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ON COMPLETELY MEET-IRREDUCIBLE ELEMENTS IN COMPACTLY GENERATED LATTICES

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Definition. An element m in a lattice is said to be *relatively maximal with respect to an element* x when $x \nleq m$, and $m < z$ implies $x \leqq z$.

An element in a lattice is said to be *relatively maximal* when it is relatively maximal with respect to some element of the lattice.

Lemma. *In any lattice,* $\pi(a) = \bigwedge(\langle a) \setminus \{a\})$ *exists for every element* a, *and either* $\pi(a) = a$ *or* $a \prec \pi(a)$.

Proof is obvious.

This lemma makes the following definition legitimate:

Definition. An element a of a lattice is said to be *completely* (alias *strictly*) *meet-irreducible* if $a \prec \pi(a)$.

Proposition. *Relatively maximal elements in a lattice coincide with completely meet-irreducible elements in the lattice.*

Proof. Suppose m is a relatively maximal element, say with respect to an element x. Then $x \leqq \pi(m)$, and consequently $m \neq \pi(m)$. By the lemma, we obtain $m \prec \pi(m)$.

To prove the converse, suppose $a \prec \pi(a)$. It is easy to see that the element a is relatively maximal with respect to $\pi(a)$. Q.E.D.

Remark. In view of the preceding proposition, we may reformulate a well-known theorem:

In a compactly generated (alias algebraic) lattice, every element is representable as a meet of a set of relatively maximal elements.

(Cf. [1].)

The set of all completely meet-irreducible (i.e. relatively maximal) elements in a lattice L will be denoted by $Rm(L)$.

Theorem. *Let* L *be a compactly generated lattice. Denote* $r(a) = \{x \in Rm(L) \mid a \leqq x\}$. *Then the following conditions are equivalent:*

(i) L *is distributive,*

(ii) *r is an embedding of the lattice L into the dual of* $\mathbf{2}^{Rm(L)}$,

(iii) $(\forall a, b \in L)\ r(a \wedge b) = r(a) \cup r(b)$ *and* $r(a \vee b) = r(a) \cap r(b)$,

(iv) $(\forall a, b \in L)\ r(a \wedge b) \subseteq r(a) \cup r(b)$.

Proof. (i) $\Rightarrow$ (iv): Take $m \in r(a \wedge b)$. In view of distributivity, $m = m \vee \vee (a \wedge b) = (m \vee a) \wedge (m \vee b)$. Since m is meet-irreducible, $m = m \vee a$ or $m = m \vee b$. Hence $m \in r(a)$ or $m \in r(b)$, and consequently $m \in r(a) \cup r(b)$.

(iv) $\Rightarrow$ (iii): Inclusions $r(a) \cup r(b) \subseteq r(a \wedge b)$ and $r(a \vee b) = r(a) \cap r(b)$ follow immediately from antitony of the operator r.

(iii) $\Rightarrow$ (ii): It remains to prove that r is injective. Suppose $r(a) = r(b)$. Inasmuch as $a = \bigwedge r(a)$ and $b = \bigwedge r(b)$, we obtain $a = b$.

(ii) $\Rightarrow$ (i): The lattice L is isomorphic to a sublattice of a Boolean lattice, and therefore distributive. Q.E.D.

Corollary. *A mapping sending each element a of a distributive lattice L to the complement of $r(a)$ is an embedding of L into* $\mathbf{2}^{Rm(L)}$.

Remark. In a distributive lattice, it is clear that all principal ideals with completely meet-irreducible top elements are prime. However, not every prime ideal in a compactly generated distributive lattice is principal. Hence we have obtained a generalization of the Birkhoff theorem for finite distributive lattices to distributive compactly generated lattices, distinct from the general case.

Reference

[1] *G. Birkhoff:* Lattice Theory. 3d ed. Amer. Math. Soc., Providence 1979.

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