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FUNCTIONAL MODEL FOR COMMUTING ISOMETRIES

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1. Introduction. Let Z be the additive group of integers and Z_+ the semigroup of non-negative integers. Let V be an isometry acting on a separable (complex) Hilbert space and let $(T_n)_{n\in Z_+}$ be the semigroup of isometries defined by $T_n = V^n$ $(n \in Z_+)$. The well-known Wold decomposition theorem says that the space H can be decomposed into the orthogonal sum $H_u \oplus H_t$ in such a way that H_u reduces every T_n to a unitary operator and the semigroup $(T_n| H_t)_{n\in Z_+}$ is unitarily equivalent to the semigroup of unilateral shifts.

For a pair of commuting isometries the situation is much more complicated. This was studied in many papers but satisfactory results were obtained only in the case when isometries V_1, V_2 on H doubly commute, i.e. $V_1V_2 = V_2V_1$, $V_1V_2^* = V_2^*V_1$ (see [7], [8]). Finally, the detailed structure of the semigroup generated by two doubly commuting isometries was given in [2].

In [9] M. Slociński suggested to study pairs of commuting isometries satisfying the following property (which we have called compatibility).

Definition 1. Let V_1, V_2 be commuting isometries on a separable Hilbert space H. We say that V_1 and V_2 are *compatible* if $V_1^nV_1^{*n}$ commutes with $V_2^mV_2^{*m}$ for every $m, n \in Z_+$ (i.e., the orthogonal projections onto the ranges of V_1^n and V_2^m commute).

Clearly, double commuting isometries are compatible.

This paper is a continuation of the work begun in [6] where the authors disproved the original Slociński's conjecture about the structure of compatible isometries. In what follows we construct a canonical functional model for general finitely generated compatible semigroups of isometries.

Let S be a commutative (additive) semigroup with a unit 0. Let $(T_s)_{s\in S}$ be a representation of S by isometries in a Hilbert space H, i.e.

$$T_s^*T_s = I\,, \quad T_{s+t} = T_sT_t\,, \quad T_0 = I \quad (s, t \in S)\,.$$

Definition 2. We call the semigroup $(T_s)_{s\in S}$ *compatible* if $T_sT_s^*(s \in S)$ form a family of commuting projections (note that $T_sT_s^*$ is the orthogonal projection onto the range of T_s).

The following proposition shows that for pairs of isometries the two notions of compatibility coincide.

Proposition 1. *Let V_1, V_2 be commuting isometries on H. Then V_1 and V_2 are compatible if and only if the semigroup $(V_1^m V_2^n)_{(m,n)\in Z_+^2}$ is compatible.*

Proof. Put $T_{(m,n)} = V_1^m V_2^n$, $(m, n) \in Z_+^2$. The "if" part is clear as

$$V_1^m V_1^{*m} = T_{(m,0)} T_{(m,0)}^*, \quad V_2^n V_2^{*n} = T_{(0,n)} T_{(0,n)}^* .$$

Assume that $V_1^m V_1^{*m}$ and $V_2^n V_2^{*n}$ commute for every $(m, n) \in Z_+^2$. The semigroup $S = Z_+^2$ is partially ordered by the relation

$$(s_1, s_2) \leqq (t_1, t_2) \quad \text{iff} \quad s_i \leqq t_i \quad \text{for} \quad i = 1, 2 .$$

Let $t, s \in Z_+^2$. If $s \leqq t$ it clearly follows that

$$T_s T_s^* T_t T_t^* = T_s T_s^* T_s T_{t-s} T_t^* = T_t T_t^* = T_t T_{t-s}^* T_s^* T_s T_s^* = T_t T_t^* T_s T_s^* .$$

If neither $s \leqq t$ nor $t \leqq s$ then provided $r = \min(s, t)$ the differences $s - r$, $t - r$ are of the form $(m, 0)$ or $(0, n)$ so that the commutativity of $T_s T_s^*$ and $T_t T_t^*$ follows from the relations

$$T_s T_s^* = T_r (T_{s-r} T_{s-r}^*)\, T_r^* , \quad T_t T_t^* = T_r (T_{t-r} T_{t-r}^*)\, T_r^*$$

by our assumption.

