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LOCALLY FINITE CONDITIONS ON LATTICE-ORDERED GROUPS

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Introduction. We take as our starting point the following theorem of Weinberg: *If G is an abelian lattice-ordered group which is finitely generated as a lattice-ordered group, then G is a free abelian group.* Weinberg's own proof of this theorem may be found in [C1]; otherwise we believe that it has not appeared in print. Against this backdrop, it is fair to ask for conditions (on an abelian lattice-ordered group, say) which determine that G has finite rank as an abelian group.

Let us state a slim version of our main theorem for abelian lattice-ordered groups. It will follow as a corollary of our work in the setting of nilpotency. But first, some notation: as usual, *l-group* stands for lattice-ordered group. When we wish to emphasize that an l-group is finitely generated with respect to group *and* lattice operations, we say that it is finitely l-generated. For each $x \in G$, $G(x)$ stands for the convex l-subgroup of G generated by x, and N_x for the intersection of all the maximal convex l-subgroups of $G(x)$. Also, if $X \subseteq G$ l-generates G, we shall write $G = \langle X \rangle_l$.

Recall the following definitions: a convex l-subgroup M of the l-group G is said to be *prime* if $a \wedge b = 0$ in G implies that $a \in M$ or $b \in M$. The set of all prime subgroups form a *root system*; that is, if M and N are incomparable primes then they have no common lower bound in the lattice of convex l-subgroups which is prime. M is a *value* of G if it is a convex l-subgroup of G and is maximal with respect to missing some $g \in G$. Values are necessarily prime. (For the claims made in this paragraph the reader might consult [BKW] if he feels in need of a proof.) In $M \subseteq G$ is a value of G it has a cover $M = \bigcap\{C \leqq G \mid C \supsetneqq M\}$. If every value is normal in its cover we say that G is *normal-valued.* If each $0 \neq g \in G$ only has finitely many values $M_1, M_2, \ldots, M_k$, meaning that $g \in \overline{M}_i \setminus M_i$, then we say that G is *finite-valued.*

Notice that N_x is invariant under all the l-automorphisms of $G(x)$; so, in particular, N_x is normal in $G(x)$. Unfortunately, in general, $G(x)/N_x$ can be quite wild; but if G is normal-valued then $G(x)/N_x$ is in fact a subdirect product of real groups.

Let us also adopt the following convention: suppose α stands for a certain property or class of l-groups. We say that G *locally* α if every finitely l-generated l-subgroup of G satisfies α.

We shall eventually prove the following theorem (and more).

Theorem 0. *For an abelian l-group the following are equivalent.*

(a) *G is locally finitely generated.*
(b) *Locally G has a finite root system of prime subgroups.*
(c) *For each* $0 < x \in G$, $G(x)/N_x$ *is representable as an l-subgroup of real-valued step functions.*

Note. $f \in R^I$ is a *step function* if $f(I)$ is finite. As in [C3], let $\mathcal{S}$ denote the class of l-groups which are l-isomorphic to an l-group of (real-valued) step functions.

Before proceeding to prove Theorem 0 we state one result which amplifies Weinberg's motivating theorem.

Proposition 1. *For an abelian l-group the following are equivalent.*
(i) *G can be lattice-ordered so that it is l-generated by two elements.*
(ii) *G can be lattice-ordered so that it is finitely l-generated.*
(iii) *G is a free abelian group of countable rank.*

Proof. (ii) $\Rightarrow$ (iii) follows from the theorem of Weinberg quoted in our introduction. (iii) $\Rightarrow$ (i) By a theorem of Weinberg [W], the free abelian l-group on two generations is free (as an abelian group) of countable rank. □

Several local conditions. In order to point the reader in the right direction, let us begin with a representation theorem for abelian l-groups of finite rank. Recall that if Γ is a root system and for each $\gamma \in \Gamma$, R_γ is a real group, then $V(\Gamma, R_\gamma)$ stands for the l-group of all functions defined on Γ so that $f(\gamma) \in R_\gamma$ and the support of f satisfies the ascending chain condition. $V(\Gamma, R_\gamma)$ is ordered (as an l-group) by defining $f > 0$ if at each one of its maximal support components $\gamma, f(\gamma) > 0$.

Theorem 2. *Suppose G is an abelian l-group which is finitely generated. Then* $G \cong \Sigma(\Gamma, T_\gamma)$, *where* Γ *is a finite root system and each* T_γ *is a finitely generated subgroup of* $\boldsymbol{R}$. (Note. $\Sigma(\Gamma, T_\gamma)$ is the l-subgroup of $V(\Gamma, T_\gamma)$ consisting of all functions which are finitely non-zero. Recall that, according to the Conrad-Harvey-Holland Theorem, every abelian l-group can be embedded in some $V(\Gamma, T_\gamma)$; see [BKW].)

