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DIFFERENTIAL GEOMETRY OF SURFACES

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In the recent years, there has been new interest in the study of submanifolds of affine spaces; see [1]. Nevertheless, only some invariants in the equiaffine theory have been considered and there are few papers in the general affine and projective geometries. The present paper is devoted to the systematic study of invariants of surfaces in 3-dimensional spaces. There is just one global theorem which is very general; one may, of course, prove better results under more special suppositions — see [5]–[9].

1. Hyperbolic surfaces in A^3_{eq}. Consider a hyperbolic surface π in the equiaffine 3-space A^3_{eq}. To each point $m \in \pi$ (in a neighborhood of a fixed point $m_0 \in \pi$) let us associate a frame $\{m; v_1, v_2, v_3\}$ such that

$$[v_1, v_2, v_3] = 1\,; \tag{1.1}$$

v_1, v_2 determine the asymptotic directions and we may write

$$\begin{aligned} dm &= \tau^1 v_1 + \tau^2 v_2\,; & dv_1 &= \tau_1^1 v_1 + \tau_1^2 v_2 + \tau^2 v_3\,,\\ dv_2 &= \tau_2^1 v_1 + \tau_2^2 v_2 + \tau^1 v_3\,, & dv_3 &= \tau_3^1 v_1 + \tau_3^2 v_2 + \tau_3^3 v_3\,. \end{aligned} \tag{1.2}$$

From (1.1) we get

$$\tau_1^1 + \tau_2^2 + \tau_3^3 = 0\,, \tag{1.3}$$

and we have the usual integrability conditions

$$d\tau^i = \tau^j \wedge \tau_j^i\,,\quad d\tau_i^j = \tau_i^k \wedge \tau_k^j \tag{1.4}$$

with

$$\tau^3 = 0\,;\quad \tau_1^3 = \tau^2\,,\quad \tau_2^3 = \tau^1\,. \tag{1.5}$$

Let $\{m; \tilde v_1, \tilde v_2, \tilde v_3\}$ be another such field of associated frames; we have

$$\tilde v_1 = \alpha_{11} v_1\,,\quad \tilde v_2 = \alpha_{22} v_2\,,\quad \tilde v_3 = \alpha_{31} v_1 + \alpha_{32} v_2 + \alpha_{33} v_3\,; \tag{1.6}$$

$$\alpha_{11}\alpha_{22}\alpha_{33} = 1\,. \tag{1.7}$$

Here we suppose π to be oriented; otherwise, we have to consider also the changes (1.6′) $\tilde v_1' = \alpha_{12} v_2$, $\tilde v_2' = \alpha_{21} v_1$, etc. From (1.2_1) and $(1.\tilde 2_1)$ we get

$$dm = \tau^1 v_1 + \tau^2 v_2 = \tilde\tau^1 \tilde v_1 + \tilde\tau^2 \tilde v_2\,,$$

i.e.,

$$\tau^1 = \alpha_{11}\tilde{\tau}^1 , \quad \tau^2 = \alpha_{22}\tilde{\tau}^2 . \tag{1.8}$$

Further, using (1.2) and $(1.\tilde{2})$ we have

$$\begin{aligned} d\tilde{v}_1 &= d\alpha_{11}.v_1 + \alpha_{11}(\tau_1^1 v_1 + \tau_1^2 v_2 + \tau^2 v_3) = \\ &= \tilde{\tau}_1^1\alpha_{11}v_1 + \tilde{\tau}_1^2\alpha_{22}v_2 + \tilde{\tau}^2(\alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3) , \\ d\tilde{v}_2 &= d\alpha_{22}.v_2 + \alpha_{22}(\tau_2^1 v_1 + \tau_2^2 v_2 + \tau^1 v_3) = \\ &= \tilde{\tau}_2^1\alpha_{11}v_1 + \tilde{\tau}_2^2\alpha_{22}v_2 + \tilde{\tau}^1(\alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3) , \end{aligned}$$

i.e.,

$$\begin{aligned} &d\alpha_{11} + \alpha_{11}\tau_1^1 = \alpha_{11}\tilde{\tau}_1^1 + \alpha_{31}\tilde{\tau}^2 , \quad \alpha_{11}\tau_1^2 = \alpha_{22}\tilde{\tau}_1^2 + \alpha_{32}\tilde{\tau}^2 , \quad \alpha_{11}\tau^2 = \alpha_{33}\tilde{\tau}^2 , \\ &\alpha_{22}\tau_2^1 = \alpha_{11}\tilde{\tau}_2^1 + \alpha_{31}\tilde{\tau}^1 , \quad d\alpha_{22} + \alpha_{22}\tau_2^2 = \alpha_{22}\tilde{\tau}_2^2 + \alpha_{32}\tilde{\tau}^1 , \quad \alpha_{22}\tau^1 = \alpha_{33}\tilde{\tau}_1 . \end{aligned} \tag{1.9}$$

From (1.8) and $(1.9_{3,6})$ we get $\alpha_{11}\alpha_{22} = \alpha_{33}$, i.e.,

$$\alpha_{11}\alpha_{22} = \alpha_{33} = \varepsilon = \pm 1 . \tag{1.10}$$

The exterior differentiation of (1.5) yields

$$\tau_1^2 \wedge \tau^1 + (\tau_1^1 + \tau_2^2) \wedge \tau^2 = 0 , \quad (\tau_1^1 + \tau_2^2) \wedge \tau^1 + \tau_2^1 \wedge \tau^2 = 0 , \tag{1.11}$$

and we have the existence of functions $A_1, \ldots, A_4$ such that

$$\tau_1^2 = A_1\tau^1 + A_2\tau^2 , \quad \tau_1^1 + \tau_2^2 = A_2\tau^1 + A_3\tau^2 , \quad \tau_2^1 = A_3\tau^1 + A_4\tau^2 . \tag{1.12}$$

From $(1.9_{2,4})$, let us calculate τ_1^2 and τ_2^1, respectively; inserting them into $(1.12_{1,3})$ and using $(1.\tilde{12})$ and (1.8), we get

$$\begin{aligned} \alpha_{11}^{-1}\alpha_{22}(\tilde{A}_1\tilde{\tau}^1 + \tilde{A}_2\tilde{\tau}^2) + \alpha_{11}^{-1}\alpha_{32}\tilde{\tau}^2 &= A_1\alpha_{11}\tilde{\tau}^1 + A_2\alpha_{22}\tilde{\tau}^2 , \\ \alpha_{22}^{-1}\alpha_{11}(\tilde{A}_3\tilde{\tau}^1 + \tilde{A}_1\tilde{\tau}^2) + \alpha_{22}^{-1}\alpha_{31}\tilde{\tau}^1 &= A_3\alpha_{11}\tilde{\tau}^1 + A_4\alpha_{22}\tilde{\tau}^2 \end{aligned}$$

and

$$\begin{aligned} A_1 &= \alpha_{11}^{-2}\alpha_{22}\tilde{A}_1 , \quad A_2 = \alpha_{11}^{-1}\tilde{A}_2 + \alpha_{11}^{-1}\alpha_{22}^{-1}\alpha_{32} , \\ A_3 &= \alpha_{22}^{-1}\tilde{A}_3 + \alpha_{11}^{-1}\alpha_{22}^{-1}\alpha_{31} , \quad A_4 = \alpha_{11}\alpha_{22}^{-2}\tilde{A}_4 . \end{aligned} \tag{1.13}$$

Thus we see that we may specialize the frames in such a way that (1.12) reduce simply to

$$\tau_1^2 = A_1\tau^1 , \quad \tau_1^1 + \tau_2^2 = 0 , \quad \tau_2^1 = A_4\tau^2 , \tag{1.14}$$

and the admissible changes of the frames reduce to

$$\tilde{v}_1 = \alpha_{11}v_1 , \quad \tilde{v}_2 = \alpha_{22}v_2 , \quad \tilde{v}_3 = \varepsilon v_3 ; \quad \alpha_{11}\alpha_{22} = \varepsilon = \pm 1 . \tag{1.15}$$

The differential consequences of (1.14) are

$$\begin{aligned} (dA_1 - 3A_1\tau_1^1) \wedge \tau^1 + \tau_3^2 \wedge \tau^2 &= 0 , \\ \tau_3^2 \wedge \tau^1 + \tau_3^1 \wedge \tau^2 &= 0 , \\ \tau_1^3 \wedge \tau^1 + (dA_4 + 3A_4\tau_1^1) \wedge \tau^2 &= 0 , \end{aligned} \tag{1.16}$$

and, using Cartan's lemma once again, we get

$$(1.17)\qquad \mathrm{d}A_1 - 3A_1\tau_1^1 = B_1\tau^1 + B_2\tau^2\,,\quad \tau_3^2 = B_2\tau^1 + B_3\tau^2\,,$$
$$\tau_3^1 = B_3\tau^1 + B_4\tau^2\,,\quad \mathrm{d}A_4 + 3A_4\tau_1^1 = B_4\tau^1 + B_5\tau^2\,.$$

The equations $(1.9_{1,2,4,5})$ reduce to

$$(1.18)\qquad \mathrm{d}\alpha_{11} + \alpha_{11}\tau_1^1 = \alpha_{11}\tilde{\tau}_1^1\,,\quad \alpha_{11}\tau_1^2 = \alpha_{22}\tilde{\tau}_1^2\,,$$
$$\alpha_{22}\tau_2^1 = \alpha_{11}\tilde{\tau}_2^1\,,\quad \mathrm{d}\alpha_{22} + \alpha_{22}\tau_2^2 = \alpha_{22}\tilde{\tau}_2^2\,;$$

fro $\mathrm{d}\tilde{v}_3$ we get $\tilde{\tau}_3^3 = 0$ and

$$(1.19)\qquad \alpha_{11}\tilde{\tau}_3^1 = \alpha_{33}\tau_3^1\,,\quad \alpha_{22}\tilde{\tau}_3^2 = \alpha_{33}\tau_3^2\,.$$

Inserting into (1.17) and using $(1.\widetilde{17})$, we have

$$(1.20)\qquad B_1 = \alpha_{11}^{-3}\alpha_{22}\tilde{B}_1\,,\quad B_2 = \alpha_{11}^{-2}\tilde{B}_2\,,\quad B_3 = \varepsilon\tilde{B}_3\,,\quad B_4 = \alpha_{22}^{-2}\tilde{B}_4\,,$$
$$B_5 = \alpha_{11}\alpha_{22}^{-3}\tilde{B}_5\,.$$

