(4) $$p=u_k(x,x')=f^k(x)\,,\quad 0<k<n\,.$$

Therefore (3) and (4) yield that $p\in x\vee^{\mathrm{Co}} x'\subseteq B$.

Now let (1.1) hold. Then $p=u_{n-k}(x',x)$ (by Lemma 1.7). From $(\delta 3)$ it follows that y is convenient to x' and (δ) and (1.1) imply

(5) $$u_j(x',x)=u^j(x')=f^j(x')\quad\text{for each}\quad j\in\{0,1,\dots,n\}\,.$$

Analogously as above, $p=f^{n-k}(x')$, $x=f^n(x')$ and therefore $p\in x\vee^{\mathrm{Co}} x'\subseteq B$.

5.16. Lemma. *Let* L, (A,f) *be as in* 5.9.

(ii) *If* $u\in L$, *then* $\{p\in A: p\leqq u\}\in\mathrm{Co}(A,f)$.

Proof. Let $u\in L$ and $B=\{p\in A: p\leqq u\}$. Assume that $x, x'\in B$, $x'=f^j(x)$, $p=f^k(x)$, where $0<k<j$. By $(\delta 2)$, there exists $y\in A$ which is convenient to x.

Then (δ) and 5.9 imply

$$(1) \qquad u_j(x, y) = u^j(x) = f^j(x) = x' ,$$

$$(2) \qquad u_k(x, y) = u^k(x) = f^k(x) = p .$$

From 1.8 we obtain

$$p = u_k(x, y) = u_{0+k}(x, y) = u_k(u_0(x, y), u_j(x, y)) = u_k(x, x') ,$$

therefore $p \leqq x \vee x' \leqq u$, i.e., $p \in B$ and hence $B \in \mathrm{Co}(A, f)$.

5.17. Lemma. *Let L, (A, f) be as in 5.9. Then the mapping φ such that $\varphi(B) = \bigvee_{b \in B} b$ for each $B \in \mathrm{Co}(A, f)$ is an isomorphism of $\mathrm{Co}(A, f)$ onto L.*

Proof. The assertion follows from 1.6, 5.15 and 5.16.

5.18. Theorem. *Let L be a lattice. Then $L \cong \mathrm{Co}(A, f)$ for some $(A, f) \in \mathscr{V}_3$ if and only if L satisfies the condition (δ).*

Proof. This follows from 5.8, 5.17 and 5.14.

6. THE GENERAL CASE

This short section will contain the main result of the present paper.

6. Theorem. *A lattice L is isomorphic to $\mathrm{Co}(A, f)$ for some (partial) monounary algebra (A, f) if and only if $L \cong \prod_{i \in I} L_i$, where each L_i (for $i \in I$) satisfies one of the conditions (α), (β), (γ), (δ).*

Proof. Let $L = \mathrm{Co}(A, f)$ and let $\{A_i\}_{i \in I}$ be the system of all connected components of (A, f). Put $L_i = \mathrm{Co}(A_i, f)$ for each $i \in I$. Then

$$(1) \qquad L \cong \prod_{i \in I} L_i .$$

Further, (R) of the first section yields that if $i \in I$, then $L_i = \mathrm{Co}(A_i, g_i)$ for some $(A_i, g_i) \in \mathscr{V}_0 \cup \mathscr{V}_1 \cup \mathscr{V}_2 \cup \mathscr{V}_3$. By 2.13, 3.7, 4.11 and 5.18 we obtain that L_i satisfies one of the conditions (α), (β), (γ), (δ).

Let us prove the converse implication. By 2.13, 3.7, 4.11 and 5.18, for each $i \in I$ there is $(A_i, f_i) \in \mathscr{V}_0 \cup \mathscr{V}_1 \cup \mathscr{V}_2 \cup \mathscr{V}_3$ such that

$$(2) \qquad L_i \cong \mathrm{Co}(A_i, f_i) .$$

Denote $(A, f) = \sum_{i \in I} (A_i, f_i)$. Then (2) implies

$$(3) \qquad \mathrm{Co}(A, f) \cong \prod_{i \in I} \mathrm{Co}(A_i, f_i) \cong \prod_{i \in I} L_i \cong L .$$

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