Proof. Let G^d be the divisible hull of G; then it is an l-group, and l-isomorphic to $V(\Gamma, S_\gamma)$, with Γ a finite root system and each $S_\gamma \leqq \boldsymbol{R}$ and of finite dimension over $\boldsymbol{Q}$. (Again, refer to [BKW] for details.) What we want is an l-automorphism of G^d, say σ so that $\sigma G = \Sigma(\Gamma, T_\gamma)$ as promised; so let us proceed by induction on the rank of G (which is the rational dimension of G^d).

There is nothing to do if the dimension of G^d is one. In general, the argument has two parts: suppose G is cardinally decomposable; so $G = A \oplus B$, both non-trivial. Then $G^d = A^d \oplus B^d$, and we can apply induction to each piece: there are l-automorphisms of A^d and B^d, α and β respectively, so that $\alpha A = \Sigma(\Gamma^\alpha, T^\gamma)$ and $\beta B = \Sigma(\Gamma^\beta, T^\gamma)$, with each T^γ a finitely generated subgroup of $\boldsymbol{R}$. Now use α and β to define σ on G^d. So we may suppose that G (and G^d) are cardinally indecomposable. In this case we have a convex l-subgroup C of G such that $G = \operatorname{lex}(C)$—meaning that C is prime in G and each $0 < g \in G \setminus C$ exceeds all $c \in C$ – and in addition, G/C is finitely generated and hence free, so $G = C \times \bar{G}$ where $\bar{G} = G/C$. Notice that $G^d = C^d \times \bar{G}^d$. Again we use induction to obtain τ, an l-automorphism of C^d so that $\tau C = \Sigma(\Gamma', T_\gamma)$ with each $T_\gamma \leqq \boldsymbol{R}$ finitely generated. Finally, left τ to an l-automorphism σ of G^d. $\square$

In the non-abelian setting we have an anlogue for Theorem 2. Recall first the so-called Finite Basis Theorem; (see for example, 2.47 in [C1].) *If in G each subset of pairwise disjoint elements is finite then there is a positive integer n so that G has n pairwise disjoint elements but not n + 1 of them. This is the case if and only if there exists a chain* $0 = G^0 \subset G^1 \subset \ldots \subset G^t = G$ *of convex l-subgroups of G such that*

(i) $G^1 = C^1_1 \oplus \ldots \oplus C^1_{n_1}$, *where each* C^1_j *is an o-group and*

(ii) $G^{i+1} = C^{i+1}_1 \oplus \ldots \oplus C^{i+1}_{n_{i+1}}$, *where each* C^{i+1}_j *is one of the* C^i_k *or else a proper lexicographic extension of twoo or more of them.*

(If (i) and (ii) above hold we say that G is a *finite lex-sum* of the o-groups $C^1_1, \ldots, C^1_{n_1}$.)

Proposition 3. *Suppose g is an l-group and* $\{G_\gamma \mid \gamma \in \Gamma\}$ *is the root system of values of G, with* G^γ *as the cover for* G_γ $(\gamma \in \Gamma)$. *Consider then the following conditions on G*:

(a) Γ *is finite and each* G^γ/G_γ *is finitely generated.*

(b) *G satisfies the* ACC *on all subgroups.*

(c) *G satisfies the* ACC *on all l-subgroups.*

(d) *G is a finite lex-sum of o-groups and each o-groups used in building the lex-sum satisfies the* ACC *on subgroups.*

(e) *G has a finite basis,* Γ *satisfies the* ACC *and each* G^γ/G_γ *is finitely generated.*

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e).

Proof. (a) $\Rightarrow$ (d). First, if G is an o-group for which (a) holds then G has a finite chain of convex subgroups $0 = K^0 \subset K^1 \subset \ldots \subset K^r = G$ with each K^{i+1}/K^i a finitely generated archimedean o-group. Use induction to conclude that G is finitely generated. Then if H is any subgroup of G the root system H of values of H is, likewise, finite with each H^γ/H_γ finitely generated, and therefore H is finitely generated. This says that G satisfies the ACC on all subgroups.

If G is any l-group satisfying (a) use the Finite Basis Theorem. As in our discussion preceding this proposition, let $G^1 = C^1_1 \oplus \ldots \oplus C^1_{n_1}$, where each C^1_j is an o-group. Then C^1_i satisfies the ACC on all subgroups, by our first paragraph and hence so does G^1. Suppose we have shown that each G^i $(i \leqq k)$ satisfies the ACC on all subgroups. $G^{k+1} = C^{k+1}_1 \oplus \ldots \oplus C^{k+1}_{n_{k+1}}$, where each C^{k+1}_j is one of the C^k_i or a proper lexicographic extension of two or more of them. In either case C^{k+1}_j satisfies the ACC on all subgroups, from which we conclude that G^{k+1} does as well; by induction we are done.

(d) $\Rightarrow$ (b) Clearly each G^i in the construction of the lex-sum possesses the ACC on all its subgroups, and hence, by extension, G does as well.

(b) $\Rightarrow$ (c) is evident.

