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NEW APPLICATIONS OF PTÁK'S EXTENSION THEOREM TO WEAK COMPACTNESS

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Our purpose is to show how Pták's double sequence method in weak compactness, developed in $[4, 5]$ (see also $[2, \S\, 24.6]$), and his Extension Theorem $[5,$ Thm. $(2,2)]$ can be used to give some new Eberlein- and Krein-type Theorems, extending also some of Valdivia's results $[6,$ Thms. 3 and 11$]$.

The notation and the terminology will be standard.

Definition. Let (E, E') be a dual pair. We say that two sets $A \subset E$ and $M \subset E'$ *interchange double limits*, and we write $A \sim M$, if for every sequence x_n from A and every sequence u_m from M the existence of both the double limits $\lim_n \lim_m \langle x_n, u_m \rangle$ and $\lim_m \lim_n \langle x_n, u_m \rangle$ ensures that they are equal.

The proof of the following Krein-type theorem is only sketched in $[5,$ Thm.$(3,2)]$; since it is somehow delicate and the result will be used in the sequel, we include a full detailed proof for the sake of completeness.

Theorem 1. *Let X be a Banach space (with unit ball U) and A a bounded set in X. Let M be a norm-generating subset of X' (i.e. $M \subset U^0$ and there exists $r > 0$ such that $p_M(x) \geqq r\|x\|$ for all x in X). Then A^{00} is weakly compact if (and only if) $A \sim M$.*

Proof. Let J be the natural injection from X into the Banach space $CB(M)$ of continuous and bounded functions on M given by $\langle J(x), u \rangle := \langle x, u \rangle$. Since $r\|x\| \leqq \|J(x)\| := p_M(x) \leqq \|x\|$, J is one-to-one, continuous and open. Then we may consider the adjoint J' from $CB(M)'$ onto X'. Define $B(x, u) := \langle x, u \rangle$ on $A \times M$; by using Pták's Extension Theorem $[5,$ Thm. $(2,2)]$, B can be extended to a separately continuous bilinear form B on $CB(A)' \times CB(M)'$ (considering on both spaces the weak-star topology). For $\hat{x}$ in $CB(A)'$ we define a linear form on X' in the following way:

$$P(\hat{x}) : v \in X' \to \langle P(\hat{x}), v \rangle := B(\hat{x}, J'^{-1}(v)) \in \mathbb{K}$$

We must check that $B(\hat{x}, \hat{u})$ does not depend on the choice of $\hat{u}$ in $J'^{-1}(v)$. Indeed: since M is total in $CB(M)'$ (w^*), for $\hat{u}$ in $CB(M)'$ there exists a net u_α from $\text{lin}\,(M)$ such that $\hat{u} = w^* - \lim_\alpha u_\alpha$, then for all x in A we have:

$$
(*) \qquad B(x, \hat{u}) = \lim_\alpha B(x, u_\alpha) = \lim_\alpha \langle x, u_\alpha \rangle = \lim_\alpha \langle J(x), u_\alpha \rangle =
$$
$$
= \lim_\alpha \langle x, J'(u_\alpha) \rangle = \langle x, J'(\hat{u}) \rangle \, .
$$

Now, if $J'(\hat{u}_1) = J'(\hat{u}_2)$ then $B(\cdot, \hat{u}_1) = B(\cdot, \hat{u}_2)$ in A and, since A is total in $CB(A)'$ (w^*), therefore in $CB(A)'$. Now we shall see that, acting on X', $P(\hat{x})$ can be identified with an element of X. Indeed: $B(\hat{x}, \cdot)$ is continuous on $CB(T)'$ (w^*), hence $B(\hat{x}, \hat{u}) = \langle f, \hat{u} \rangle$ for some f in $CB(M)$. Suppose that f is not in $J(X)$; since $J(X)$ is closed in $CB(M)$ (because J is open), by Hahn-Banach theorem there exists $\hat{u}_0$ in $CB(M)'$ such that $\langle J(x), \hat{u}_0 \rangle = 0$ for all x in X and $\langle f, \hat{u}_0 \rangle = 1$. But if we take a net x_α in $\text{lin}\,(A)$ such that $\hat{x} = w^* - \lim x_\alpha$, then

$$
1 = \langle f, \hat{u}_0 \rangle = B(\hat{x}, \hat{u}_0) = \lim_\alpha B(x_\alpha, \hat{u}_0) = (\text{by}(*)) =
$$
$$
= \lim_\alpha \langle x_\alpha, J'(\hat{u}_0) \rangle = \lim_\alpha \langle J(x_\alpha), \hat{u}_0 \rangle = 0
$$

a contradiction. Thus $f = J(x)$ for some x in X and, in this case, for all v in X' we have:

$$
\langle P(\hat{x}), v \rangle = B(\hat{x}, J'^{-1}(v)) = \langle f, J'^{-1}(v) \rangle = \langle J(x), J'^{-1}(v) \rangle = \langle x, v \rangle \, .
$$