In the next section we give two examples the latter of which forms a canonical model for compatible semigroups of isometries. The main theorem (Section 4) states that any finitely generated compatible semigroup $(T_s)_{s\in S}$ of isometries is a direct sum of semigroups which are unitarily equivalent to the semigroup $(W_s)_{s\in S}$ from Example 2. The appropriate measurable space X and a projection valued measure Q are constructed in a standard way in Section 3.

We conclude our introduction with an example of commuting isometries which are not compatible.

Example. Let H be a Hilbert space with an orthonormal basis $(e_i)_{i=1}^\infty \cup (f_i)_{i=1}^\infty$. Define isometries $V, W \in B(H)$ by the relations

$$\begin{aligned} Ve_i &= e_{i+1}, \quad Vf_i = f_{i+1}, \\ We_i &= \tfrac{1}{2}(e_i + e_{i+1} + f_i - f_{i+1}), \\ Wf_i &= \tfrac{1}{2}(f_{i+1} + f_{i+2} + e_{i+1} - e_{i+2}). \end{aligned}$$

It easily follows that $VW = WV$, $|Ve_i| = |Wf_i| = 1$, $(We_i, We_j) = (Wf_i, Wf_j) = 0$ for all $i \neq j$, and $(We_i, Wf_j) = 0$ for all i, j. Thus V and W are commuting isometries which are not compatible as VV^* and WW^* do not commute.

2. Two examples. We introduce the following notation. Let S be a commutative semigroup and G its "division" group, i.e. $G = \{[s - t]: s, t \in S\}$ where $[s - t]$

denotes the class of equivalence $\sim$ containing $s-t$, and $(s-t)\sim(u-v)$ if $s+v=u+t$ $(s,t,u,v\in S)$. A non-empty subset $X\subset G$ will be called a *diagram* if $\varphi\in X$, $s\in S$ imply $\varphi+s\in X$.

Denote by $\mathscr{X}$ the set of all diagrams. For $\varphi\in G$ define $E_\varphi=\{X\in\mathscr{X},\ \varphi\in X\}$. Clearly, $E_\varphi\subset E_{\varphi+s}$ $(\varphi\in G,\ s\in S)$. Let $\mathscr{S}$ be the σ-algebra generated by the sets E_φ, $\varphi\in G$.

Example 1. Let μ be a positive measure on $(\mathscr{X},\mathscr{S})$, $\mu(\mathscr{X})=1$. Let K be the set of all functions $f\colon G\to L^2(\mu)$, $\varphi\mapsto f_\varphi\in L^2(\mu)$ such that $\operatorname{supp} f_\varphi\subset E_\varphi$ $(\varphi\in G)$ and

$$\sum_{\varphi\in G}|f_\varphi|^2_{L^2(\mu)}<\infty\,.$$

Then K with the inner product

$$(f,g)_K=\sum_{\varphi\in G}\int_{\mathscr{X}} f_\varphi\overline{g_\varphi}\,\mathrm{d}\mu$$

becomes a Hilbert space.

Define $T_s\in B(K)$ by $(T_sf)_\varphi=f_{\varphi-s}$ $(s\in S,\ \varphi\in G)$. Clearly, $(T_s)_{s\in S}$ is a commutative semigroup of isometries. It is easy to check that $(T_s^*f)_\varphi=f_{\varphi+s}\chi_{E_\varphi}$ and $(T_sT_s^*f)_\varphi= =f_\varphi\chi_{E_{\varphi-s}}$ where χ_A denotes the characteristic function of a set A. So the projections $T_sT_s^*$, $T_tT_t^*$ commute for every $s,t\in S$, hence $(T_s)_{s\in S}$ is compatible.