Lemma 1.1. *Consider a hyperbolic surface* $\pi \subset A^3_{\mathrm{eq}}$. *Locally, we may associate to it frames* $\{m; v_1, v_2, v_3\}$ *such that we have* (1.1) *and* (1.2) *with* (1.3) + (1.5) + + (1.14) + (1.17). *If* $\{m; \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ *is another field of frames with the same properties, we have* (1.15) *and*

$$(1.21)\qquad I := A_1A_4 = \varepsilon\tilde{A}_1\tilde{A}_4\,,$$

$$(1.22)\qquad I_1 := B_3 = \varepsilon\tilde{B}_3\,,\quad I_2 := B_2B_4 = \tilde{B}_2\tilde{B}_4\,,\quad I_3 := B_1B_5 = \tilde{B}_1\tilde{B}_5\,,$$
$$I_4 := B_1B_4^2 = \varepsilon\tilde{B}_1\tilde{B}_4^2\,,\quad I_5 := B_2^2B_5 = \varepsilon\tilde{B}_2^2\tilde{B}_5\,.$$

For the form

$$(1.23)\qquad \mathrm{d}S^2 := 2\tau^1\tau^2$$

we have

$$(1.24)\qquad \mathrm{d}S^2 = \varepsilon\,\mathrm{d}\tilde{S}^2\,.$$

This lemma determines a set of *equiaffine* invariants of the 4th order of our surfaces. The form $\mathrm{d}S^2$ from (1.23) induces an invariant (up to the sign) hyperbolic metric on π. Let us calculate its *Gauss curvature*. We are going to use the following assertion: let $\mathrm{d}\Sigma^2$ be a hyperbolic 2-dimensional metric, and let us write $\mathrm{d}\Sigma^2 = \sigma^1\sigma^2$; then there is exactly one 1-form σ such that $\mathrm{d}\sigma^1 = \sigma^1 \wedge \sigma$, $\mathrm{d}\sigma^2 = \sigma \wedge \sigma^2$, and the Gauss curvature $\varkappa$ is given by $\mathrm{d}\sigma = \frac{1}{2}\varkappa\sigma^1 \wedge \sigma^2$. In our case $\sigma^1 = \sqrt{2}.\tau^1$, $\sigma^2 = = \sqrt{2}.\tau^2$, $\sigma = \tau_1^1$, and we get

$$(1.25)\qquad \varkappa = A_1A_4 - B_3 = I - I_1\,.$$

This equation may be called the *theorema egregium*.

2. Comparison with Blaschke's notation. Let our surface be given (locally) by $m = m(u, v)$, u and v being asymptotic parameters. According to [2], equation (119) on p. 122, we have

$$(2.1)\qquad F^2 = (m_u, m_v, m_{uv})\,;$$

we take $F > 0$ (here we write m instead of x). Then the equations [2] (2) on p. 132 read

(2.2) $$F m_{uu} = F_u m_u + A m_v\,, \quad F m_{vv} = D m_u + F_v m_v\,.$$

Take the frames

(2.3) $$v_1 = F^{-1/2} m_u\,, \quad v_2 = F^{-1/2} m_v\,, \quad v_3 = F^{-1} m_{uv}\,.$$

Then we have (1.1) and

(2.4) $$\begin{aligned} dm &= \tau^1 v_1 + \tau^2 v_2\,, \\ dv_1 &= \tfrac{1}{2} F^{-3/2}(F_u \tau^1 - F_v \tau^2)\, v_1 + F^{-3/2} A \tau^1 v_2 + \tau^2 v_3\,, \\ dv_2 &= F^{-3/2} D \tau^2 v_1 - \tfrac{1}{2} F^{-3/2}(F_u \tau^1 - F_v \tau^2)\, v_2 + \tau^1 v_3\,, \\ dv_3 &= F^{-3}\{(AD + F F_{uv} - F_u F_v)\, \tau^1 + F D_u \tau^2\}\, v_1 + \\ &\quad + F^{-3}\{F A_v \tau^1 + (AD + F F_{uv} - F_u F_v)\, \tau^2\}\, v_2 \end{aligned}$$

with

(2.5) $$\tau^1 = F^{1/2}\, du\,, \quad \tau^2 = F^{1/2}\, dv\,.$$

Comparing with (1.14) and (1.17), we get

(2.6) $$A_1 = F^{-3/2} A\,, \quad A_2 = F^{-3/2} D\,;$$

(2.7) $$\begin{aligned} B_1 &= F^{-3}(F A_u - 3 F_u A) = F(F^{-3} A)_u\,, \quad B_2 = F^{-2} A_v\,, \\ B_3 &= F^3(AD + F F_{uv} - F_u F_v)\,, \quad B_4 = F^{-2} D_u\,, \\ B_5 &= F^{-3}(F D_v - 3 F_v D) = F(F^{-3} D)_v\,. \end{aligned}$$

Thus the Pick invariant I (1.21) equals to

(2.8) $$I = F^{-3} A D\,;$$

compare with [2] (4) on p. 132 or [2] ($c3_1$) on p. 164. Blaschke's curvatures H and K are then

(2.9) $$H = -B_3\,, \quad K = B_3^2 - B_2 B_4\,;$$

see [2] ($c3_3$), p. 164. The invariant form [2] (1), p. 131, is exactly our form (1.23). Notice: the theorema egregium (1.25) is $\varkappa = I + H$ (writing $\varkappa$ instead of Blaschke's S; see [2] (5), p. 132).

Let us determine the equation of the *Lie quadric*. In the local coordinates (X', Y', Z') given by $P = m + X' m_u + Y' m_v + Z' F^{-1} m_{uv}$ (see [2] (48) on p. 222 and (2_2) on p. 132), the equation of the Lie quadric is

(2.10) $$H Z'^2 - 2Z' + 2F X' Y' = 0\,;$$

see [2] (49) on p. 223. From (2.3) we have

(2.11) $$P = m + X' F^{1/2} v_1 + Y' F^{1/2} v_2 + Z' v_3\,.$$

Thus we easily get

Lemma 2.1. *To the hyperbolic surface $\pi \subset A^3_{\mathrm{eq}}$ let us associate a field of frames:*

$\{m; v_1, v_2, v_3\}$ *as described in Lemma* 1.1. *At a fixed point* m_0, *introduce the local coordinates* (X, Y, Z) *by*

$$P = m_0 + Xv_1 + Yv_2 + Zv_3 \, ; \quad v_i = v_i(m_0) \, . \tag{2.12}$$

Then the Lie quadric is the quadric given by

$$2(Z - XY) + B_3 Z^2 = 0 \, . \tag{2.13}$$

3. Hyperbolic surfaces in A^3. To each point $m \in \pi \subset A^3$ let us associate a frame $\{m; v_1, v_2, v_3\}$ such that we have (1.2). Of course, (1.1) does not hold, and we cannot use (1.3). Thus we use the equations (1.5) as our starting point; let us write them once again:

$$\tau^3 = 0 \, ; \quad \tau_3^1 = \tau^2 \, , \quad \tau_2^3 = \tau^1 \, . \tag{3.1}$$

The differential consequences are

$$\begin{aligned} &\tau_1^2 \wedge \tau^1 + \tfrac{1}{2}(\tau_1^1 + \tau_2^2 - \tau_3^3) \wedge \tau^2 = 0 \, , \\ &\tfrac{1}{2}(\tau_1^1 + \tau_2^2 - \tau_3^3) \wedge \tau^1 + \tau_2^1 \wedge \tau^2 = 0 \, , \end{aligned} \tag{3.2}$$

and we get the existence of functions $A_1, \ldots, A_4$ such that

$$\begin{aligned} &\tau_1^2 = A_1\tau^1 + A_2\tau^2 \, , \quad \tau_1^1 + \tau_2^2 - \tau_3^3 = 2(A_2\tau^1 + A_3\tau^2) \, , \\ &\tau_2^1 = A_3\tau^1 + A_4\tau^2 \, . \end{aligned} \tag{3.3}$$

The admissible changes of the frames are

$$\tilde{v}_1 = \alpha_{11}v_1 \, , \quad \tilde{v}_2 = \alpha_{22}v_2 \, , \quad \tilde{v}_3 = \alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3 \, . \tag{3.4}$$

We have

$$\begin{aligned} \mathrm{d}m &= \tau^1 v_1 + \tau^2 v_2 = \tilde{\tau}^1\alpha_{11}v_1 + \tilde{\tau}^2\alpha_{22}v_2 \, , \\ \mathrm{d}\tilde{v}_1 &= \mathrm{d}\alpha_{11} . v_1 + \alpha_{11}(\tau_1^1 v_1 + \tau_1^2 v_2 + \tau^2 v_3) = \\ &= \tilde{\tau}_1^1\alpha_{11}v_1 + \tilde{\tau}_1^2\alpha_{22}v_2 + \tilde{\tau}^2(\alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3) \, , \\ \mathrm{d}\tilde{v}_2 &= \mathrm{d}\alpha_{22} . v_2 + \alpha_{22}(\tau_2^1 v_1 + \tau_2^2 v_2 + \tau^1 v_3) = \\ &= \tilde{\tau}_2^1\alpha_{11}v_1 + \tilde{\tau}_2^2\alpha_{22}v_2 + \tilde{\tau}^1(\alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3) \, , \\ \mathrm{d}\tilde{v}_3 &= \mathrm{d}\alpha_{31} . v_1 + \mathrm{d}\alpha_{32} . v_2 + \mathrm{d}\alpha_{33} . v_3 + \alpha_{31}(\tau_1^1 v_1 + \tau_1^2 v_2 + \tau^2 v_3) + \\ &\quad + \alpha_{32}(\tau_2^1 v_1 + \tau_2^2 v_2 + \tau^1 v_3) + \alpha_{33}(\tau_3^1 v_1 + \tau_3^2 v_2 + \tau_3^3 v_3) = \\ &= \tilde{\tau}_3^1\alpha_{11}v_1 + \tilde{\tau}_3^2\alpha_{22}v_2 + \tilde{\tau}_3^3(\alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3) \, . \end{aligned} \tag{3.5}$$

From (3.5) and the terms at v_3 in $(3.5_{2,3})$ we obtain

$$\tau^1 = \alpha_{11}\tilde{\tau}^1 \, , \quad \tau^2 = \alpha_{22}\tilde{\tau}^2 \tag{3.6}$$

and $\alpha_{11}\tau^2 = \alpha_{33}\tilde{\tau}^2$, $\alpha_{22}\tau^1 = \alpha_{33}\tilde{\tau}^1$. This implies

$$\alpha_{33} = \alpha_{11}\alpha_{22} \, . \tag{3.7}$$

Using (3.3) and $(3.\tilde{3})$, we get

$$\begin{aligned} &A_1 = \alpha_{11}^{-2}\alpha_{22}\tilde{A}_1 \, , \quad A_2 = \alpha_{11}^{-1}\tilde{A}_2 + \alpha_{11}^{-1}\alpha_{22}^{-1}\alpha_{32} \, , \\ &A_3 = \alpha_{22}^{-1}\tilde{A}_3 + \alpha_{11}^{-1}\alpha_{22}^{-1}\alpha_{31} \, , \quad A_4 = \alpha_{11}\alpha_{22}^{-2}\tilde{A}_4 \, . \end{aligned} \tag{3.8}$$