In this way, we can define the linear map $P: CB(A)' \to X$ by setting $\langle P(\hat{x}), v \rangle = B(\hat{x}, J'^{-1}(v))$ for v in X'. P is weak*-to-weak continuous since $J'^{-1}(v)$ is in $CB(M)'$ and B is separately continuous. On the other hand, for x in A and every v in X' we have:

$$
\langle P(x), v \rangle = B(x, J'^{-1}(v)) = (\text{by } (*)) = \langle x, v \rangle
$$

hence P acts on A, and being linear also on $\text{acx}(A)$ (the absolutely convex hull of A), as the identity. Finally, since A is contained in the unit ball of $CB(A)'$ which is weakly-star compact and absolutely convex, $\text{acx}(A)$ is contained in a weakly compact set, namely, the image under P of the unit ball of $CB(A)'$. This implies that A^{oo} is weakly compact. The "only if" part is immediate.   QED

In what follows, (E, E') stands for a dual pair. $\mathcal{M}$ will be a family of weakly relatively countably compact sets of E', whose union is $\sigma(E', E)$-total in E'. τ will denote the topology on E of uniform convergence on $\mathcal{M}$. We recall that τ can be strictly finer than the Mackey topology (e.g. if E and E' are both the space φ of finitely non-zero sequences and e_n is the sequence of unit vectors, then $M = \{e_n\}$ is $\sigma(\varphi, \varphi)$-relatively compact but $\text{acx}(M)$ is not $\sigma(\varphi, \varphi)$-relatively compact since it contains the finite sections of any vector in the unit ball of ℓ^1 and these sections converge to the vector in the $\sigma(\ell^1, \varphi)$-topology).

The following is an extension of Krein Theorem $[2, \S\ 24.5.(4)\ \text{and}\ 6.(1)]$.

Theorem 2. *Assume that $E(\tau)$ is quasi-complete. Then for a τ-bounded set $A \subset E$ the following are equivalent:*

(1) *A is $\sigma(E, E')$-relatively compact,*

(2) *$A \sim M$ for all M in $\mathscr{M}$,*

(3) *$A^{00} \sim M^{00}$ for all M in $\mathscr{M}$,*

(4) *A^{00} is $\sigma(E, E')$-compact.*

Proof. (1) implies (2) is exactly as in $[2, \S\, 24.6.(1)]$. (2) implies (3) (this does not depend on the quasi-completeness hypothesis): Let $p_M(x) := \sup\{|\langle x, u\rangle| : u \in M\}$ be the polar seminorm associated to M. Consider the null space $N := \{x : p_M(x) = 0\}$. The quotient E/N is normed space under the norm $\|\hat{x}\| := p_M(x)$ where $\hat{x}$ is the coset of x in E/N. Let X be the completion of $E/N(\|\cdot\|)$. Then $\hat{A}$ is a bounded subset of the Banach space X, M is a norm-generating subset of $X' \cong N^{\perp}$ (whose unit ball is M^{00}) and, since $\langle \hat{x}, u\rangle_{(X,X')} = \langle x, u\rangle_{(E,E')}$, we have that $A \sim M$. By using Theorem 1, we obtain: $(\hat{A})^{00}$ (bipolar in X) is $\sigma(X, X')$-compact, then $(\hat{A})^{00} \sim M^{00}$ (again by Theorem 1) and, since the coset of A^{00} is contained in $(\hat{A})^{00}$, it follows that $A^{00} \sim M^{00}$. (3) implies (4): Again, exactly as in $[2, \S\, 24.6.(1)]$ we have that if z is a $\sigma(E'^{*}, E')$-closure point of A^{00} in E'^{*} (the algebraic dual of E') then, under (3), z must be weakly continuous on M^{00} for every M in $\mathscr{M}$. By the completion theorem $[2, \S\, 21.9.(2)]$ we have that z is in the completion $F(\tilde{\tau})$ of $E(\tau)$. If F' denotes the dual of $F(\tilde{\tau})$, we have that B, the closure of A^{00} in $F(\sigma(F, F'))$ is $\sigma(F, F')$-compact. Now we have that:

$$ B = \overline{A^{00}}^{F(\tilde{\tau})} = \overline{A^{00}}^{E(\tau)} = A^{00} \subset E $$

(first equality: A^{00} is convex; second equality: $E(\tau)$ is quasi-complete). Thus $A^{00} = B$ is, indeed, $\sigma(E, F')$-compact and, since $E' \subset F'$, therefore $\sigma(E, E')$-compact. (4) implies (1) is immediate. QED

Definition. A $\sigma(E, E')$-bounded subset A of E is said to be *weakly (relatively) q-partially compact* if the following holds: Whenever z is a $\sigma(E'^{*}, E')$-adherent point of A in E'^{*} then for each weakly relatively compact sequence u_n from E' there exists x_0 in A (in E) with