Example 2. Let $(\mathscr{X},\mathscr{S},\mu)$ be as above. Let us denote

$$K_0=\{f\colon G\to L^2(\mu),\ \varphi\mapsto f_\varphi \text{ s.t. } \operatorname{supp} f_\varphi\subset E_\varphi\ (\varphi\in G)\,,$$
$$f_\varphi\neq 0 \text{ for only a finite number of elements } \varphi\in G\}$$

and let $c_\varphi(\varphi\in G)$ be a family of bounded measurable complex functions on $\mathscr{X}$ which are positive definite in the following sense:

$$\sum_{\varphi,\psi\in G}\int_{\mathscr{X}} f_\varphi(X)\overline{f_\psi(X-\varphi+\psi)}\,c_{\varphi-\psi}(X)\,\mathrm{d}\mu(X)\geqq 0$$

for every function $f\in K_0$, and normalized by the condition $c_0=1$. By $X-\varphi+\psi$ $(X\in\mathscr{X},\ \varphi,\psi\in G)$ we denote the diagram $X-\varphi+\psi=\{\xi-\varphi+\psi,\ \xi\in X\}$. Then K_0 is a linear space with a positive semidefinite bilinear form

$$\langle f,g\rangle=\sum_{\varphi,\psi}\int_{\mathscr{X}} f_\varphi(X)\overline{g_\psi(X-\varphi+\psi)}\,c_{\varphi-\psi}(X)\,\mathrm{d}\mu(X)\,.$$

Denote $K_1=\{f\in K_0,\ \langle f,f\rangle=0\}$ and let K be the completion of K_0/K_1. Defining $(W_s^0f)_\varphi=f_{\varphi-s}$ $(s\in S,\ \varphi\in G)$ we obtain isometries on K_0 which leave the kernel K_1 invariant. So they determine in a natural way the semigroup $(W_s)_{s\in S}$ of commuting isometries on K which is clearly compatible.

Remarks. 1. Example 2 includes Example 1 for $c_\varphi=\delta_{0\varphi}$ (the Kronecker delta).

2. Taking the Dirac measure $\mu=\delta_G$ concentrated on $G\in\mathscr{X}$ we obtain commuting unitary operators. Conversely, any commutative semigroup of unitary operators

(which are doubly commuting, hence compatible) with a cyclic vector $h \in H$, $|h| = 1$, can be obtained for $\mu = \delta_G$, $c_{s-t} = (U_s h, U_t h)$.

3. Let $D \in \mathscr{X}$ be any diagram, $\mu = \delta_D$, $c_\varphi = \delta_{0\varphi}$. Then Example 2 gives the $(T_s)_{s\in S}$ of isometries on the Hilbert space H_D spanned by an orthonormal semigroup family of vectors $\{e_\varphi, \varphi \in D\}$ defined by $T_s e_\varphi = e_{\varphi+s}$.

4. Let $D \in \mathscr{X}$ be a diagram with an automorphism $\alpha \in G$ (i.e. $D + \alpha = D$), let $c_{k\alpha} = c_k (k \in Z)$ and $c_\varphi = 0$ otherwise. For $S = Z_+^2$, $G = Z^2$, $\alpha(i, j) = (i + 1, j - 1)$ for $(i, j) \in D = \{(r, s), r + s \geqq 0\}$, we obtain Example 2 of [6].

5. Let V_1, V_2 be doubly commuting isometries with a cyclic vector. Then one can take $S = Z_+^2$, $G = Z^2$ and a measure concentrated on the four-point set $\{Z \times Z, Z \times Z_+, Z_+ \times Z, Z_+ \times Z_+\}$. These four diagrams correspond to the four parts in the Wold decomposition of two doubly commuting isometries given in [7], [8].

3. Construction of spectral measure. The aim of this section is to show that for any finitely generated compatible semigroup of isometries we can construct a projection-valued measure Q defined on the measurable space $(\mathscr{X}, \mathscr{S})$ introduced in the previous section.

