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$B(X)$ IS GENERATED IN STRONG OPERATOR TOPOLOGY BY TWO OF ITS ELEMENTS

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Let X be a real or complex Banach space and τ a topology on $B(X)$. We say that $B(X)$ is *τ-generated* by its subset G if it coincides with the smallest τ-closed subalgebra of $B(X)$ containing G. In particular we say that $B(X)$ is *strongly generated* by G if τ is the strong operator topology. Similarly we say that $B(X)$ is generated by G if τ is the norm topology and $B(X)$ is algebraically generated by G if τ is the discrete topology.

As a motivation for the result presented here we mention a similar theorem known for X being a separable Hilbert space (see [1], [2], cf also [3] and the references therein) and the results given in [6]–[8]. In [6] it is shown that for any Hilbert space H there are two subalgebras $A_1, A_2 \subset B(H)$ with square zero and therefore commutative such that the set $G = A_1 \cup A_2$ algebraically generates $B(H)$. In [7] this result is extended to those Banach spaces X which are „n-th powers", $n > 1$, i.e. which can be decomposed into direct sums $X = X_1 \oplus \ldots \oplus X_n$ of closed subspaces X_i which are mutually isomorphic. In [8] it is shown that for every Banach space X the algebra $B(X)$ is strongly generated by two commutative subalgebras (in fact by two subalgebras of square zero) if $\dim X > 1$. Our present result improves the last one for separable Banach space so that instead of commutativity we have a single generation. Our proof is based upon the following result due to Ovsepian and Pełczyński ([4], Theorem 1) on the existence of total bounded biorthogonal systems in separable Banach spaces:

Theorem [O–P]. *Let x be a separable Banach space. Then there is a sequence (x_i) of elements in X and a sequence (f_i) of functionals in X such that*

(1) $f_m(x_n) = \delta_{m,n}$ *(the Kronecker symbol) for* $m, n = 1, 2, \ldots$.

(2) *The linear span of (x_i) is dense in X in the norm topology.*

(3) *If $f_n(x) = 0$ for all n then $x = 0$.*

(4) $\sup_n \|x_n\| \|f_n\| = M < \infty$.

By [5] we can assume $M < 1 + \varepsilon$ for a given positive ε. In what follows we shall assume that the sequences (x_i) and (f_i) satisfying (1)–(4) are normalized so that they satisfy

(5) $$\|x_i\| = 1 \quad \text{and} \quad \|f_i\| \leqq M \quad \text{for all} \quad i\,.$$

We can now formulate our result.

Theorem. *Let X be a real or complex Banach space. There exist operators R and S in $B(X)$ such that $B(X)$ is strongly generated by $\{R, S\}$.*

Proof. For operators R and S in $B(X)$ denote by $\mathrm{Alg}\,(R, S)$ the subalgebra of $B(X)$ algebraically generated by R and S. It consists of linear combinations of the products of powers of R and S. We have to construct two operators R and S so that $\mathrm{Alg}(R, S)$ is strongly dense in $B(X)$. This means that for a given T in $B(X)$, $y_1, \ldots, y_n$ linearly independent elements in X and a positive ε, there is an operator B in $\mathrm{Alg}(R, S)$ such that

(6) $$\|By_i - Ty_i\| < \varepsilon \quad \text{for} \quad 1 \leqq i \leqq n\,.$$

In order to have (6) it is sufficient to find operators B_i in $\mathrm{Alg}(R, S)$, $1 \leqq i \leqq n$, such that $\|B_i y_i - Ty_i\| < \varepsilon$ and $B_i y_j = 0$ for $j \neq i$, because then (6) is satisfied by $B = \sum_{i=1}^{n} B_i$. Thus for proving strong density of $\mathrm{Alg}(R, S)$ in $B(X)$ we have to show that for given linearly independent $y_0, y_1, \ldots, y_n$ in X, positive ε and an element z in X there is an operator B in $\mathrm{Alg}(R, S)$ such that

(7) $$\|By_0 - z\| < \varepsilon \quad \text{and} \quad By_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, n\,.$$

Let (x_i) and (f_i) be a sequence of elements and functionals satisfying conditions (1)–(5) and put

(8) $$R = \sum_{i=1}^{\infty} 2^{-i}(f_i \otimes x_{i+1}) \quad \text{and} \quad S = \sum_{i=1}^{\infty} 2^{-i}(f_{i+1} \otimes x_i)\,,$$

where $f \otimes x$ is the one-dimensional operator on X given by $u \mapsto f(u)\,x$. By (5) we have $\|R\|, \|S\| \leqq M$ and so $R, S \in B(X)$.