(c) $\Rightarrow$ (d) Suppose, by way of contradiction, that $a_1, a_2, \ldots$ is an infinite, pairwise disjoint set in G then $\langle a_1\rangle_l \subset \langle a_1, a_2\rangle_l \subset \ldots \subset \langle a_1, \ldots, a_n\rangle_l \subset \ldots i$ an infinite ascending chain of l-subgroups. Therefore G does have a finite basis, and so the Finite Basis Theorem applies. Since each o-group used in constructing the lex-sum is an l-homomorphic image of a convex l-subgroup of G it too satisfies the ACC on all subgroups.

(d) $\Rightarrow$ (e) is clear. □

Two immediate comments on the preceding proposition:

I. (e) does not imply (d). Take the free group F on two generators. F contains a subgroup of infinite rank, namely the commutator subgroup. Now totally order F by its descending central chain. Its Γ clearly satisfies the ACC and each G^γ/G_γ is finitely generated. However F does not satisfy the ACC on its subgroups.

II. We do not know whether (b) implies (a). If this implication is valid for o-groups then it follows in general. But the problem seems to be precisely with the o-groups. We shall see presently that for nilpotent o-groups (b) $\Rightarrow$ (a).

Indeed, let us now turn to a study of the following seven conditions:

(1) Locally G satisfies the ACC on all subgroups.

(2) Locally G satisfies the ACC on all l-subgroups.

(3) G is locally finitely generated; (*for emphasis*, this says that each finitely l-generated l-subgroup of G is finitely generated as a group).

(4) Locally g has a finite root system of prime subgroups.

(5) Locally G has a finite basis.

(6) G is locally finite-valued.

(7) For each $0 > x \in G$, $G(x)/N_x \in \mathscr{S}$.

The following should all be evident:

i) (1) and (2) are equivalent, in view of Proposition 3, and (1) implies (3).

ii) Using the proof of Proposition 3 – specifically, of (c) $\Rightarrow$ (d) – one gets that (2) $\Rightarrow$ (5).

iii) Clearly (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

In order to go any further we need to investigate the situation for archimedean l-groups.

Proposition 4. *Suppose G is archimedean. Then G is locally finitely generated if and only if for each* $0 < x \in G$, $G(x) \in \mathscr{S}$.

Proof. Suppose first that G is locally finitely generated. We show first that G is hyper-archimedean: if $0 < a, b \in G$ then $H = \langle a, b\rangle_l$ has finite rank, and therefore H is (l-isomorphic to) a finite cardinal sum of real o-groups; in particular, there is a natural number m such that $ma \wedge b = (m + 1)\, a \wedge b$, which is condition (6) of Theorem 1.1 in [C3] characterizing hyperarchimedeaneity.

Next, suppose $0 < x \in G$. Since G is hyper-archimedean, we know that $G = G(x) \oplus x'$, and we may suppose without loss of generality that $G(x) \subseteq \boldsymbol{R}^I$ and that x is identified with $s(i) = 1$ for each $i \in I$. Now take $y > 0$ in $G(x)$; if y does have infinite range we have indices $i_1, i_2, \ldots, i_k, \ldots \in J$ so that $y(i_1) < y(i_2) < \ldots \ldots < y(i_k) < \ldots$. Choose natural numbers m_1 and n_1 such that $m_1\, y(i_1) < n_1 < m_1\, y(i_2)$; then $(n_1 x - m_1 y) \vee 0$ has support $I_1 = \{i \in I \mid y(i) = y(i_1)\}$ and constant non-zero range $n_1 - m_1\, y(i_1)$. Then choose $m_2, n_2 > 0$ so that $m_2\, y(i_2) < n_2 < m_2\, y(i_3)$; as before, $(n_2 x - m_2 y) \vee 0$ has support $I_2 = \{i \in I \mid y(i) = y(i_2)\}$ and constant non-zero range $n_2 - m_2\, y(i_2)$. If we continue in this manner we get an infinite independent set living in $\langle x, y \rangle_l$; this contradicts our hypothesis. Hence $G(x) \in \mathscr{S}$.

Conversely, suppose each $G(x) \in \mathscr{S}$ $(x > 0)$. Clearly, g is hyperarchimedean, and it suffices to prove that the l-subgroup H of G generated by $a_1, \ldots, a_n$ has finite rank. (Without loss of generality, take each $a_i > 0$.) Let $x = a_1 \vee \ldots \vee a_n$; since $G(x) \in \mathscr{S}$ we may think of $G(x)$ as an l-group of real-valued functions in $\boldsymbol{R}^I$ so that, once more, $x(i) = 1$, for each $i \in I$. In particular, each a_i is a step function. Arguing as in the proof of the reverse implication, each a_i is the sum of disjoint functions in H which are constant on their support. This makes it clear that H has finite rank. □

As corollary of Proposition 4 we obtain, refering once again to our several conditions:

Proposition 5. *If* (6) *holds for G then G is normal-valued and for each* $0 < x \in G$, $G(x)/N_x \in \mathscr{S}$.

Proof. If G is locally finite-valued, then for each $p < x$, $y \in G$, x and y are finite-valued in $\langle x, y \rangle_l$. Using their decompositions into disjoint special elements one can prove that $x = x_1 + x_2$ with $x_1 \wedge x_2 = 0$, $x_1 \in G(y)$ and $x_2 \gg x_2 \wedge y$. This means that G is a pairwise-splitting l-group – see [M2], Theorem 2 – and hence normal-valued.