Let S be a commutative semigroup with N generators, $N < \infty$, and G its division group. Let $(T_s)_{s\in S}$ be its representation by isometries in a Hilbert space H. As any such semigroup is an epimorphic image of the free commutative semigroup with N generators, we may assume without loss of generality that S is the free commutative semigroup. In the sequel we shall assume $S = Z_+^N$ and $G = Z^N$ $(N < \infty)$ although it is possible to do all the considerations for a general commutative semigroup with N generators.

For $\varphi \in G$, $\varphi = s - t$ $(s, t \in S)$ put $T_\varphi = T_t^* T_s$. Clearly, T_φ does not depend on the choice of s and t as $T_{t+r}^* T_{s+r} = T_t^* T_s = T_\varphi$ for each $r \in S$. Define further

$$Q_\varphi = T_\varphi^* T_\varphi = T_s^* T_t T_t^* T_s \quad (\varphi = s - t,\ s,\ t \in S) .$$

Proposition 2. $(T_s)_{s\in S}$ *is compatible if and only if* $(Q_\varphi)_{\varphi\in G}$ *is a family of commuting projections.*

Proof. The "if" part is easy to see from the identity

$$Q_{0-t} = T_t T_t^* \quad (t \in S) .$$

Suppose that $(T_s)_{s\in S}$ is a compatible semigroup. Let $\varphi = s - t \in G$, $s, t \in S$. Clearly,

$$Q_\varphi = T_s^* T_t T_t^* T_s = Q_\varphi^* ,$$

$$Q_\varphi^2 = T_s^*(T_t T_t^*)(T_s T_s^*)(T_t T_t^*)\, T_s = T_s^* T_t T_t^* T_s = Q_\varphi .$$

If $\varphi, \psi \in G$, $\varphi = s - t$, $\psi = u - v$ $(s, t, u, v \in S)$ then also $\varphi = (s + u) - (t + u)$, $\psi = (u + s) - (v + s)$, hence

$$Q_\varphi Q_\psi = T_{s+u}^*(T_{t+u} T_{t+u}^*)(T_{s+u} T_{u+s}^*)(T_{v+s} T_{v+s}^*)\, T_{u+s} =$$
$$= T_{s+u}^*(T_{v+s} T_{v+s}^*)(T_{s+u} T_{u+s}^*)(T_{t+u} T_{t+u}^*)\, T_{u+s} = Q_\psi Q_\varphi .$$

Corollary. *If $(T_s)_{s\in S}$ is a compatible semigroup of isometries on a Hilbert space H and $\varphi \in G$, $\varphi = s - t$ $(s, t \in S)$, then $T_\varphi = T_t^* T_s$ is a partial isometry with the initial space $T_s^* T_t H$ and range $T_t^* T_s H$.*

For $\varphi \in G$ let us denote $E_\varphi = \{X \in \mathscr{X}, \varphi \in X\}$, $E^\varphi = \mathscr{X} - E$. For finite sequences $\varphi = \{\varphi_1, \ldots, \varphi_p\}$, $\psi = \{\psi_1, \ldots, \psi_q\}$ of elements of G we shall write

$$E_\varphi^\psi = \bigcap_{i=1}^{p} E_{\varphi_i} \cap \bigcap_{j=1}^{q} E^{\psi_j}.$$

We shall call such sets elementary.

Lemma 1. *The set $\mathscr{S}_0$ of all finite disjoint unions of elementary sets forms an algebra generated by the sets E_φ, $\varphi \in G$.*

Proof. For all finite sequences $\varphi, \psi, \sigma, \tau$ of elements of G we have $E_\varphi^\psi \cap E_\sigma^\tau = E_{\varphi\cup\sigma}^{\psi\cup\tau}$. If $\bigcup_i E_i$, $\bigcup_j E_j'$ are two finite disjoint unions of elementary sets then

$$\left(\bigcup_i E_i\right) \cap \left(\bigcup_j E_j'\right) = \bigcup_{i,j} (E_i \cap E_j')$$

is a disjoint union of elementary sets, hence $\mathscr{S}_0$ is closed under intersections.