First we show that all operators $f_m \otimes x_n$, $1 \leqq m, n < \infty$, are in $\mathrm{Alg}(R, S)$. To this end we prove the formula

(9) $$f_m \otimes x_n = 2^p R^{n-1}(4SR - RS)\, S^{m-1}\,, \quad p = \binom{m}{2} + \binom{n}{2}\,,$$
$$1 \leqq m, n < \infty\,,$$

where $\binom{k}{2}$ is 0 if $k = 1$. Using (1) we obtain immediately from (8)

(10) $$Rx_n = 2^{-n}x_{n+1} \quad \text{and} \quad Sx_n = 2^{-n+1}x_{n-1}\,, \quad n = 1, 2, \ldots\,,$$

where $x_0 = 0$. This implies

(11) $$(4SR - RS)\, x_1 = x_1 \quad \text{and} \quad (4SR - RS)\, x_m = 0 \quad \text{for} \quad m \neq 1\,.$$

Denote by $A_{m,n}$ the right-hand side operator in formula (9). We see by (10) and (11) that $A_{m,n}x_k = 0$ for $k \neq m$ and

$$A_{m,n}x_m = 2^p R^{n-1}(4SR - RS)\,2^{-m+1} \dots 2^{-1}x_1 = 2^q R^{n-1}x_1 = x_n\,,$$

$$p = \binom{m}{2} + \binom{n}{2}, \quad q = \binom{n}{2}.$$

We infer that $A_{m,n} = f_m \otimes x_n$ since both operators agree on linearly dense sequence (x_i) and formula (9) holds true.

Let $y_0, y_1, \dots, y_n$ be linearly independent elements in X, let z be in X and let ε be a positive number. We shall be done if we find an operator B in $\mathrm{Alg}(R, S)$ so that relations (7) are satisfied. To this end observe that the sequences $\{f_i(y_0)\}_{i=1}^{\infty}$, $\{f_i(y_1)\}_{i=1}^{\infty}, \dots, \{f_i(y_n)\}_{i=1}^{\infty}$ are linearly independent elements in l^{∞}, otherwise there would exist non-zero coefficients $\lambda_0, \dots, \lambda_n$ such that $0 = \sum_{j=0}^{n} \lambda_j f_i(y_j) = f_i(\sum_{j=0}^{n} \lambda_j y_j)$ for all i and by (3) we would have $\sum_{j=0}^{n} \lambda_j y_j = 0$ which is a contradiction. The linear independence of these sequences implies that there are indices $i_0, i_1, \dots, i_n$ such that the finite sequences

$$\{f_{i_k}(y_0)\}_{k=0}^{n},\ \{f_{i_k}(y_1)\}_{k=0}^{n},\ \dots,\ \{f_{i_k}(y_n)\}_{k=0}^{n}$$

are linearly independent. This means that there are coefficients $c_0, c_1, \dots, c_n$ such that

(12) $$\sum_{k=0}^{n} c_k f_{i_k}(y_0) = 1 \quad \text{and} \quad \sum_{k=0}^{n} c_k f_{i_k}(y_j) = 0\,, \quad 1 \leqq j \leqq n\,.$$

Define now $A = \sum_{k=0}^{n} c_k(f_{i_k} \otimes x_1)$.

By (9) we have $A \in \mathrm{Alg}(R, S)$ and by (12) we conclude

(13) $$Ay_0 = \sum_{k=0}^{n} c_k f_{i_k}(y_0)\, x_1 = x_1 \quad \text{and} \quad Ay_i = \sum_{k=0}^{n} c_k f_{i_k}(y_i)\, x_1 = 0\,, \quad 1 \leqq i \leqq n\,.$$

For a given positive ε and z in X we find by (2) a finite linear combination

(14) $$z_0 = \sum_{j=1}^{m} b_j x_j$$

such that

(15) $$\|z - z_0\| < \varepsilon\,.$$

Define $B = \sum_{j=1}^{m} b_j(f_1 \otimes x_j)\, A$. This is again an operator in $\mathrm{Alg}(R, S)$. By (13) and (14) we have

$$By_0 = \sum_{j=1}^{m} b_j f_1(Ay_0)\, x_j = \sum_{j=1}^{m} b_j x_j = z_0$$

and

$$By_i = \sum_{j=1}^{m} b_j f_1(Ay_i)\, x_j = 0 \quad \text{for} \quad i = 1, 2, \ldots, n\,.$$

Thus by (15) we obtain (7) and the conclusion follows.

Remarks. 1° Without assuming that X is a separable space our conclusion fails to be true. This follows from the fact that for any operators R, S in $B(X)$ and for any element x in X the orbit $0(x) = \{Tx\colon T \in \mathrm{Alg}(R, S)\}$ is separabe. Thus for non-separable X there is an element z in X such that $\mathrm{dist}\,(z, 0(x)) = 1$, and there is no operator T in $\mathrm{Alg}(R, S)$ satisfying $\|Tx - (f \otimes z)\,x\| < 1$ if $f(x) = 1$.

2° It is still an open problem whether $B(X)$ can be separable for an infinite-dimensional Banach space (for this information the authors are indebted to Tadeusz Figiel), and for most familiar Banach spaces the algebra $B(X)$ is known to be non-separable. Thus for many cases (perhaps for all cases) our theorem cannot be improved by replacing strong generation by generation. In particular the algebra $B(H)$ cannot be generated by two operators if H is an infinite-dimensional Hilbert space.

3° In our theorem we can replace strong generation by weak generation (genera tion in the weak operator topology). This follows from the fact that for linear subspaces of $B(X)$ their closures in strong and in weak operator topologies coincide.

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