We see that we may choose the frames in such a way that

$$(3.9)\qquad \tau_1^2 = A_1\tau^1\,, \quad \tau_1^1 + \tau_2^2 - \tau_3^3 = 0\,, \quad \tau_2^1 = A_4\tau^2\,,$$

and the admissible changes of the frames are then

$$(3.10)\qquad \tilde{v}_1 = \alpha_{11}v_1\,, \quad \tilde{v}_2 = \alpha_{22}v_2\,, \quad \tilde{v}_3 = \alpha_{33}v_3\,; \quad \alpha_{33} = \alpha_{11}\alpha_{22}\,.$$

The differential consequences of (3.9) are

$$(3.11)\qquad \begin{aligned} \{dA_1 + A_1(\tau_2^2 - 2\tau_1^1)\} \wedge \tau^1 + \tau_3^2 \wedge \tau^2 &= 0\,,\\ \tau_3^2 \wedge \tau^1 + \tau_3^1 \wedge \tau^2 &= 0\,,\\ \tau_3^1 \wedge \tau^1 + \{dA_4 + A_4(\tau_1^1 - 2\tau_2^2)\} \wedge \tau^2 &= 0\,, \end{aligned}$$

and we have

$$(3.12)\qquad \begin{aligned} dA_1 + A_1(\tau_2^2 - 2\tau_1^1) &= B_1\tau^1 + B_2\tau^2\,,\\ \tau_3^2 = B_2\tau^1 + B_3\tau^2\,, \quad \tau_3^1 &= B_3\tau^1 + B_4\tau^2\,,\\ dA_4 + A_4(\tau_1^1 - 2\tau_2^2) &= B_4\tau^1 + B_5\tau^2\,. \end{aligned}$$

Because of $\alpha_{31} = \alpha_{32} = 0$, the equations (3.5) yield

$$(3.13)\qquad \begin{aligned} \tau_3^1 = \alpha_{22}^{-1}\tilde{\tau}_3^1\,, \quad \tau_3^2 &= \alpha_{11}^{-1}\tilde{\tau}_3^2\,,\\ \tau_1^1 = \tilde{\tau}_1^1 - \alpha_{11}^{-1}\,d\alpha_{11}\,, \quad \tau_2^2 &= \tilde{\tau}_2^2 - \alpha_{22}^{-1}\,d\alpha_{22}\,; \end{aligned}$$

using (3.12) and $(3.\tilde{12})$ we get

$$(3.14)\qquad \begin{aligned} B_1 = \alpha_{11}^{-3}\alpha_{22}\tilde{B}_1\,, \quad B_2 = \alpha_{11}^{-2}\tilde{B}_2\,, \quad B_3 &= \alpha_{33}^{-1}\tilde{B}_3\,,\\ B_4 = \alpha_{22}^{-2}\tilde{B}_4\,, \quad B_5 &= \alpha_{11}\alpha_{22}^{-3}\tilde{B}_5\,, \end{aligned}$$

and we have

$$(3.15)\qquad \begin{aligned} \tau^1\tau^2 = \alpha_{33}\tilde{\tau}^1\tilde{\tau}^2\,; \quad I = \alpha_{33}^{-1}\tilde{I}\,, \quad I_1 &= \alpha_{33}^{-1}\tilde{I}_1\,,\\ I_2 = \alpha_{33}^{-2}\tilde{I}_2\,, \quad I_3 = \alpha_{33}^{-2}\tilde{I}_3\,, \quad I_4 = \alpha_{33}^{-3}\tilde{I}_4\,, \quad I_5 &= \alpha_{33}^{-3}\tilde{I}_5\,; \end{aligned}$$

for the definition of I, I_α see (1.21) + (1.22).

Thus we get the following

Lemma 3.1. *Consider a hyperbolic surface $\pi \subset A^3$. Locally, we may associate to it frames $\{m; v_1, v_2, v_3\}$ such that we have (1.2) with (3.1) + (3.9) + (3.12). If $\{m; \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ is another set of frames with the same properties, we have* (3.10) and $(3.8_{1,4})$ + (3.14).

Eliminating α_{11}, α_{22} from $(3.8_{1,4})$ + (3.14), we get all affine invariants up to order 4 of our surface. In particular, we obtain

Proposition 3.1. *The forms*

$$(3.16)\qquad I\,dS^2,\ I_1\,dS^2,\ I_2(dS^2)^2,\ I_3(dS^2)^2,\ I_4(dS^2)^3,\ I_5(dS^2)^3$$

are not only equiaffine but also affine invariants of our surface π.

4. Hyperbolic surfaces in P^3. In the projective space, our frames consist from four analytic points $\{m_0 = m, m_1, m_2, m_3\}$ such that

(4.1) $$[m_0, m_1, m_2, m_3] = 1\,,$$

and we have

(4.2) $$\mathrm{d}m_\alpha = \tau_\alpha^\beta m_\beta\,; \quad \alpha, \beta \ldots = 0, \ldots, 3\,;$$

with the integrability conditions

(4.3) $$\mathrm{d}\tau_\alpha^\beta = \tau_\alpha^\gamma \wedge \tau_\gamma^\beta\,.$$

Let the frames be chosen in such a way that the straight lines $\{m_0, m_1\}$, $\{m_0, m_2\}$ are the asymptotic tangents. Writing, as usual,

(4.4) $$\tau^1 := \tau_0^1\,, \quad \tau^2 := \tau_0^2\,,$$

(4.1) implies

(4.5) $$\tau_0^0 + \tau_1^1 + \tau_2^2 + \tau_3^3 = 0$$

and we have the equations

(4.6) $$\tau_0^3 = 0\,; \quad \tau_1^3 = \tau^2\,, \quad \tau_2^3 = \tau^1$$

as our starting point.

The differential consequences are

(4.7) $$\tau_1^2 \wedge \tau^1 + (\tau_1^1 + \tau_2^2) \wedge \tau^2 = 0\,, \quad (\tau_1^1 + \tau_2^2) \wedge \tau^1 + \tau_2^1 \wedge \tau^2 = 0\,,$$

and it is possible to show that we may choose the frames in such a way that

(4.8) $$\tau_1^2 = A_1\tau^1\,, \quad \tau_1^1 + \tau_2^2 = 0\,, \quad \tau_2^1 = A_4\tau^2\,.$$

The exterior differentiation yields the relations

(4.9) $$\begin{aligned} \{\mathrm{d}A_1 + A_1(\tau_0^0 - 3\tau_1^1)\} \wedge \tau^1 + (\tau_3^2 - \tau_1^0) \wedge \tau^2 &= 0\,,\\ (\tau_3^2 - \tau_1^0) \wedge \tau^1 + (\tau_3^1 - \tau_2^0) \wedge \tau^2 &= 0\,,\\ (\tau_3^1 - \tau_2^0) \wedge \tau^1 + \{\mathrm{d}A_4 + A_4(\tau_0^0 + 3\tau_1^1)\} \wedge \tau^2 &= 0 \end{aligned}$$

and the existence of functions $B_1, \ldots, B_5$ such that

(4.10) $$\begin{aligned} \mathrm{d}A_1 + A_1(\tau_0^0 - 3\tau_1^1) &= B_1\tau^1 + B_2\tau^2\,,\\ \tau_3^2 - \tau_1^0 = B_2\tau^1 + B_3\tau^2\,, \quad \tau_3^1 - \tau_2^0 &= B_3\tau^1 + B_4\tau^2\,,\\ \mathrm{d}A_4 + A_4(\tau_0^0 + 3\tau_1^1) &= B_4\tau^1 + B_5\tau^2\,. \end{aligned}$$

Let $\{\tilde{m}_0, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3\}$ be another frame satisfying the equations (4.6) + (4.8) + + (4.10). Then

(4.11) $$\begin{aligned} \tilde{m}_0 = \alpha_{00}m_0\,, \quad \tilde{m}_1 = \alpha_{10}m_0 + \alpha_{11}m_1\,, \quad \tilde{m}_2 = \alpha_{20}m_0 + \alpha_{22}m_2\,,\\ \tilde{m}_3 = \alpha_{30}m_0 + \alpha_{31}m_1 + \alpha_{32}m_2 + \alpha_{33}m_3 \end{aligned}$$

with

(4.12) $$\alpha_{00}\alpha_{11}\alpha_{22}\alpha_{33} = 1\,.$$

From

$$d\tilde{m}_0 = d\alpha_{00}.m_0 + \alpha_{00}(\tau_0^0 m_0 + \tau^1 m_1 + \tau^2 m_2) =$$
$$= \tilde{\tau}_0^0 \alpha_{00} m_0 + \tilde{\tau}^1(\alpha_{10} m_0 + \alpha_{11} m_1) + \tilde{\tau}^2(\alpha_{20} m_0 + \alpha_{22} m_2)$$

we get

(4.13) $$\tau^1 = \alpha_{00}^{-1}\alpha_{11}\tilde{\tau}^1, \quad \tau^2 = \alpha_{00}^{-1}\alpha_{22}\tilde{\tau}^2;$$

(4.14) $$d\alpha_{00} + \alpha_{00}\tau_0^0 = \alpha_{00}\tilde{\tau}_0^0 + \alpha_{10}\tilde{\tau}^1 + \alpha_{20}\tilde{\tau}^2.$$