Let E_φ^ψ be an elementary set. Then

$$\mathscr{X} - E_\varphi^\psi = \bigcup\{E_{(\varphi-\varphi')\cup\psi'}^{(\psi-\psi')\cup\varphi'},\ \varphi' \subset \varphi,\ \psi' \subset \psi,\ \varphi' \cup \psi' \neq \emptyset\},$$

so the complement of an elementary set belongs to $\mathscr{S}_0$.

If $\bigcup_i E_i$ is a finite disjoint union of elementary sets then

$$\mathscr{X} - \bigcup_i E_i = \bigcap_i (\mathscr{X} - E_i) \in \mathscr{S}_0 .$$

So $\mathscr{S}_0$ is closed under taking complements, hence it is an algebra.

Now we return to the given compatible semigroup $(T_s)_{s\in S}$ of isometries. Define

$$Q(\mathscr{X}) = I,$$

$$Q(E_\varphi^\psi) = \prod_{i=1}^{p} Q_{\varphi_i} \prod_{j=1}^{q} (I - Q_{\psi_j})$$

for all finite sequences $\varphi = \{\varphi_1, \ldots, \varphi_p\}$, $\psi = \{\psi_1, \ldots, \psi_q\}$ of elements of G,

$$Q\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} Q(E_i)$$

of any finite disjoint union $\bigcup_i E_i$ of elementary sets. The correctness of the definition follows in a standard way. It is easily seen that Q is finitely additive on elementary sets. If some set $E \in \mathscr{S}_0$ can be written in two ways as a disjoint union of elementary sets

$$E = \bigcup_{i=1}^{m} E_i = \bigcup_{j=1}^{n} E_j',$$

then

$$\sum_{i=1}^{m} Q(E_i) = Q(\bigcup_{i=1}^{m} E_i) = Q(\bigcup_{i,j} (E_i \cap E'_j)) = Q(\bigcup_{j=1}^{n} E'_j) = \sum_{j=1}^{n} Q(E'_j),$$

hence the value of $Q(E)$ does not depend on the way of expressing $E \in \mathscr{S}_0$.

We conclude that Q is an additive projection-valued function on $\mathscr{S}_0$. Now we are going to show that Q extends to a σ-additive projection-valued function on the σ-algebra $\mathscr{S}$. We shall need the following lemma.

Lemma 2. *If* $A_n \in \mathscr{S}_0$ $(n = 1, 2, \ldots)$, $A_1 \supset A_2 \supset \ldots \supset \bigcap_{n=1}^{\infty} A_n = \emptyset$, *then* $\lim_{n\to\infty} Q(A_n) = 0$ *in the strong operator topology.*

Proof. Suppose on the contrary that there exists a unit vector $h \in H$ such that $Q(A_n)\, h \nrightarrow 0$ $(n \to \infty)$. Let us denote

$$I_k = \{i \in \boldsymbol{Z},\ |i| \leqq k\}^N,$$

$$\mathscr{X}_k = \{X \cap I_k,\ X \in \mathscr{X}\},$$

$$\pi_k\colon \mathscr{X} \to \mathscr{X}_k, \quad \pi_k(X) = X \cap I_k \quad (k = 1, 2, \ldots)$$

(recall that $G = \boldsymbol{Z}^N$).

Clearly, for any $a \in \mathscr{X}_k$ we have $\pi_k^{-1}(\{a\}) \in \mathscr{S}_0$ and

$$\mathscr{X} = \bigcup_{a \in \mathscr{X}_k} \pi_k^{-1}(\{a\}).$$

The union is finite and disjoint so that

$$\sum_{a \in \mathscr{X}_k} Q(\pi_k^{-1}(\{a\})) = I$$

and we can find $x_k \in \mathscr{X}_k$ such that

$$Q(A_n)\, Q(\pi_k^{-1}(\{x_k\}))\, h \nrightarrow 0 \quad (n \to \infty).$$

We can even choose $x_k \in \mathscr{X}_k$ inductively in such a way that $x_k \cap I_{k-1} = x_{k-1}$. Taking now $X = \bigcup_{k=1}^{\infty} x_k \in \mathscr{X}$ we have $X \cap I_k = x_k$ $(k = 1, 2, \ldots)$ whence

$$Q(A_n)\, Q(\pi_k^{-1}(\{X \cap I_k\}))\, h \nrightarrow 0 \quad (n \to \infty).$$