$(**)$ $$ \langle z - x_0, u_n\rangle = 0 \quad n = 1, 2, \ldots. $$

Remark. By replacing such a given sequence u_n for the sequence $v_1, v_2, v_3, \ldots :=$ $:= u_1, u_2, u_1, u_3, u_2, u_1, u_4, u_3, u_2, u_1, \ldots$ one can see that $(**)$ is equivalent to the (formally) weaker condition $\lim_n \langle z - x_0, u_n\rangle = 0$. This also applies to the concept of weak (r) partial compactness introduced by Day (in which z is assumed to be a $\sigma(E'^{*}, E')$-adherent point of a countable subset of A, but the sequences u_n are taken from absolutely convex, $\sigma(E', E)$-compact sets, see $[2, \S\, 24.3]$). Bearing this in mind, we have $[2, \S\, 24.3.(2), (3), (4)]$ that if A is weakly (r.) countably-, pseudo- or convex-compact then A is weakly (relatively) q-partially compact. This is also the case of the sets studied by Valdivia $[6]$. A stronger condition is true, indeed:

Proposition. *Let A be a subset of E such that every $\sigma(E, E')$-continuous real-valued function on E is bounded on A. Then for every sequence u_n from E' and every z, $\sigma(E'^*, E')$-adherent point of A in E'^*, there exists x_0 in E such that $(**)$ holds.*

Proof. The functions $f_n(x) := |\langle x - z, u_n \rangle|$ $n = 1, 2, \ldots$ are weakly continuous on E, therefore $f(x) := \sum_{n=1}^{\infty} f_n(x)/2^n(1 + f_n(x))$ is also weakly continuous (Weierstrass M-test). Since we may choose x such that $f_1(x), \ldots, f_m(x)$ are small enough for every fixed m, it is clear that $\inf \{f(x): x \in A\} = 0$. If there exists x_0 in E such that $f(x_0) = 0$, we are done. If $f(x) > 0$ for all x in E, then $g(x) := 1/f(x)$ would be a weakly continuous real valued function on E, but unbounded on A: a contradiction. QED

Our Eberlein-type theorem for weak relative q-partial compactness is the following:

Theorem 3. *Let $E(\tau)$ be quasi-complete and A a τ-bounded weakly relatively q-partially compact subset of E, then A is weakly relatively compact.*

Proof. According to Theorem 2, we have to prove that $A \sim M$ for every $M \in \mathcal{M}$. Assume that x_n from A and u_m are sequences such that $\langle x_n, u_m \rangle \to_m \alpha_n \to_n \alpha$ and $\langle x_n, u_m \rangle \to_n \beta_m \to_m \beta$. Since A is $\sigma(E, E')$-bounded, the sequence x_n has a $\sigma(E'^*, E')$-adherent point z in E'^* and $\beta_m = \langle z, u_m \rangle$ for all m. Since M is $\sigma(E', E)$-relatively countably compact, u_m is $\sigma(E', E)$-relatively compact and $\alpha_n = \langle x_n, u_0 \rangle$ for all n and some u_0 $\sigma(E', E)$-adherent point of u_m in E'. Now take the point x_0 in E corresponding to $(**)$ for the sequence $u_0, u_1, u_2, \ldots$ then we have

$$|\alpha - \beta| \leq |\alpha - \langle x_n, u_0 \rangle| + |\langle x_n - z, u_0 \rangle| + |\langle x_0, u_0 - u_m \rangle| + |\langle x_0, u_m \rangle - \beta|$$

and these summands can be made arbitrarily small for suitable m and n. QED

Corollary 1. *A Banach space E is reflexive if and only if every $\sigma(E'^*, E')$-closure point of its unit ball is $\sigma(E', E)$-continuous when restricted to separable subspaces of E'.*

Our Theorem extends results previously given by Pták [3], Dieudonné [1] and, by using the Proposition above, Valdivia [6, Thm. 3]. Also note that the Theorem remains valid if we assume Day's weak relative partial compactness, since we use only that z is $\sigma(E'^*, E')$-adherent to x_n. In this way we also extend [2, § 24.3.(6)]. The purpose of keeping the condition $(**)$ for every adherent z is to give, finally, an extension of [6, Thm. 11].

Corollary 2. *Let $E(\tau)$ be quasi-complete and A a convex weakly relatively q-partially compact subset of E, then A is weakly relatively compact.*

Proof. We have to prove that A is τ-bounded. We shall make use of an idea of [6, Thm. 8]: If $M \in \mathcal{M}$, let G be the subspace of E' of elements bounded on M. Let A^* be the $\sigma(E'^*, E')$-closure of A in E'^*. If A^* is not contained in G we can find z in A^* and u_n from M such that $\langle z, u_n \rangle \to_n \infty$, take the point x_0 in E satisfying

$(**)$ to obtain a contradiction with the fact that M is $\sigma(E', E)$-bounded. Thus A^* is contained in G and, since A^* is $\sigma(G, E')$-compact and convex and M^0 (the polar in G) is a barrel, M^0 absorbs A^* and this implies that sup $\{p_M(x): x \in A\} < +\infty$. QED

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