Further,

$$d\tilde{m}_1 = d\alpha_{10}.m_0 + d\alpha_{11}.m_1 + \alpha_{10}(\tau_0^0 m_0 + \tau^1 m_1 + \tau^2 m_2) +$$
$$+ \alpha_{11}(\tau_1^0 m_0 + \tau_1^1 m_1 + \tau_1^2 m_2 + \tau^2 m_3) =$$
$$= \tilde{\tau}_1^0 \alpha_{00} m_0 + \tilde{\tau}_1^1(\alpha_{10} m_0 + \alpha_{11} m_1) + \tilde{\tau}_1^2(\alpha_{20} m_0 + \alpha_{22} m_2) +$$
$$+ \tilde{\tau}(\alpha_{30} m_0 + \alpha_{31} m_1 + d_{32} m_2 + \alpha_{33} m_3),$$

$$d\tilde{m}_2 = d\alpha_{20}.m_0 + d\alpha_{22}.m_2 + \alpha_{20}(\tau_0^0 m_0 + \tau^1 m_1 + \tau^2 m_2) +$$
$$+ \alpha_{22}(\tau_2^0 m_0 + \tau_2^1 m_1 + \tau_2^2 m_2 + \tau^1 m_3) =$$
$$= \tilde{\tau}_0^2 \alpha_{00} m_0 + \tilde{\tau}_2^1(\alpha_{10} m_0 + \alpha_{11} m_1) + \tilde{\tau}_2^2(\alpha_{20} m_0 + \alpha_{22} m_2) +$$
$$+ \tilde{\tau}^1(\alpha_{30} m_0 + \alpha_{31} m_1 + \alpha_{32} m_2 + \alpha_{33} m_3),$$

$$d\tilde{m}_3 = d\alpha_{30}.m_0 + d\alpha_{31}.m_1 + d\alpha_{32}.m_2 + d\alpha_{33}.m_3 +$$
$$+ \alpha_{30}(\tau_0^0 m_0 + \tau^1 m_1 + \tau^2 m_2) + \alpha_{31}(\tau_1^0 m_0 + \tau_1^1 m_1 + \tau_1^2 m_2 + \tau^2 m_3) +$$
$$+ \alpha_{32}(\tau_2^0 m_0 + \tau_2^1 m_1 + \tau_2^2 m_2 + \tau^1 m_3) +$$
$$+ \alpha_{33}(\tau_3^0 m_0 + \tau_3^1 m_1 + \tau_3^2 m_2 + \tau_3^3 m_3) =$$
$$= \tilde{\tau}_3^0 \alpha_{00} m_0 + \tilde{\tau}_3^1(\alpha_{10} m_0 + \alpha_{11} m_1) + \tilde{\tau}_3^2(\alpha_{20} m_0 + \alpha_{22} m_2) +$$
$$+ \tilde{\tau}_3^3(\alpha_{30} m_0 + \alpha_{31} m_1 + \alpha_{32} m_2 + \alpha_{33} m_3).$$

Comparing the terms at m_3 in $d\tilde{m}_1$ and $d\tilde{m}_2$, we get

(4.15) $$\alpha_{11}\tau^2 = \alpha_{22}\tilde{\tau}^2, \quad \alpha_{22}\tau^1 = \alpha_{33}\tilde{\tau}^1;$$

(4.15) and (4.13) imply $\alpha_{00}\alpha_{33} = \alpha_{11}\alpha_{22}$ and, because of (4.12),

(4.16) $$\alpha_{00}\alpha_{33} = \alpha_{11}\alpha_{22} = \varepsilon = \pm 1.$$

Comparing the terms at m_2 in $d\tilde{m}_1$ and at m_1 in $d\tilde{m}_2$, we get (using (4.13) and (4.8))

$$\alpha_{10}\alpha_{00}^{-1}\alpha_{22}\tilde{\tau}^2 + \alpha_{11}^2 A_1 \alpha_{00}^{-1}\tilde{\tau}^1 = \alpha_{22}\tilde{A}_1\tilde{\tau}^1 + \alpha_{32}\tilde{\tau}^2,$$
$$\alpha_{20}\alpha_{00}^{-1}\alpha_{11}\tilde{\tau}^1 + \alpha_{22}^2 A_4 \alpha_{00}^{-1}\tilde{\tau}^2 = \alpha_{11}\tilde{A}_4\tilde{\tau}^2 + \alpha_{31}\tilde{\tau}^1,$$

i.e.,

(4.17) $$A_1 = \alpha_{00}\alpha_{11}^{-2}\alpha_{22}\tilde{A}_1, \quad A_4 = \alpha_{00}\alpha_{11}\alpha_{22}^{-2}\tilde{A}_4;$$

(4.18) $$\alpha_{31} = \alpha_{00}^{-1}\alpha_{11}\alpha_{20}, \quad \alpha_{32} = \alpha_{00}^{-1}\alpha_{22}\alpha_{10}.$$

Comparing the remaining coefficients in $d\tilde{m}_1$, $d\tilde{m}_2$, $d\tilde{m}_3$ we get

(4.19) $$\alpha_{10}\tau^1 + d\alpha_{11} + \alpha_{11}\tau_1^1 = \alpha_{11}\tilde{\tau}_1^1 + \alpha_{31}\tilde{\tau}^2,$$
$$\alpha_{20}\tau^2 + d\alpha_{22} + \alpha_{22}\tau_2^2 = \alpha_{22}\tilde{\tau}_2^2 + \alpha_{32}\tilde{\tau}^1,$$
$$d\alpha_{31} + \alpha_{30}\tau^1 + \alpha_{31}\tau_1^1 + \alpha_{32}\tau_2^1 + \alpha_{33}\tau_3^1 = \alpha_{11}\tilde{\tau}_3^2 + \alpha_{31}\tilde{\tau}_3^3,$$

$$d\alpha_{32} + \alpha_{30}\tau^2 + \alpha_{31}\tau_1^2 + \alpha_{32}\tau_2^2 + \alpha_{33}\tau_3^2 = \alpha_{22}\tilde{\tau}_3^2 + \alpha_{32}\tilde{\tau}_3^3 ,$$
$$d\alpha_{33} + \alpha_{31}\tau^2 + \alpha_{32}\tau^1 + \alpha_{33}\tau_3^3 = \alpha_{33}\tilde{\tau}_3^3 ,$$
$$d\alpha_{10} + \alpha_{10}\tau_0^0 + \alpha_{11}\tau_1^0 = \alpha_{00}\tilde{\tau}_1^0 + \alpha_{10}\tilde{\tau}_1^1 + \alpha_{20}\tilde{\tau}_1^2 + \alpha_{30}\tilde{\tau}^2 ,$$
$$d\alpha_{20} + \alpha_{20}\tau_0^0 + \alpha_{22}\tau_2^0 = \alpha_{00}\tilde{\tau}_2^0 + \alpha_{10}\tilde{\tau}_2^1 + \alpha_{20}\tilde{\tau}_2^2 + \alpha_{30}\tilde{\tau}^1 .$$

Using these relations and taking into account (4.18) and (4.17), we get

$$(4.20)\quad B_1 = \alpha_{00}^2\alpha_{11}^{-3}\alpha_{22}\tilde{B}_1 + 4\alpha_{00}\alpha_{11}^{-3}\alpha_{22}\alpha_{10}\tilde{A}_1 , \quad B_2 = \alpha_{00}^2\alpha_{11}^{-2}\tilde{B}_2 - 2\alpha_{00}\alpha_{11}^{-2}\alpha_{20}\tilde{A}_1 ,$$
$$B_4 = \alpha_{00}^2\alpha_{22}^{-2}\tilde{B}_4 - 2\alpha_{00}\alpha_{22}^{-2}\alpha_{10}\tilde{A}_4 , \quad B_5 = \alpha_{00}^2\alpha_{11}\alpha_{22}^{-3}\tilde{B}_5 + 4\alpha_{00}\alpha_{11}\alpha_{22}^{-3}\alpha_{20}\tilde{A}_4$$

from $(4.10_{1,4})$ and $(4.\tilde{1}0_{1,4})$. Introduce the functions

$$(4.21)\qquad C_1 := A_4B_1 + 2A_1B_4 , \quad C_2 := A_1B_5 + 2A_4B_2 ;$$

then

$$(4.22)\qquad C_1 = \alpha_{00}^3\alpha_{11}^{-2}\alpha_{22}^{-1}\tilde{C}_1 , \quad C_2 = \alpha_{00}^3\alpha_{11}^{-1}\alpha_{22}^{-2}\tilde{C}_2 ,$$

and we have eliminated α_{10}, α_{20} from (4.20).

Lemma 4.1. *Consider a hyperbolic surface* $\pi \subset P^3$. *Locally, we may associate to it frames* $\{m = m_0, m_1, m_2, m_3\}$ *such that we have* (4.1) *and* (4.2) *with* (4.5) + + (4.6) + (4.8). *The admissible changes of the frames are then* (4.11) *with* (4.16) + + (4.18).

Proposition 4.1. *The forms*

$$(4.23)\qquad I\, dS^2 , \quad C_1(\tau^1)^2\,\tau^2 , \quad C_2\,\tau^1(\tau^2)^2$$

are not only affine but also projective invariants of our surface. We get the projective scalar invariants up to order 4 *by eliminating* α_{00}, α_{11} *from*

$$(4.24)\quad A_1 = \varepsilon\alpha_{00}\alpha_{11}^{-3}\tilde{A}_1 , \quad A_4 = \alpha_{00}\alpha_{11}^3\tilde{A}_4 , \quad C_1 = \varepsilon\alpha_{00}^3\alpha_{11}^{-1}\tilde{C}_1 , \quad C_2 = \alpha_{00}^3\alpha_{11}\tilde{C}_2 .$$

It is known that the area element $I\tau^1 \wedge \tau^2$ is a projective invariant; see [2], p. 174, Aufgabe 8.