On the other hand, $X \notin A_m$ for some $m \in \{1, 2, \ldots\}$. Choosing $k \in \{1, 2, \ldots\}$ big enough such that all $\varphi, \psi \in G$ involved in the expression of $A_m \in \mathscr{S}_0$ as a finite disjoint union of elementary sets are contained in I_k we find that

$$A_m \cap \pi_k^{-1}(\{X \cap I_k\}) = \emptyset,$$

hence $Q(A_m)\, Q(\pi_k^{-1}(\{X \cap I_k\})) = 0$, a contradiction.

It follows from Lemma 2 that the projection-valued function Q is σ-additive on the algebra $\mathscr{S}_0$, so it can be uniquely extended (see [4] for ordinary complex measure) to a spectral measure Q defined on the σ-algebra $\mathscr{S}$ generated by the sets E_φ, $\varphi \in G$.

Fix now a unit vector $h \in H$ and define a positive measure μ on $(\mathcal{X}, \mathcal{S})$ by

$$\mu(A) = |Q(A)\, h|^2\,, \quad A \in \mathcal{S}\,.$$

Further we introduce a family $(\mu_\varphi)_{\varphi \in G}$ of complex measures defined by

$$\mu_\varphi(A) = (T_\varphi\, Q(A)\, h, h)\,, \quad \varphi \in G\,, \quad A \in \mathcal{S}\,.$$

Clearly, $\mu_0 = \mu$, μ_φ is absolutely continuous with respect to μ, $|\mu_\varphi(A)| \leqq \mu(A)$ because T_φ is a partial isometry. Hence there exist functions $c_\varphi \in L^1(\mu)$, $\varphi \in G$, such that

$$c_\varphi = \mathrm{d}\mu_\varphi/\mathrm{d}\mu\,, \quad |c_\varphi| \leqq 1\,.$$

Lemma 3. *If $(T_s)_{s \in S}$ is a compatible semigroup of isometries then*

$$Q_\psi T_\varphi = T_\varphi Q_{\psi+\varphi}\,, \tag{1}$$

$$T^*_\psi T_\varphi = T_{\varphi-\psi} Q_\varphi \tag{2}$$

for every $\varphi, \psi \in G$.

Proof. Let $\varphi = s - t$, $\psi = u - v$ $(\psi + \varphi = (s + u) - (t + v)$, $\varphi - \psi = (s + v) - (t + u))$. Using the commutativity of projections $T_s T^*_s$ $(s \in S)$ we obtain

$$\begin{aligned} Q_\psi T_\varphi &= (T^*_u T_v T^*_v T_u)\, T^*_t T_s = T^*_{u+t}(T_{v+t} T^*_{v+t})\,(T_{u+t} T^*_{t+u})\, T_{s+u} = \\ &= T^*_{u+t}(T_{v+t} T^*_{v+t})\, T_{s+u} = T^*_{u+t}(T_{s+u} T^*_{s+u})\,(T_{v+t} T^*_{v+t})\, T_{s+u} = \\ &= T^*_t T_s (T^*_{s+u} T_{v+t} T^*_{v+t} T_{s+u}) = T_\varphi Q_{\psi+\varphi}\,. \end{aligned}$$

Analogously

$$\begin{aligned} T^*_\psi T_\varphi &= T^*_u T_v T^*_t T_s = T^*_{u+t} T_v (T_t T^*_t)\,(T_s T^*_s)\, T_s = \\ &= T^*_{u+t} T_{v+s} (T^*_s T_t T^*_t T_s) = T_{\varphi-\psi} Q_\varphi\,. \end{aligned}$$

From equality (1) in Lemma 3 we derive the following

Proposition 3. *If $(T_s)_{s \in S}$ is a compatible semigroup of isometries then*

$$Q(A)\, T_\varphi = T_\varphi Q(A + \varphi)$$

for every $A \in \mathcal{S}$, $\varphi \in G$ where $A + \varphi = \{X + \varphi, X \in A\}$.