Next, if (6) holds for G it also does for any convex l-subgroup of G and any l-homomorphic image of one of these. So each $G(x)/N_x$ satisfies (6); since it is archimedean as well it follows that in $G(x)/N_x$ every finitely l-generated l-subgroup is a finite cardinal sum of real o-groups, which, in turn, are finitely generated. Thus $G(x)/N_x$ is locally finitely generated and, by Proposition 4, $G(x)/N_x \in \mathscr{S}$. □

We are, at last, ready for the main theorem, of which Theorem 0 is a consequence.

Theorem 6. *For a nilpotent l-group G the conditions* (1) *through* (7) *are equivalent.*

In the proof we will use the following fact about finitely generated nilpotent groups; it is due to Baer [Ba]: if G is a finitely generated nilpotent group it satisfies the ACC on all subgroups; hence every subgroup is finitely generated. (Also, in any central chain the factors are finitely generated.) In fact, we can do better: suppose that G is *polycyclic*; that is, G has a weak composition series with cyclic factors: $G = G_0 \supset G_1 \supset \ldots \supset G_k = \{0\}$, each G_{i+1} normal in G_i and G_i/G_{i+1} cyclic, with no possible refinements. Then the torsion-free factors in such a composition series are invariants of G and hence their number as well. (For details see [R], pp. 664.) Since every finitely generated nilpotent group is polycyclic we get:

Lemma 7. *If G is a nilpotent, finitely generated o-group its chain of convex subgroups is finite. The length depends only on the number of generators and the nilpotency class.*

Proof of Theorem 6. First (3) $\Rightarrow$ (1) If G is locally finitely generated and H is finitely l-generated in G, then H is finitely generated, and, by Baer's result, has the ACC on all subgroups. We now know that (1), (2) and (3) are equivalent for nilpotent l-groups.

(1) $\Rightarrow$ (4): This follows from Proposition 3 and Lemma 7. In fact, we obtain in Proposition 3 that (b) $\Rightarrow$ (a), as promised earlier.

Finally (7) $\Rightarrow$ (1): suppose H is a finitely l-generated l-subgroup of G. In view of Lemma 7 there is a bound on the length of the roots of H. So it makes sense to speak of the length τ of the longest root of H. We will proceed by induction on π. Pick a set of generators $\{y_1, \ldots, y_n\}$, all positive; let $a = y_1 \vee \ldots \vee y_n$ and consider $G(a)/N_a = S_a$. The image of H in S_a is a finite cardinal sum of real o-groups. This means that in H a is finite-valued; splitting it into the sum of its special components we get $H = H_1 \oplus \ldots \oplus H_k$, where each H_i satisfies (7) and has a connected root system of primes. Without loss of generality assume $H = H_1$.

The situation is as follows: H is l-generated by $\{y_1, \ldots, y_n\}$ and $H = \operatorname{lex}(M)$, where M is the maximal convex l-subgroup of H. The trick is to prove that M is finitely l-generated; then apply induction: M satisfies the ACC on all subgroups. Then since H/M has the same property we may conclude that the ACC holds for H; that will prove (7) $\Rightarrow$ (1).

So let us prove that M is finitely l-generated. First arrange the y_i so that $y_1, \ldots, y_t$ are maximally independent modulo M. (Yes, $t = n$ is a possibility.) Now let T be the subgroup of H generated by $\{y_1, \ldots, y_n\}$, and S be the normal subgroup of T generated by all commutators $[y_i, y_j]$ $(1 \leqq i, j < t)$ and y_k $(t < k \leqq n)$, if any. By Baer's result S is finitely generated. A typical element of T is a "linear" combination $f(y_1, \ldots, y_n)$ of the y_i $(1 \leqq i \leqq n)$. In view of the choice of t such an $f(y_1, \ldots, y_n)$ is in M if and only if the sum of the coefficients of each y_i is zero for each $i = 1, \ldots, t$; that is, if and only if $f(y_1, \ldots, y_n) \in S$. From this it follows that S l-generates M, and so M is finitely l-generated, as promised. □

Having concluded the proof of Theorem 6 let us list some examples which will indicate how some of the above implications can fail in general.

An o-group which satisfies (4) but not (1). Let $\boldsymbol{R}_\pi$ be the subgroup of $\boldsymbol{R}$ of all "polynomials" in π with integer coefficients and exponents; and $f \in \boldsymbol{R}_\pi$ has the form $f = \sum_{i=1}^{n} a_i \pi^{t_i}$, with $a_i, t_i \in \boldsymbol{Z}$. Form a splitting extension $G = \boldsymbol{R}_\pi \overleftarrow{\times} \boldsymbol{Z}$ so that the integer k on top conjugates an element of $\boldsymbol{R}_\pi$ by multiplying it by π^k. G has exactly one non-trivial proper convex subgroup, namely $\boldsymbol{R}_\pi$; it can be generated by two elements, but $\boldsymbol{R}_\pi$ is not finitely generated. Hence G does not satisfy the ACC on subgroups. Notice that G is locally finitely generated, so (3) does not imply (1), in general, either.