5. Canonical lines. Consider a hyperbolic surface $\pi \subset A^3_{eq}$, and consider the equations (1.5) + (1.14) + (1.17). Let $m = m(u, v)$, u and v being the asymptotic parameters, and take (locally)

$$(5.1)\qquad \tau^1 = r\, du , \quad \tau^2 = s\, dv ; \quad r = r(u, v) > 0 , \quad s = s(u, v) > 0 .$$

From

$$(5.2)\qquad d\tau^1 = \tau^1 \wedge \tau_1^1 , \quad d\tau^2 = -\tau^2 \wedge \tau_1^1$$

we get

$$(5.3)\qquad \tau_1^1 = s^{-1}s_u\, du - r^{-1}r_v\, dv .$$

From (5.1) we obtain

$$(5.4)\qquad m_u = rv_1 , \quad m_v = sv_2$$

and

(5.5) $$m_{uu} = \left(r_u + rs^{-1}s_u\right) v_1 + A_1 r^2 v_2 \,,$$
$$m_{vv} = A_4 s^2 v_1 + \left(s_v + r^{-1} s r_v\right) v_2 \,, \quad m_{uv} = rsv_3 \,.$$

Consequently,

(5.6) $$m_{uu} = \left(r^{-1} r_u + s^{-1} s_u\right) m_u + A_1 r^2 s^{-1} m_v \,,$$
$$m_{vv} = A_4 r^{-1} s^2 m_u + \left(r^{-1} r_v + s^{-1} s_v\right) m_v \,.$$

Working in the projective extension of A^3_{eq}, we have the fundamental equations (5.6) in the form

(5.7) $$m_{uu} = \theta_u m_u + \beta m_v + p_{11} m \,, \quad m_{vv} = \gamma m_u + \theta_v m_v + p_{22} m \,;$$

compare with [4] (I_{bis}) on p. 90. Thus, in our case,

(5.8) $$\theta = \log rs \,, \quad \beta = r^2 s^{-1} A_1 \,, \quad \gamma = r^{-1} s^2 A_4 \,, \quad p_{11} = p_{22} = 0 \,.$$

From [4] (10_3) on p. 93 we have

(5.9) $$a_{12} = rs \,.$$

According to [4], § 27 on p. 155, the *canonical line* with the parameter λ (in the case $\beta\gamma \neq 0$!) is the straight line through the points m and

(5.10) $$m_{uv} + \frac{1}{2}\left(\frac{\partial \log a_{12}^{-1}\beta\gamma}{\partial v} m_u + \frac{\partial \log a_{12}^{-1}\beta\gamma}{\partial u} m_v\right) +$$
$$+ \lambda\left(\frac{\partial \log \beta^2\gamma}{\partial v} m_u + \frac{\partial \log \beta\gamma^2}{\partial u} m_v\right);$$

by $\partial \log f/\partial u$ we simply mean $f^{-1} f_u$, etc. Using (5.8) + (5.9), we easily prove

Lemma 5.1. *Let $\pi \subset A^3_{\text{eq}}$ be a hyperbolic surface. Then its canonical line n_λ with the parameter λ (if it exists!) is determined by the point m and the vector*

(5.11) $$v_\lambda := Iv_3 + \tfrac{1}{2}\{(A_1 B_5 + A_4 B_2) v_1 + (A_1 B_4 + A_4 B_1) v_2\} + \lambda(C_2 v_1 + C_1 v_2).$$

The line n_λ is a projective invariant of our surface.

From (1.17),

(5.12) $$\mathrm{d}I = (A_1 B_4 + A_4 B_2)\,\tau^1 + (A_1 B_5 + A_4 B_2)\,\tau^2 \,.$$

Thus the line n_0 is determined by the vector v_3 if and only if $\mathrm{d}J = 0$. Because v_3 determines the direction of the (equi)affine normal of π, we have re-proved the known assertion: the (equi)affine normal coincides with the projective Fubini normal at each point of π if and only if I = const. on π. See also [3], p. 111, Aufgabe 3.

Let us suppose $I \neq 0$ on π. Then, see (4.24),

(5.13) $$K_1 := A_1^{-5} A_4^{-4} C_1^3 \,, \quad K_2 := A_1^{-4} A_4^{-5} C_2^3$$

are the fundamental projective invariants of the 4th order of our surface. Using

Blaschke's notation (2.2) and (2.8), we easily see that

$$(5.14)\qquad K_1 = \frac{F^3}{A^2 D}\left\{\left(\log \frac{AD^2}{F^3}\right)_u\right\}^3, \quad K_2 = \frac{F^3}{AD^2}\left\{\left(\log \frac{A^2 D}{F^3}\right)_v\right\}^3.$$

In the Fubini-Čech notation (5.7) we have

$$(5.15)\qquad K_1 = \beta^{-2}\gamma^{-1}\{(\log \beta\gamma^2)_u\}^3, \quad K_2 = \beta^{-1}\gamma^{-1}\{(\log \beta^2\gamma)_v\}^3,$$

and we see that K_1 and K_2 are even invariants with respect to the projective deformations of our surface.

6. Elliptic surfaces in A^3_{eq}. Let $\pi \subset A^3_{\text{eq}}$ be an elliptic surface. To each point $m \in \pi$ let us associate a frame $\{m; e_1, e_2, e_3\}$ such that

$$(6.1)\qquad [e_1, e_2, e_3] = 1,$$

and we have the fundamental equations

$$(6.2)\qquad dm = \omega^1 v_1 + \omega^2 v_2, \quad de_i = \omega_i^j e_j$$

with the usual integrability conditions and the condition

$$(6.3)\qquad \omega_1^1 + \omega_2^2 + \omega_3^3 = 0;$$

compare with (1.1)–(1.4). It is easy to see to see that the frames may be chosen in such a way that

$$(6.4)\qquad \omega^3 = 0; \quad \omega_1^3 = \omega^1, \quad \omega_2^3 = \omega^2.$$

The differential consequences are

$$(6.5)\qquad \begin{aligned} (2\omega_1^1 - \omega_3^3) \wedge \omega^1 + (\omega_1^2 + \omega_2^1) \wedge \omega^2 &= 0, \\ (\omega_1^2 + \omega_2^1) \wedge \omega^1 + (2\omega_2^2 - \omega_3^3) \wedge \omega^2 &= 0, \end{aligned}$$

and we get the existence of functions $a_1, \ldots, a_4$ such that

$$(6.6)\qquad \begin{aligned} 2\omega_1^1 - \omega_3^3 = a_1\omega^1 + a_2\omega^2, \quad \omega_1^2 + \omega_2^1 = a_2\omega^1 + a_3\omega^2, \\ 2\omega_2^2 - \omega_3^3 = a_3\omega^1 + a_4\omega^2. \end{aligned}$$

Let $\{m; \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ be another field of frames associated to our surface, let us suppose that it satisfies the equations $(6.\tilde{1})$–$(6.\tilde{4})$ and

$$(6.7)\qquad \tilde{e}_1 = a_{11}e_1 + a_{12}e_2, \quad \tilde{e}_2 = a_{21}e_1 + a_{22}e_2, \quad \tilde{e}_3 = a_{31}e_1 + a_{32}e_2 + a_{33}e_3$$

with

$$(6\,8)\qquad (a_{11}a_{22} - a_{12}a_{21})\, a_{33} = 1.$$

We have

$$(6.9)\qquad dm = \omega^1 e_1 + \omega^2 e_2 = \tilde{\omega}^1(a_{11}e_1 + a_{12}e_2) + \tilde{\omega}^2(a_{21}e_1 + a_{22}e_2),$$

i.e.,

$$(6.10)\qquad \omega^1 = a_{11}\tilde{\omega}^1 + a_{21}\tilde{\omega}^2, \quad \omega^2 = a_{12}\tilde{\omega}^1 + a_{22}\tilde{\omega}^2.$$

Further,

$$(6.11)\qquad \begin{aligned} d\tilde{e}_1 &\equiv (a_{11}\omega^1 + a_{12}\omega^2)\, e_3 \equiv \tilde{\omega}^1 a_{33} e_3, \\ d\tilde{e}_2 &\equiv (a_{21}\omega^1 + a_{22}\omega^2)\, e_3 \equiv \tilde{\omega}^2 a_{33} e_3 \quad (\text{mod } e_1, e_2) \end{aligned}$$

and, by virtue of (6.10),

$$a_{11}^2 + a_{12}^2 = a_{21}^2 + a_{22}^2 = a_{33}\,, \quad a_{11}a_{21} + a_{12}a_{22} = 0\,. \tag{6.12}$$

Thus $a_{33} > 0$, and there is a function φ such that

$$\begin{aligned} a_{11} &= \sqrt{a_{33}}.\cos\varphi\,, \quad a_{12} = -\sqrt{a_{33}}.\sin\varphi\,, \\ a_{21} &= \varepsilon\sqrt{a_{33}}.\sin\varphi\,, \quad a_{22} = \varepsilon\sqrt{a_{33}}.\cos\varphi\,; \quad \varepsilon = \pm 1\,. \end{aligned} \tag{6.13}$$

Inserting into (6.8) we get $\varepsilon a_{33}^2 = 1$, i.e.,

$$\varepsilon = 1\,, \quad a_{33} = 1\,. \tag{6.14}$$

After elementary calculations (comparing the terms at e_j in $\mathrm{d}\tilde{e}_i$), we get

$$\begin{aligned} &\cos^2\varphi.\omega_1^1 - \sin\varphi\cos\varphi.(\omega_2^1 + \omega_1^2) + \sin^2\varphi.\omega_2^2 = \\ &\quad = \tilde{\omega}_1^1 + (a_{31}\cos\varphi - a_{32}\sin\varphi)\,\tilde{\omega}^1\,, \\ &-\mathrm{d}\varphi + \sin\varphi\cos\varphi.(\omega_1^1 - \omega_2^2) - \sin^2\varphi.\omega_2^1 + \cos^2\varphi.\omega_1^2 = \\ &\quad = \tilde{\omega}_1^2 + (a_{31}\sin\varphi + a_{32}\cos\varphi)\,\tilde{\omega}^1\,, \\ &\mathrm{d}\varphi + \sin\varphi\cos\varphi.(\omega_1^1 - \omega_2^2) + \cos^2\varphi.\omega_2^1 - \sin^2\varphi.\omega_1^2 = \\ &\quad = \tilde{\omega}_2^1 + (a_{31}\cos\varphi - a_{32}\sin\varphi)\,\tilde{\omega}^2\,, \\ &\sin^2\varphi.\omega_1^1 + \sin\varphi\cos\varphi.(\omega_2^1 + \omega_1^2) + \cos^2\varphi.\omega_2^2 = \\ &\quad = \tilde{\omega}_2^2 + (a_{31}\sin\varphi + a_{32}\cos\varphi)\,\tilde{\omega}^2\,, \\ &a_{31}(\cos\varphi.\tilde{\omega}^1 + \sin\varphi.\tilde{\omega}^2) + a_{32}(-\sin\varphi.\tilde{\omega}^1 + \cos\varphi.\tilde{\omega}^2) + \omega_3^3 = \tilde{\omega}_3^3\,. \end{aligned} \tag{6.15}$$

Considering the analogous equations $(6.\tilde{6})$, we finally get

$$\begin{aligned} \tilde{a}_1 + \tilde{a}_3 &= \cos\varphi(a_1 + a_3) - \sin\varphi(a_2 + a_4) - 4(a_{31}\cos\varphi - a_{32}\sin\varphi)\,, \\ \tilde{a}_2 + \tilde{a}_4 &= \sin\varphi(a_1 + a_3) + \cos\varphi(a_2 + a_4) - 4(a_{31}\sin\varphi + a_{32}\cos\varphi)\,. \end{aligned} \tag{6.16}$$

Hence we have

Lemma 6.1. *Let $\pi \subset A_{\mathrm{eq}}^3$ be an elliptic surface. Locally, we may associate to it frames $\{m; e_1, e_2, e_3\}$ such that we have (6.1) and (6.2) with (6.3,) (6.4) and*

$$\begin{aligned} 2\omega_1^1 - \omega_3^3 = -a_3\omega^1 + a_2\omega^2\,, \quad \omega_1^2 + \omega_2^1 = a_2\omega^1 + a_3\omega^2\,, \\ 2\omega_2^2 - \omega_3^3 = a_3\omega^1 - a_2\omega^2\,. \end{aligned} \tag{6.17}$$