Proof. Let $\varphi \in G$. We prove that the set

$$\mathcal{A} = \{A \in \mathcal{S},\; Q(A)\, T_\varphi = T_\varphi Q(A + \varphi)\}$$

forms a σ-algebra. As $E_\psi \in \mathcal{A}$ for any $\psi \in G$ by equality (1) in Lemma 3, the equality $\mathcal{A} = \mathcal{S}$ follows.

For any $A, B \in \mathcal{A}$ we have

$$\begin{aligned} Q(\mathcal{X} - A)\, T_\varphi &= (I - Q(A))\, T_\varphi = T_\varphi(I - Q(A + \varphi)) = \\ &= T_\varphi\, Q((\mathcal{X} - A) + \varphi)\,, \end{aligned}$$

$$Q(A \cap B)\, T_\varphi = Q(A)\, Q(B)\, T_\varphi = T_\varphi Q(A + \varphi)\, Q(B + \varphi) = $$
$$= T_\varphi Q(A \cap B + \varphi)\,,$$

and for $A = \bigcup_{i=1}^{\infty} A_i$, a disjoint union of sets $A_i \in \mathscr{A}$, we conclude

$$Q(\bigcup_{i=1}^{\infty} A_i)\, T_\varphi = \sum_{i=1}^{\infty} Q(A_i)\, T_\varphi = \sum_{i=1}^{\infty} T_\varphi Q(A_i + \varphi) = T_\varphi Q(\bigcup_{i=1}^{\infty} A_i + \varphi)\,.$$

The proof is complete.

4. Unitary equivalence. Let $(T_s)_{s\in S}$ be a compatible semigroup of isometries on a Hilbert space H, $h \in H$, $|h| = 1$. Let μ and $(\mu_\varphi)_{\varphi\in G}$ be the measures constructed in the preceding section and let $(c_\varphi)_{\varphi\in G}$ be the corresponding measurable functions, $c_\varphi \in L^1(\mu)$. We shall show that $(T_s)_{s\in S}$ restricted to the smallest reducing subspace containing the given $h \in H$ are unitarily equivalent to the semigroup $(W_s)_{s\in S}$ constructed in Example 2.

For disjoint sets $A_1, \ldots, A_n \in \mathscr{S}$ and arbitrary complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ define

$$U_0(\sum_{i=1}^{n} \alpha_i \chi_{A_i}) = \sum_{i=1}^{n} \alpha_i\, Q(A_i)\, h\,.$$

As

$$|\sum_{i=1}^{n} \alpha_i \chi_{A_i}|^2_{L^2(\mu)} = \sum_{i=1}^{n} |\alpha_i|^2\, \mu(A_i) = \sum_{i=1}^{n} |\alpha_i|^2\, |Q(A_i)\, h|^2 = |\sum_{i=1}^{n} \alpha_i\, Q(A_i)\, h|^2\,,$$

the operator U_0 is an isometry defined on a dense subset of $L^2(\mu)$, hence it can be uniquely extended to an isometry $U_0: L^2(\mu) \to H$. As in Example 2, denote

$$K_0 = \{f: G \to L^2(\mu),\ \varphi \mapsto f_\varphi,\ \sup f_\varphi \subset E_\varphi,\ f_\varphi \neq 0$$
$$\text{for only a finite number of elements } \varphi \in G\}\,.$$

For our convenience, we write formally $f = \sum_{\varphi\in G} f_\varphi e_\varphi$ for $f \in K_0$. Define the operator $U: K_0 \to H$ by

$$Uf = U(\sum_{\varphi\in G} f_\varphi e_\varphi) = \sum_{\varphi\in G} T_\varphi U_0 f_\varphi \quad (f \in K_0)\,.$$