An o-group satisfying (5) **but not** (4). The free group on two generators, ordered via its lower central chain. (Recall the comment following Proposition 3.)

(6) **does not imply** (5). Let $G = \boldsymbol{Z} \operatorname{Wr} \boldsymbol{Z}$ with the cardinal order on the bottom. This can be generated by two elements, is finite-valued, but does have an infinite set of pairwise disjoint elements. G is normal-valued, but not representable. We do not known whether (6) $\Rightarrow$ (5) for representable l-groups.

(7) **does not imply** (6). In $\boldsymbol{Z} \operatorname{Wr} \boldsymbol{Z}$, the unrestricted wreath product, let G be the l-subgroup of those $\{-x_n -; k) \in \boldsymbol{Z} \operatorname{Wr} \boldsymbol{Z}$ for which $(- x_n -)$ is eventually a constant at both ends; that is, there is a natural number n_0 such that $s_n = x_n$, for all $n \geqq n_0$ and $n' \leqq -n_0$. G is l-generated by $x = (- 0 -; 1)$, $y = (-1 -; 0)$ and $z = (- z_n -; 0)$, where $z_n = 0$ for $n \neq 0$ and $z_0 = 1$. It is evident that (7) holds but G is not finite-valued. Once again, this is normal-valued but not representable. We do not know what happens if G is a representable l-group, but strongly suspect that (7) does not imply (6) even so.

(6) **does not imply** (3). Let $\boldsymbol{Q}/2$ stand for the group of rational numbers whose denominators are powers of 2. Set $H = \boldsymbol{Q}/2 \times^{\leftarrow} \boldsymbol{Z}$ with the following group operation:

$$(\alpha, m) + (\beta, n) = (\alpha + 2^m \beta, m + n)\,.$$

Then let $K = \boldsymbol{Z} \operatorname{wr} H$ and G be the l-subgroup of K l-generated by $x = (-x_{(\alpha,m)}-; 0, 1)$, where $x_{(0,0)} = 1$ and $x_{(\alpha,m)} = 0$ if $(\alpha, m) \neq (0, 0)$, and $y = (- 0 -; 1, 0)$. Let $\tilde{K} = \{(-x_{(\alpha,m)} -; 0, 0) \mid x_{(\alpha,m)} \in \boldsymbol{Z}\}$. By l-generating G with x and y we will produce *positive* elements of $\tilde{K}$ which have support elements (α, m) which are arbitrarily close in the rational coordinate α. No finite subset of G will do this using the group operations along; hence, G is not finitely generated.

We believe that (4) implies (3); however, most arguments we have tried seem to involve some rather intractable problems – at least intractable to us – concerning finitely generated groups.

We should like to present one more characterization of the l-groups which satisfy condition (7): $G(x)/N_x \in \mathscr{S}$ for each $0 < x \in G$. It involves the notion of a u-constant, which is motivated by the following observation: suppose $G \in \mathscr{S}$ and has a strong order unit u (which we promptly identify with the function $u(i) = 1$, where $i \in I$ and G is an l-group real-valued, step functions on I). Then $0 < g \in G$ is constant on its support – say $g(i) = r(0 < r \in \boldsymbol{R})$ whenever $g(i) \neq 0$ – if and only if for each pair of natural numbers m and n, $g_{mn} = (mu - ng) \vee 0$ is either an order unit or disjoint from g.

So for any l-group G having a strong order unit u – that is, $G = G(u)$ – we call $0 < g \in G$ a *u-constant* (or simply a *constant* when the unit is understood) if for each $m, n \in \boldsymbol{N}$ either

(i) $M + g_{mn} > M$ for all the maximal convex l-subgroups of G or

(ii) $g_{mn} \wedge g \in M$ for all the maximal convex l-subgroups of G, where, as before

$g_{mn} = (mn - ng) \vee 0$. Then, assuming all the maximal convex l-subgroups of G normal, the following is more or less obvious:

Lemma 8. *$g > 0$ is a u-constant if and only if $g \in N_u$ or else in G/N_u, with u represented as the constant function 1, $N_u + g$ is constant on its support.*

Lemma 8 says that $0 < g \in G = G(u)$ is a u-constant if and only if $N_u + g$ is an $(N_u + u)$-constant, (under the assumption of normality for all the maximal convex l-subgroups). So if $G/N_u \in \mathscr{S}$ then, if $x > 0$, each coset $N_u + x$ is the sum of pairwise-disjoint $(N_u + u)$-constants; that is $x = \sum_{j=1}^{m} g_j + a$, where $a \in N_u$ and each g_j is a constant; moreover, without too much pain we can arrange it so that the g_j are pairwise disjoint.