If $\{m; \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is another field of frames with the same properties, we have

$$\tilde{e}_1 = \cos\varphi.e_1 - \sin\varphi.e_2\,, \quad \tilde{e}_2 = \sin\varphi.e_1 + \cos\varphi.e_2\,, \quad \tilde{e}_3 = e_3\,. \tag{6.18}$$

Thus the straight line $n = \{m + te_3;\ t \in \mathbb{R}\}$ is an equiaffine invariant of our surface; let us call it the *equiaffine normal* of π. Further, the equations (6.10) read

$$\omega^1 = \cos\varphi.\tilde{\omega}^1 + \sin\varphi.\tilde{\omega}^2\,, \quad \omega^2 = -\sin\varphi.\tilde{\omega}^1 + \cos\varphi.\tilde{\omega}^2\,, \tag{6.19}$$

and the form

$$\mathrm{d}s^2 := (\omega^1)^2 + (\omega^2)^2 \tag{6.20}$$

is the invariant *equiaffine metric* of π. Using (6.18), i.e., $a_{31} = a_{32} = 0$, the equations (6.25) yield

(6.21) $$\tilde{a}_2 = \cos 3\varphi . a_2 - \sin 3\varphi . a_3 , \quad \tilde{a}_3 = \sin 3\varphi . a_2 + \cos 3\varphi . a_3 ,$$

i.e.,

(6.22) $$\tilde{a}_2^2 + \tilde{a}_3^2 = a_2^2 + a_3^2 .$$

The equiaffine invariant

(6.23) $$J := \tfrac{1}{2}(a_2^2 + a_3^2)$$

is called the *Pick invariant.*

Define

(6.24) $$\omega := \tfrac{1}{2}(\omega_1^2 - \omega_2^1) ;$$

the equations (6.17) and (6.3) yield

(6.25) $$\omega_1^2 = \tfrac{1}{2}(a_2\omega^1 + a_3\omega^2) + \omega , \quad \omega_2^1 = \tfrac{1}{2}(a_2\omega^1 + a_3\omega^2) - \omega ,$$
$$\omega_1^1 = -\omega_2^2 = -\tfrac{1}{2}(a_3\omega^1 - a_2\omega^2) , \quad \omega_3^3 = 0 .$$

The differential consequences of (6.17) are

(6.26) $$-(\mathrm{d}a_3 + 3a_2\omega - 3\omega_3^1) \wedge \omega^1 + (\mathrm{d}a_2 - 3a_3\omega + \omega_3^2) \wedge \omega^2 = 0 ,$$
$$(\mathrm{d}a_2 - 3a_3\omega + \omega_3^2) \wedge \omega^1 + (\mathrm{d}a_3 + 3a_2\omega + \omega_3^1) \wedge \omega^2 = 0 ,$$
$$(\mathrm{d}a_3 + 3a_2\omega + \omega_3^1) \wedge \omega^1 - (\mathrm{d}a_2 - 3a_3\omega - 3\omega_3^2) \wedge \omega^2 = 0 ,$$

and we get the existence of functions $b_1, \ldots, b_5$ such that

(6.27) $$-\mathrm{d}a_3 - 3a_2\omega + 3\omega_3^1 = b_1\omega^1 + b_2\omega^2 ,$$
$$\mathrm{d}a_2 - 3a_3\omega + \omega_3^2 = b_2\omega^1 + b_3\omega^2 ,$$
$$\mathrm{d}a_3 + 3a_2\omega + \omega_3^1 = b_3\omega^1 + b_4\omega^2 ,$$
$$-\mathrm{d}a_2 + 3a_3\omega + 3\omega_3^2 = b_4\omega^1 + b_5\omega^2 ,$$

i.e.,

(6.28) $$\omega_3^1 = \tfrac{1}{4}(b_1 + b_3)\,\omega^1 + \tfrac{1}{4}(b_2 + b_4)\,\omega^2 ,$$
$$\omega_3^2 = \tfrac{1}{4}(b_2 + b_4)\,\omega^1 + \tfrac{1}{4}(b_3 + b_5)\,\omega^2 ,$$
$$\mathrm{d}a_2 - 3a_3\omega = \tfrac{1}{4}(3b_2 - b_4)\,\omega^1 + \tfrac{1}{4}(3b_3 - b_5)\,\omega^2 ,$$
$$\mathrm{d}a_3 + 3a_2\omega = \tfrac{1}{4}(3b_3 - b_1)\,\omega^1 + \tfrac{1}{4}(3b_4 - b_2)\,\omega^2 .$$

Using these formulas, we reduce the system (6.26) to

(6.29) $$(\mathrm{d}a_2 - 3a_3\omega) \wedge \omega^1 + (\mathrm{d}a_3 + 3a_2\omega) \wedge \omega^2 = \tfrac{1}{4}(b_5 - b_1)\,\omega^1 \wedge \omega^2 ,$$
$$(\mathrm{d}a_3 + 3a_2\omega) \wedge \omega^1 - (\mathrm{d}a_2 - 3a_3\omega) \wedge \omega^2 = -\tfrac{1}{2}(b_2 + b_4)\,\omega^1 \wedge \omega^2 .$$

The exterior differentiation of (6.28) yields

(6.30) $$(\mathrm{D}b_1 + \mathrm{D}b_3) \wedge \omega^1 + (\mathrm{D}b_2 + \mathrm{D}b_4) \wedge \omega^2 =$$
$$= \{\tfrac{1}{2}a_2(b_1 - b_5) + a_3(b_2 + b_4)\}\,\omega^1 \wedge \omega^2 ,$$
$$(\mathrm{D}b_2 + \mathrm{D}b_4) \wedge \omega^1 + (\mathrm{D}b_3 + \mathrm{D}b_5) \wedge \omega^2 =$$
$$= \{\tfrac{1}{2}a_3(b_1 - b_5) - a_2(b_2 + b_4)\}\,\omega^1 \wedge \omega^2 ,$$

$$(3\mathrm{D}b_2 - \mathrm{D}b_4) \wedge \omega^1 + (3\mathrm{D}b_3 - \mathrm{D}b_5) \wedge \omega^2 = 12a_3\varkappa\omega^1 \wedge \omega^2 ,$$
$$(3\mathrm{D}b_3 - \mathrm{D}b_1) \wedge \omega^1 + (3\mathrm{D}b_4 - \mathrm{D}b_2) \wedge \omega^2 = -12a_2\varkappa\omega^1 \wedge \omega^2$$

with

(6.31) $$\mathrm{D}b_1 = \mathrm{d}b_1 - 4b_2\omega , \quad \mathrm{D}b_2 = \mathrm{d}b_2 + (b_1 - 3b_3)\,\omega ,$$
$$\mathrm{D}b_3 = \mathrm{d}b_3 + 2(b_2 - b_4)\,\omega , \quad \mathrm{D}b_4 = \mathrm{d}b_4 + (3b_3 - b_5)\,\omega ,$$
$$\mathrm{D}b_5 = \mathrm{d}b_5 + 4b_4\omega ,$$

(6.32) $$\varkappa = \tfrac{1}{2}(a_2^2 + a_3^2) - \tfrac{1}{8}(b_1 + 2b_3 + b_5) .$$

It is easy to see that

(6.33) $$\mathrm{d}\omega^1 = -\omega^2 \wedge \omega , \quad \mathrm{d}\omega^2 = \omega^1 \wedge \omega ,$$

(6.34) $$\mathrm{d}\omega = -\varkappa\omega^1 \wedge \omega^2 .$$

Thus $\varkappa$ is the *Gauss curvature* of the equiaffine metric $\mathrm{d}s^2$ from (6.20). Because of (6.32) and (6.32),

(6.35) $$H := \tfrac{1}{8}(b_1 + 2b_3 + b_5)$$

is an equiaffine invariant of π; let us call it the *equiaffine mean curvature*. Under this notation, the equation (6.32) reads

(6.36) $$\varkappa = J + H ,$$

and it may be called the *theorema egregium*.

7. Invariants of elliptic surfaces. There are many ways how to obtain the invariants (of order 4) of our surface. One of them is to continue the calculations of the beginning of the last section and to follow the procedure as indicated in the first section. In what follows, I am going to explain other possibilities.

Given an elliptic surface $\pi \subset A^3_{\mathrm{eq}}$, let a field of frames $\{m; e_1, e_2, e_3\}$ be chosen as indicated in Lemma 6.1. Let ${}^cA^3_{\mathrm{eq}}$ be the complexification of A^3_{eq}, and let us consider the frames

(7.1) $$v_1 = \alpha(e_1 - \mathrm{i}e_2) , \quad v_2 = \alpha(e_1 + ie_2) , \quad v_3 = \beta e_3$$

with

(7.2) $$\beta = 2\alpha^2 , \quad \beta^2 = -\mathrm{i} .$$

Then

(7.3) $$[v_1, v_2, v_3] = 1 .$$

From (7.1), we have

(7.4) $$e_1 = \tfrac{1}{2}\alpha^{-1}(v_1 + v_2) , \quad e_2 = \tfrac{1}{2}\mathrm{i}\alpha^{-1}(v_1 - v_2) , \quad e_3 = \beta^{-1}v_3 .$$

Further,

(7.5) $$\mathrm{d}m = \tau^1 v_1 + \tau^2 v_2$$

with

(7.6) $$\tau^1 = \tfrac{1}{2}\alpha^{-1}(\omega^1 + \mathrm{i}\omega^2) , \quad \tau^2 = \tfrac{1}{2}\alpha^{-1}(\omega^1 - \mathrm{i}\omega^2) .$$

It is easy to see that

$$(7.7)\qquad dv_1 = \tau_1^1 v_1 + \tau_1^2 v_2 + \tau^2 v_3\,,\quad dv_2 = \tau_2^1 v_1 + \tau_2^2 v_2 + \tau^1 v_3\,,$$
$$dv_3 = \tau_3^1 v_1 + \tau_3^2 v_2 + \tau_3^3 v_3$$

with

$$(7.8)\qquad \begin{aligned} \tau_1^1 &= \tfrac{1}{2}(\omega_1^1 + \omega_2^2) + \tfrac{1}{2}\mathrm{i}(\omega_1^2 - \omega_2^1) = \bar{\tau}_2^2\,,\\ \tau_1^2 &= \tfrac{1}{2}(\omega_1^1 - \omega_2^2) - \tfrac{1}{2}\mathrm{i}(\omega_1^2 + \omega_2^1) = \bar{\tau}_2^1\,,\quad \tau_3^3 = 0\,,\\ \tau_3^1 &= \tfrac{1}{2}\alpha^{-1}\beta(\omega_3^1 + \mathrm{i}\omega_3^2)\,,\quad \tau_3^2 = \tfrac{1}{2}\alpha^{-1}\beta(\omega_3^1 - \mathrm{i}\omega_3^2)\,. \end{aligned}$$