For $f = \chi_A e_\varphi$, $g = \chi_B e_\psi$ $(A, B \in \mathscr{S},\ A \subset E_\varphi,\ B \subset E_\psi,\ \varphi, \psi \in G)$ we then obtain

$$(Uf, Ug) = (T_\varphi\, Q(A)\, h,\ T_\psi\, Q(B)\, h) = (T_\psi^* T_\varphi\, Q(A)\, h,\ Q(B)\, h) =$$
$$= (T_{\varphi-\psi} Q_\varphi\, Q(A)\, h,\ Q(B)\, h) = (Q(B)\, T_{\varphi-\psi}\, Q(A)\, h,\ h) =$$
$$= (T_{\varphi-\psi} Q((B + \varphi - \psi) \cap A)\, h,\ h) = \mu_{\varphi-\psi}((B + \varphi - \psi) \cap A) =$$
$$= \int_{(B+\varphi-\psi)\cap A} c_{\varphi-\psi}\, d\mu = \int_{\mathscr{X}} \chi_A(X)\, \chi_B(X - \varphi + \psi)\, c_{\varphi-\psi}(X)\, d\mu(X) =$$
$$= \langle f, g \rangle_{K_0}$$

where $\langle \cdot, \cdot \rangle_{K_0}$ is the bilinear form introduced in Example 2.

Clearly, the last equality holds also for $f = f_\varphi e_\varphi$, $g = g_\psi e_\psi$, $\operatorname{supp} f_\varphi \subset E_\varphi$, $\operatorname{supp} g_\psi \subset E_\psi(\varphi, \psi \in G)$, and $f_\varphi, g_\psi \in L^2(\mu)$. For $f, g \in K_0$, $f = \sum_{\varphi \in G} f_\varphi e_\varphi$, $g = \sum_{\psi \in G} g_\psi e_\psi$ we then have

$$(Uf, Ug) = \sum_{\varphi,\psi \in G} (Uf_\varphi e_\varphi, Ug_\psi e_\psi) = \sum_{\varphi,\psi \in G} \langle f_\varphi, g_\psi \rangle_{K_0} =$$
$$= \sum_{\varphi,\psi \in G} \int_{\mathscr{X}} f_\varphi(X) \overline{g_\psi(X - \varphi + \psi)}\, c_{\varphi-\psi}(X)\, \mathrm{d}\mu(X) =$$
$$= \langle f, g \rangle_{K_0} .$$

This shows that the functions $(c_\varphi)_{\varphi \in G}$ are positive definite in the sense of Example 2, and $U: K_0 \to H$ is an isometry. As $K_1 = \{f \in K_0, \langle f, f \rangle_{K_0} = 0\} \subset \operatorname{Ker} U$, the isometry U can be uniquely extended to an isometry $U: K \to H$, K being the completion of K_0/K_1.

Let $s \in S$, $f = \chi_A e_\varphi$, $A \in \mathscr{S}$, $A \subset E_\varphi$, $\varphi \in G$. By Lemma 3 we obtain

$$T_s U f = T_s T_\varphi U_0 \chi_A = T_{\varphi+s} Q_\varphi\, Q(A)\, h = T_{\varphi+s}\, Q(A)\, h =$$
$$= T_{\varphi+s} U_0 \chi_A = U \chi_A e_{\varphi+s} = U W_s f .$$

This implies $T_s U = U W_s$ on K_0 whence the same intertwining relation holds on K.

So U maps K isometrically onto the smallest subspace of H containing h and reducing all the isometries T_s $(s \in S)$.

We have proved the following main theorem:

Theorem. *Let $(T_s)_{s \in S}$ be a compatible semigroup of isometries on a Hilbert space H. Then H can be decomposed into an orthogonal sum $H = \bigoplus_\alpha H_\alpha$ of subspaces reducing all the isometries T_s $(s \in S)$ such that for every α the semigroup $(T_s \mid H_\alpha)_{s \in S}$ is unitarily equivalent to the semigroup $(W_s)_{s \in S}$ defined in Example 2 for some measure $\mu^{(\alpha)}$ and a positive definite function $(c_\varphi^{(\alpha)})_{\varphi \in G}$.*

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