Writing $a = a^+ - a^-$, we get that $a^- \leqq g_1 + \ldots + g_m$, because $a^+ \wedge a^- = 0$ and $x > 0$. By the Riesz-Interpolation Property: $a^- = a_1 + \ldots + a_n$, with $0 \leqq a_j \leqq g_j$. Replacing each g_j by $g_j - a_j$ and observing that a^+ is (trivially) a constant, we get half of

Proposition 9. *Suppose $G = G(u)$, $u > 0$, and each maximal convex l-subgroup of G is normal in G. Then $G/N_u \in \mathscr{S}$ if and only if each $0 < x \in G$ is a sum of constants.*

Proof (of sufficiency). If each $0 < x \in G$ is a sum of constants then (by Lemma 8) $N_u + x$ is a sum of constant functions, whence $G/N_u \in \mathscr{S}$. □

From a global point of view we can be a little better. If condition (7) holds each $G(x)/N_x$ is hyper-archimedean, so, using Lemma 3 from [M2], we can deduce that G is a pairwise-splitting l-group. The notion of pairwise-splittings allows us to prove

Theorem 10. *If G is normal-valued then condition (7) holds if and only if for each $0 < u \in G$ each $0 < x \in G(u)$ can be written as a sum of pairwise disjoint u-constants.*

Proof. The sufficiency has been done. So suppose G satisfies condition (7), and let us return to the argument preceding the statement of Proposition 9; what we have in $G(u)$ is: $0 < x = \sum_{j=1}^{m} g_j + a$, with each g_i u-constant, $0 \leqq a \in N_u$ and $g_i \wedge g_j = 0$ for $i \neq j$. Now notice that no value of a can exceed a value of one of the g_j's; (if so we would violate the pairwise disjointness of the g_j). Use pairwise-splitting and split a by g_1: $a = a_1 + b_1$, which in view of the preceding remark means that $b_1 \wedge g_1 = 0$. So we rewrite $x = g_1' + g_2 + \ldots + g_m + b_1$, with $g_1' = g_1 + a_1$, and $b_1 \wedge g_1 = 0$ and g_1' is still a u-constant. By induction (repeat the above procedure with all the remaining g_j) we obtain: $x = \sum_{j=1}^{m} g_j' + c$ where each g_j' is u-constant, $c \wedge g_j' = 0$, $(1 \leqq j \leqq m)$, $g_i' \wedge g_j' = 0$, if $i \neq j$, and $C \in N_u$. □

The class of l-groups satisfying condition (7) will be denoted Loc $(\mathscr{S})$. Elsewhere we will prove that Loc $(\mathscr{S})$ is a (hereditary) torsion class. (For a definition of torsion classes we refer the reader to [M1] or [M3].)

Abelian l-groups in Loc($\mathscr{S}$) and vector lattices. Or: back to square one. This work was, after all, motivated by a statement about abelian l-groups. We know now that conditions (1) through (7) are equivalent for abelian l-groups. Let us begin here with a companion theorem for (real) vector lattices.

Theorem 11. *Suppose V is a vector lattice. Then the following are equivalent:*
(1v) *V is locally of finite dimension.*
(2v) *Locally V has a finite root system of prime subspaces.*
(3v) *Locally V has a finite basis.*
(4v) *V is locally finite-valued.*
(5v) *For each $0 < x \in V$, $V(x)/N_x$ is hyper-archimedean.*
(6v) *Each finitely l-generated l-subspace W is l-isomorphic to $\Sigma(\Gamma_W, \boldsymbol{R}_\gamma)$, where Γ_W is a finite root system and $R_\gamma = \boldsymbol{R}$ for each $\gamma \in \Gamma_W$.*

Proof. The implications (1v) $\Rightarrow$ (2v) $\Rightarrow$ (3v) $\Rightarrow$ (4v) and (6v) $\Leftrightarrow$ (1v) should be clear. Moreover, Propositions 4 and 5 (rather, their proofs) are easily adapted to obtain (4v) $\Rightarrow$ (5v). Finally, a vector lattice U is hyper-archimedean if and only if $U \in \mathscr{S}$; (refer to [C3]). It then follows that (5) $\Rightarrow$ (1). □

In addition, the following remarks are in order at this point. Suppose G a finitely generated abelian l-group. According to Theorem 2, $G \cong \Sigma(\Gamma, T_\gamma)$, where Γ is a finite root system and each T_γ is a finitely generated, real o-group. Then $V = \Sigma(\Gamma, R_\gamma)$ (each $R_\gamma = \boldsymbol{R}$) is at one the *v-hull* and the *a-closure* of G. (For definitions of these concepts we refer the reader to [BKW], [B1] and [C2].)