Using (6.25), we obtain

$$(7.9)\qquad \tau_1^2 = -\alpha(a_3 + \mathrm{i}a_2)\,\tau^1\,,\quad \tau_2^1 = -\alpha(a_3 - \mathrm{i}a_2)\,\tau^2\,,$$
$$\tau_1^1 = \mathrm{i}\omega\,,\quad \tau_2^2 = -\mathrm{i}\omega\,,\quad \tau_3^3 = 0\,.$$

Thus we see that the frames $\{m; v_1, v_2, v_3\}$ satisfy (1.2) + (1.3) + (1.5) + (1.14) with

$$(7.10)\qquad A_1 = -\alpha(a_3 + \mathrm{i}a_2)\,,\quad A_4 = -\alpha(a_3 - \mathrm{i}a_2)\,.$$

From (1.17) we get

$$(7.11)\qquad \begin{aligned} B_1 &= \tfrac{1}{8}\beta(b_1 - 6b_3 + b_5) - \tfrac{1}{2}\mathrm{i}\beta(b_2 - b_4)\,,\quad B_5 = \bar{B}_1\,,\\ B_2 &= \tfrac{1}{8}\beta(b_1 - b_5) - \tfrac{1}{4}\mathrm{i}\beta(b_2 + b_4)\,,\quad B_4 = \bar{B}_2\,,\\ B_3 &= \tfrac{1}{8}\beta(b_1 + 2b_3 + b_5)\,. \end{aligned}$$

Now, Lemma 1.1 determines the fundamental invariants of a hyperbolic surface. We have, see (1.21) + (1.22) and (6.23) + (6.35),

$$(7.12)\qquad I = A_1 A_4 = \beta J\,,$$

$$(7.13)\qquad \begin{aligned} I_1 &= B_3 = -\beta H\,,\\ I_2 = B_2 B_4 &= -\tfrac{1}{64}\mathrm{i}J_2\,,\quad I_3 = B_1 B_5 = -\tfrac{1}{64}\mathrm{i}J_3\,,\\ I_4 + I_5 &= \tfrac{1}{256}\beta^3 J_4\,,\quad I_4 - I_5 = \tfrac{1}{64}\mathrm{i}\beta^3 J_5 \end{aligned}$$

with

(7.14)

$$J_2 = (b_1 - b_5)^2 + 4(b_2 + b_4)^2\,,\quad J_3 = (b_1 - 6b_3 + b_5)^2 + 16(b_2 - b_4)^2\,,$$
$$J_4 = (b_1 - 6b_3 + b_5)\{(b_1 - b_5)^2 - 4(b_2 + b_4)^2\} + 16(b_1 - b_5)(b_2^2 - b_4^2)\,,$$
$$J_5 = (b_1 - 6b_3 + b_5)(b_1 - b_5)(b_2 + b_4) + (b_2 - b_4)\{4(b_2 + b_4)^2 - (b_1 - b_5)^2\}.$$

Blaschke's curvature K from (2.9) is then

$$(7.15)\qquad K = -\mathrm{i}K'$$

with

$$(7.16)\qquad K' = \tfrac{1}{16}\{(b_1 + b_3)(b_3 + b_5) - (b_2 + b_4)^2\}\,.$$

Let us remark that

$$(7.17)\qquad H^2 - K' = \tfrac{1}{64}J_2 \geqq 0\,.$$

Proposition 7.1. *Let $\pi \subset A^3_{\mathrm{eq}}$ be an elliptic surface, and let $\{m; e_1, e_2, e_3\}$ be a field* *of associated frames as described in Lemma* 6.1; *the functions* $b_1, \ldots, b_5$ *let be given by* (6.28). *Then* J (6.23), H (6.35) *and* $J_2, \ldots, J_5$ (7.14) *are equiaffine invariants of our surface.*

The affine and projective invariants may be determined by using Propositions 3.1 and 4.1, respectively.

There is still another way to determine the invariants. Given a function

$$F = F(a_2, a_3, b_1, \ldots, b_5)\,, \tag{7.18}$$

then

$$\mathrm{d}F = \frac{\partial F}{\partial a_2}(\mathrm{d}a_2 - 3a_3\omega) + \frac{\partial F}{\partial a_3}(\mathrm{d}a_3 + 3a_2\omega) + \sum_{i=1}^{5}\frac{\partial F}{\partial b_i}\mathrm{D}b_i + \Phi\omega \tag{7.19}$$

with

$$\Phi = 3a_3\frac{\partial F}{\partial a_2} - 3a_2\frac{\partial F}{\partial a_3} + 4b_2\frac{\partial F}{\partial b_1} + (3b_3 - b_1)\frac{\partial F}{\partial b_2} + + 2(b_4 - b_2)\frac{\partial F}{\partial b_3} + (b_5 - 3b_3)\frac{\partial F}{\partial b_4} - 4b_4\frac{\partial F}{\partial b_5}\,, \tag{7.20}$$

the 1-forms $\mathrm{D}b_i$ being defined by (6.31). Because of (6.29) + (6.30), the 1-forms $\mathrm{d}a_2 - 3a_3\omega$, $\mathrm{d}a_3 + 3a_2\omega$ and $\mathrm{D}b_i$ are linear combinations of ω^1, ω^2. Thus we get

Proposition 7.2. *The function* F (7.18) *is an equiaffine invariant of our surface if and only if* $\Phi = 0$, Φ *being defined by* (7.20). *The condition* $F = 0$ *has an equiaffine signification if and only if* $\Phi = 0$ *is a consequence of* $F = 0$.

Let us determine the projective invariants of our surface; the affine case is similar and simpler. First of all, let us consider the hyperbolic case. To a surface $\pi \subset P^3$, associate frames as described in Lemma 4.1. Especially, we have the equations (4.10). After prolongation, we get

$$\begin{aligned} \mathrm{D}B_1 \wedge \tau^1 + \mathrm{D}B_2 \wedge \tau^2 &= 3A_1(B_3 - A_1A_4)\,\tau^1 \wedge \tau^2\,,\\ \mathrm{D}B_2 \wedge \tau^1 + \mathrm{D}B_3 \wedge \tau^2 &= -A_1B_4\tau^1 \wedge \tau^2\,,\\ \mathrm{D}B_3 \wedge \tau^1 + \mathrm{D}B_4 \wedge \tau^2 &= A_4B_2\tau^1 \wedge \tau^2\,,\\ \mathrm{D}B_4 \wedge \tau^1 + \mathrm{D}B_5 \wedge \tau^2 &= 3A_4(A_1A_4 - B_3)\,\tau^1 \wedge \tau^2 \end{aligned} \tag{7.21}$$

with

$$\begin{aligned} \mathrm{D}B_1 &= \mathrm{d}B_1 + 2B_1(\tau_0^0 - 2\tau_1^1) + 4A_1\tau_1^0\,,\\ \mathrm{D}B_2 &= \mathrm{d}B_2 + 2B_2(\tau_0^0 - \tau_1^1) - 2A_1\tau_2^0\,,\\ \mathrm{D}B_3 &= \mathrm{d}B_3 + 2B_3\tau_0^0 - 2\tau_3^0\,,\\ \mathrm{D}B_4 &= \mathrm{d}B_4 + 2B_4(\tau_0^0 + \tau_1^1) - 2A_4\tau_1^0\,,\\ \mathrm{D}B_5 &= \mathrm{d}B_5 + 2B_5(\tau_0^0 + 2\tau_1^1) + 4A_4\tau_2^0\,. \end{aligned} \tag{7.22}$$

Consider a function

$$G = G(A_1, A_4, B_1, \ldots, B_5)\,. \tag{7.23}$$

Then

$$(7.24)\quad \mathrm{d}G = \frac{\partial G}{\partial A_1}\{\mathrm{d}A_1 + A_1(\tau_0^0 - 3\tau_1^1)\} + \frac{\partial G}{\partial A_4}\{\mathrm{d}A_4 + A_4(\tau_0^0 + 3\tau_1^1)\} + \\ + \sum_{i=1}^{5}\frac{\partial G}{\partial B_i}\,\mathrm{D}B_i + \Psi_1\tau_0^0 + \Psi_2\tau_1^1 + \Psi_3\tau_1^0 + \Psi_4\tau_2^0 + \Psi_5\tau_3^0$$

with

$$(7.25)\qquad \Psi_1 = -\left(A_1\frac{\partial G}{\partial A_1} + A_4\frac{\partial G}{\partial A_4} + \sum_{i=1}^{5} B_i\frac{\partial G}{\partial B_i}\right),$$

$$\Psi_2 = 3\left(A_1\frac{\partial G}{\partial A_1} - A_4\frac{\partial G}{\partial A_4}\right) + 2\left(2B_1\frac{\partial G}{\partial B_1} + B_2\frac{\partial G}{\partial B_2} - B_4\frac{\partial G}{\partial B_4} - 2B_5\frac{\partial G}{\partial B_5}\right),$$

$$\Psi_3 = -2\left(2A_1\frac{\partial G}{\partial B_1} - A_4\frac{\partial G}{\partial B_4}\right),\quad \Psi_4 = 2\left(A_1\frac{\partial G}{\partial B_2} - 2A_4\frac{\partial G}{\partial B_5}\right),\quad \Psi_5 = 2\frac{\partial G}{\partial B_3},$$

and we easily get

Proposition 7.3. *The function G from* (7.23) *is a projective invariant of a hyperbolic surface if and only if* $\Psi_i = 0$; $i = 1, \ldots, 5$.

It is just a simple exercise to obtain the elliptic version of this proposition. To do that, we have to calculate $b_1, \ldots, b_5$ as functions of $B_1, \ldots, B_5$ from (7.11) and a_2, a_3 from (7.10). Further, we define $G(A_1, A_4, B_1, \ldots) := F(a_2, a_3, b_1, \ldots)$, and use the conditions (7.25).