To close this article we shall give some reasonably detailed descriptions of finitely l-generated abelian l-groups. We shall only sketch out proofs, or else state the appropriate helpful lemmas. First, some notation: say that G is *n l-generated* if it can be l-generated by n elements. G is *positively n l-generated* if it can be l-generated by n positive elements. Notice straightaway that "n l-generated" need not imply "positively n l-generated: $\boldsymbol{Z} \oplus \boldsymbol{Z}$ is 1 l-generated but not positively. We shall also say that G is *exactly n l-generated* if it can be n l-generated but not $n - 1$ l-generated. The next two lemmas are quite easy; we leave the proofs to the reader. *For the remainder of this article – unless the contrary is expressly stated – every l-group is abelian and finite-valued.*

Lemma 12. *Suppose G is positive n l-generated. Then $G = \langle a_1, a_2, \ldots, a_n \rangle_l$ with each $a_i > 0$ and $2a_i < a_{i+1}$.*

Lemma 13. *If G_1 and G_2 are positively n-generated then so is $G_1 \oplus G_2$.*

We're aiming for the following theorem. Its proof is by induction on the number of generators, so we settle the initial case separately.

Theorem 14. *For $n \geqq 2$, if G is n l-generated then it is also positively n-generated.*

First, let us dispose of 2 l-generated l-groups; we characterize them completely in the proposition below. Note that it also proves Theorem 14 for $n = 2$.

Proposition 15. *Suppose G is 2 l-generated. Then it is positively 2 l-generated* *and, in fact, a finite cardinal sum of copies of* (a) $\boldsymbol{Z} \overleftarrow{\times} \boldsymbol{Z}$, (b) $(\boldsymbol{Z} \oplus \boldsymbol{Z}) \overleftarrow{\times} \boldsymbol{Z}$ *or* (c) *any* 2 *generated real o-group.*

Sketch of the proof. The trick is to verify the following four facts: if Γ is the root system of values of G then

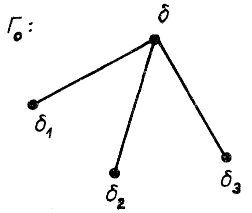
(i) the length of every root of Γ is at most 2;

(ii) if $\gamma_1 < \gamma_2$ in Γ then the real components at γ_1 and γ_2 are both cyclic;

(iii) if $\gamma \in \Gamma$ is both maximal and minimal then γ's component has rank not exceeding 2;

(iv) no $\gamma \in \Gamma$ exceeds more than two incomparable elements of Γ.

(i), (ii) and (iii) are easy to establish. As for (iv), if it fails then G has an l-homomorphic image whose root system is



We identify G with the l-homomorphic image; so $G = \Sigma(\Gamma_0, T_\gamma)$ with each T_γ cyclic. $G = \langle x, y \rangle_l$ and without loss of generality x "lives" at δ, so take $x > 0$. There are two cases to check: (I) $|y| \ll x$ – in which case $G \cong \boldsymbol{Z} \overleftarrow{\times} \boldsymbol{Z}$ or $(\boldsymbol{Z} \oplus \boldsymbol{Z}) \overleftarrow{\times} \boldsymbol{Z}$ and neither of which can occur – or (II) $y > 0$ also lives at δ. A bit of patient computation rules this out as well.

Now invoke Lemma 13 and the proof is done, because it is clear that $\boldsymbol{Z} \overleftarrow{\times} \boldsymbol{Z}$, $(\boldsymbol{Z} \oplus \boldsymbol{Z}) \overleftarrow{\times} \boldsymbol{Z}$ and all 2 generated, real o-groups are positively 2 l-generated. □

(Note. $\boldsymbol{Z}$ is among the 2 generated, real o-groups.)

In order to complete the proof of Theorem 14 we need one more lemma.

Lemma 16. *Suppose G is finitely generated and P is a prime subgroup so that $G =$* lex (P) *and G/P has rank k. If P is exactly m l-generated then G is exactly $m + k$ l-generated.*

Sketch of a proof. Let $G = \langle y_1, \dots, y_q \rangle_l$. As in the proof of Proposition 15, some of the y_i must live outside P, so that we may assume, without loss of generality that $y_1 > \dots > y_t > 0$ $(t \leqq q)$ and all fail to be in P. What as to be established – and it is not hard – is that

(i) $t - k$, if we also insist that the cosets $y_i + P$ $(1 \leqq i \leqq t)$ be maximally independent, and

(ii) $P = \langle y_{k+1}, \dots, y_g \rangle_l$.

Then it is clear why $g \geqq m + k$. □

Now use Lemmas 13 and 16 and Proposition 15 to prove Theorem 14 by induction on n. With a little more effort one can get the next result, which calculates via the "weights" of roots the size of the least l-generating set. Suppose G has a finite root system Γ and that each $\gamma \in \Gamma$ has a finitely generated component T_γ. If $\gamma_1 > \ldots > \gamma_m$ is a root of Γ and t_i is the rank if T_i we call the *weight* of $\gamma_1, \ldots, \gamma_m$

$$wt(\gamma_1, \ldots, \gamma_m) = \begin{cases} t_1 + \ldots + t_m, & \text{if } \gamma_{m-1} \text{ does not exceed three or more copies of } \boldsymbol{Z}, \\ t_1 + \ldots + t_m + 1, & \text{otherwise.} \end{cases}$$

Proposition 17. *Suppose G is exactly n l-generated. Then* $n = \max \{wt(\gamma_1, \ldots, \gamma_m) \mid \gamma_1 > \ldots > \gamma_m$ *is a root of* $\Gamma\}$, *where* $G \cong \Sigma(\Gamma, T_\gamma)$.

Sketch of a proof. Induction on n. If $n = 1$ then $G = \boldsymbol{Z}$ or $\boldsymbol{Z} \oplus \boldsymbol{Z}$, and the result is obvious. For the rest use Lemmas 13 and 16. □

If the reader will return for a moment to the proof of sufficiency in Proposition 4, it will be fairly evident upon inspection that in that argument we have in fact proved: *if G is archimedean and each 2 l-generated l-subgroup of G has finite rank then* $G \in \mathrm{Loc}(\mathscr{S})$. Carrying this idea through the argument for Proposition 5 we obtain – observe: G is *not necessarily abelian*:

Proposition 5ª. *Suppose every 2 l-generated l-subgroup of G is finite-valued. Then G is normal-valued and lies in* $\mathrm{Loc}(\mathscr{S})$.

All of which gives, for nilpotent groups, a little appendix to Theorem 6:

Theorem 6ª. *For a nilpotent l-group G conditions* (1) *through* (7) *are equivalent to: every 2 l-generated l-subgroup of G is finite-valued.*

It seems reasonable to ask whether condition (6): G is locally finite-valued, is equivalent to ($n \in \boldsymbol{N}$):

(6n) every n l-generated l-subgroup of G is finite-valued.

Theorem 6ª says that this is true for every $n \in \boldsymbol{N}$ for nilpotent groups. We do not know what happens in general; as stated before, we do not know whether (7) implies (6) under some reasonable assumption or other, (such as representability). In addition – and we end the article with these remarks – there are the following curious facts.

Proposition 18. *Suppose G is an abelian l-group and $G = \langle x, y \rangle_l$. If x and y are finite-valued then G is finite-valued.*

Sketch of proof. Let $a = x^+ + x^- + y^+ + y^-$, and by decomposing a – which must be finite valued – into its special components, argue that it's sufficient to assume a is special (i.e. G has a connected root system) and $G = \mathrm{lex}(N)$, where N is the maximal convex l-subgroup of G, and $G = G(a)$. Then, as in the proofs of Proposition 15 and Lemma 16, one of the generators must "live" outside N and be positive: say $0 < x \in G \setminus N$. If $|y| \ll x$ we're happy. If, on the other hand, $y \notin N$ then x and y must be dependent mod N, and we can replace y by another generator y_1 such that $|y_1| \ll x$. In any event, this happily leads us to conclude that $G \cong \boldsymbol{Z} \overleftarrow{\times} \boldsymbol{Z}$ or else $(\boldsymbol{Z} \oplus \boldsymbol{Z}) \overleftarrow{\times} \boldsymbol{Z}$, either one finite-valued. □

For contrast, here are two examples to indicate the limitations of Proposition 18. The first one shows that it is false for n l-generated abelian l-groups, with $n \geqq 3$; the second that it is false for 2 l-generated l-groups if G is not abelian.

I. In $\prod_{n=1}^{\infty} R_n$ (with $R_n = \boldsymbol{R}$) we let H be the l-subgroup l-generated by $a = (1, 1, 1, \ldots)$ and $b = (1, 2, 3, \ldots)$. Let $G = H \overleftarrow{\times} \boldsymbol{Z}$, and take $x_1 = (0, 1)$, $x_2 = (a, 1)$ and $x_3 = (b, 1)$; then $G = \langle x_1, x_2, x_3 \rangle_l$ and all three are finite-valued. However G is not a finite-valued l-group.

II. Let $W = \boldsymbol{Z} \operatorname{Wr} \boldsymbol{Z}$, the full wreath product of $\boldsymbol{Z}$ with itself. Let $f(n) = 1$, $n \in \boldsymbol{Z}$, and $g(n) = n$, if $n \geqq 0$, and $g(n) = 0$, if $n < 0$. Now set $a(f, 1)$ and $b = (b, 1)$; $G = \langle a, b \rangle_l$. First, it is clear that a and b are special in G. However, a straightforward calculation shows that: $[a, b] = (h, 0)$, where $h(n) = 1$ if $n \geqq 0$, 0 if $n < 0$, and $[a \ b]^{ma} = (h(n + m), 0)$. It should now be clear that G is not finite-valued.

By way of recapitulation, we shall leave the reader with a short list of those questions which we could not answer and seemed (to us) worth pursuing.

A. If G has locally a finite root system, then is it locally finitely generated? If not so in general, then under what reasonable hypotheses?

B. If G is representable and in Loc $(\mathscr{S})$, is it locally finite-valued?

C. If G is representable and locally finite-valued does it have locally a finite basis?

D. If G is locally finitely generated is it normal-valued? (That is, does condition (3) imply normal-valuedness. All the other do.)

E. If G is a finitely generated o-group which satisfies the ACC on all subgroups, must its chain of convex subgroups be finite?

F. Under what conditions is an l-group $G = \langle a, b \rangle_l$, with $a \gg b > 0$, finite-valued? (Certainly if G is abelian, but as the last example shows, not in general.)

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