8. Characterization of quadratic surfaces. Let $\pi \subset A^3_{\mathrm{eq}}$ be an elliptic surface satisfying an equiaffine condition

$$(8.1)\qquad F(a_2, a_3, b_1, \ldots, b_5) = 0\,.$$

In what follows, let us write

$$(8.2)\qquad R \equiv S$$

instead of

$$(8.3)\quad R = S + (\cdot)\,a_2 + (\cdot)\,a_3 + (\cdot)\,b_2 + (\cdot)\,b_4 + (\cdot)(b_1 - 3b_3) + (\cdot)(b_5 - 3b_3).$$

Applying Cartan's lemma to (6.29), we see that

$$(8.4)\quad \mathrm{d}a_2 - 3a_3\omega = a_{21}\omega^1 + a_{22}\omega^2\,,\quad \mathrm{d}a_3 + 3a_2\omega = a_{31}\omega^1 + a_{32}\omega^2$$

with

$$(8.5)\qquad a_{31} - a_{22} \equiv 0\,,\quad a_{21} + a_{32} \equiv 0\,.$$

Similarly, from (6.30),

(8.6)

$$\mathrm{d}b_1 - 4b_2\omega = b_{11}\omega^1 + b_{12}\omega^2\,,\quad \mathrm{d}b_2 + (b_1 - 3b_3)\,\omega = b_{21}\omega^1 + b_{22}\omega^2\,,$$

$$\mathrm{d}b_3 + 2(b_2 - b_4)\,\omega = b_{31}\omega^1 + b_{32}\omega^2\,,\quad \mathrm{d}b_4 + (3b_3 - b_5)\,\omega = b_{41}\omega^1 + b_{42}\omega^2\,,$$

$$\mathrm{d}b_5 + 4b_4\omega = b_{51}\omega^1 + b_{52}\omega^2$$

with

$$(8.7)\qquad b_{21}+b_{41}-b_{12}-b_{32}\equiv 0\,,\quad b_{31}+b_{51}-b_{22}-b_{42}\equiv 0\,,$$
$$3b_{31}-b_{51}-3b_{22}+b_{42}\equiv 0\,,\quad 3b_{41}-b_{21}-3b_{32}+b_{12}\equiv 0\,,$$

i.e.,

$$(8.8)\qquad b_{21}\equiv b_{12}\,,\quad b_{31}\equiv b_{22}\,,\quad b_{41}\equiv b_{32}\,,\quad b_{51}\equiv b_{42}\,.$$

From (8.1) we get

$$(8.9)\qquad \frac{\partial F}{\partial a_2}\,\mathrm{d}a_2+\frac{\partial F}{\partial a_3}\,\mathrm{d}a_3+\sum_{i=1}^{5}\frac{\partial F}{\partial b_i}\,\mathrm{d}b_i=0\,.$$

Let $m_0\in\pi$ be an arbitrary point. There exists a coordinate neighborhood $U\subset\pi$ of m_0 such that the equiaffine metric (6.20) may be written as

$$(8.10)\qquad \mathrm{d}s^2=r^2(\mathrm{d}x^2+\mathrm{d}y^2)\,,\quad r=r(x,y)>0$$

in U, i.e.,

$$(8.11)\qquad \omega^1=r\,\mathrm{d}x\,,\quad \omega^2=r\,\mathrm{d}y\,.$$

This and (6.33) yield

$$(8.12)\qquad \omega=-r^{-1}\left(\frac{\partial r}{\partial y}\,\mathrm{d}x-\frac{\partial r}{\partial x}\,\mathrm{d}y\right),$$

while (8.4) + (8.6) imply

$$(8.13)\qquad \frac{\partial a_2}{\partial x}=ra_{21}\,,\quad \frac{\partial a_2}{\partial y}\equiv ra_{22}\,,\quad \frac{\partial a_3}{\partial x}\equiv ra_{31}\,,\quad \frac{\partial a_3}{\partial y}\equiv ra_{32}\,;$$
$$\frac{\partial b_i}{\partial x}\equiv rb_{i1}\,,\quad \frac{\partial b_i}{\partial y}\equiv rb_{i2}\,;\quad i=1,\ldots,5\,.$$

Inserting (8.13_{1-4}) into (8.5) we get

$$(8.14)\qquad \frac{\partial a_3}{\partial x}-\frac{\partial a_2}{\partial y}\equiv 0\,,\quad \frac{\partial a_2}{\partial x}+\frac{\partial a_3}{\partial y}\equiv 0\,.$$

From (8.7) we conclude

$$(8.15)\qquad b_{12}-3b_{32}-b_{21}+3b_{41}\equiv 0\,,\quad b_{51}-3b_{31}-b_{42}+3b_{22}\equiv 0\,;$$

inserting there from (8.13), we obtain

$$(8.16)\qquad \frac{\partial(b_1-3b_3)}{\partial y}-\frac{\partial b_2}{\partial x}+3\frac{\partial b_4}{\partial x}\equiv 0\,,\quad \frac{\partial(b_5-3b_3)}{\partial x}-\frac{\partial b_4}{\partial y}+3\frac{\partial b_2}{\partial y}\equiv 0\,.$$

Let us take into account the condition (8.9). Because of Proposition 7.2, we have

$$(8.17)\qquad \frac{\partial F}{\partial a_2}\,a_{2\alpha}+\frac{\partial F}{\partial a_3}\,a_{3\alpha}+\sum_{i=1}^{5}\frac{\partial F}{\partial b_i}\,b_{i\alpha}\equiv 0\,;\quad \alpha=1,2\,.$$

Using (8.5) + (8.8), these equations may be rewritten as

$$(8.18)\qquad \frac{\partial F}{\partial a_2}\,a_{21}+\frac{\partial F}{\partial a_3}\,a_{22}+\frac{\partial F}{\partial b_1}(b_{11}-3b_{31}+3b_{22})+$$

$$+\frac{\partial F}{\partial b_2} b_{21} + \frac{\partial F}{\partial b_3} b_{22} + \frac{\partial F}{\partial b_4} b_{41} + \frac{\partial F}{\partial b_5} b_{42} \equiv 0\,,$$

$$\frac{\partial F}{\partial a_2} a_{22} - \frac{\partial F}{\partial a_3} a_{21} + \frac{\partial F}{\partial b_1} b_{21} + \frac{\partial F}{\partial b_2} b_{22} + \frac{\partial F}{\partial b_3} b_{41} +$$

$$+\frac{\partial F}{\partial b_4} b_{42} + \frac{\partial F}{\partial b_5} (b_{52} - 3b_{32} + 3b_{41}) \equiv 0$$

and, because of (8.13), we get them in the final form

(8.19)
$$\frac{\partial F}{\partial a_2}\frac{\partial a_2}{\partial x} + \frac{\partial F}{\partial a_3}\frac{\partial a_2}{\partial y} + \frac{\partial F}{\partial b_1}\left\{\frac{\partial(b_1 - 3b_3)}{\partial x} + 3\frac{\partial b_2}{\partial y}\right\} +$$

$$+\frac{\partial F}{\partial b_2}\frac{\partial b_2}{\partial x} + \frac{\partial F}{\partial b_3}\frac{\partial b_2}{\partial y} + \frac{\partial F}{\partial b_4}\frac{\partial b_4}{\partial x} + \frac{\partial F}{\partial b_5}\frac{\partial b_4}{\partial y} \equiv 0\,,$$

$$\frac{\partial F}{\partial a_2}\frac{\partial a_2}{\partial y} - \frac{\partial F}{\partial a_3}\frac{\partial a_2}{\partial x} + \frac{\partial F}{\partial b_1}\frac{\partial b_2}{\partial x} + \frac{\partial F}{\partial b_2}\frac{\partial b_2}{\partial y} +$$

$$+\frac{\partial F}{\partial b_3}\frac{\partial b_4}{\partial x} + \frac{\partial F}{\partial b_4}\frac{\partial b_4}{\partial y} + \frac{\partial F}{\partial b_5}\left\{\frac{\partial(b_5 - 3b_3)}{\partial y} + 3\frac{\partial b_4}{\partial x}\right\} \equiv 0\,.$$

Write

(8.20)
$$\mathscr{f} = (a_2, a_3, b_2, b_4, b_1 - 3b_3, b_5 - 3b_3)^{\mathrm{T}}\,;$$

the system (8.14) + (8.16) + (8.19) is then of the form

(8.21)
$$\mathscr{A}\frac{\partial \mathscr{f}}{\partial x} + \mathscr{B}\frac{\partial \mathscr{f}}{\partial y} + \mathscr{C}\mathscr{f} = 0\,.$$

The symbol of (8.21) being defined by

(8.22)
$$\sigma(\xi, \eta) = \|\mathscr{A}\xi + \mathscr{B}\eta\|\,, \quad (\xi, \eta) \in \mathbb{R}^2\,,$$

it is easy to see that

(8.23)
$$\det \sigma(\xi, \eta) = -(\xi^2 + \eta^2)\,\mathscr{D}$$

with

(8.24)
$$\mathscr{D} = \begin{vmatrix} \xi & -3\xi & -\eta & 0 \\ -3\eta & \eta & 0 & -\xi \\ R_1 & R_2 & R_3 & 0 \\ S_1 & S_2 & 0 & S_4 \end{vmatrix}\,,$$

$$R_1 = \frac{\partial F}{\partial b_2}\xi + \left(3\frac{\partial F}{\partial b_1} + \frac{\partial F}{\partial b_3}\right)\eta\,, \quad R_2 = \frac{\partial F}{\partial b_4}\xi + \frac{\partial F}{\partial b_5}\eta\,, \quad R_3 = \frac{\partial F}{\partial b_1}\xi\,,$$

$$S_1 = \frac{\partial F}{\partial b_1}\xi + \frac{\partial F}{\partial b_2}\eta\,, \quad S_2 = \left(\frac{\partial F}{\partial b_3} + 3\frac{\partial F}{\partial b_5}\right)\xi + \frac{\partial F}{\partial b_4}\eta\,, \quad S_4 = \frac{\partial F}{\partial b_5}\eta\,.$$

Theorem. *Let $\pi \subset A^3_{eq}$ be an analytic elliptic surface satisfying the condition* (8.1). *Let $\mathscr{D}$ in* (8.24) *vanish if and only if $\xi = \eta = 0$. Then there are only two possibilities*: (i) *π is (a piece of) a quadratic surface*; (ii) *the set*

$$N := \{m \in \pi;\ J = J_2 = J_3 = 0 \text{ at } m\} \tag{8.25}$$

consists of isolated points.

Proof. Let $m_0 \in N$ be not isolated; for the definition of J_2 and J_3 see (7.14), J being the Pick invariant. Around m_0, take a coordinate neighborhood U as above, and consider the system (8.21). Because of our supposition, it is elliptic, and [10], Theorem 5.4.1 implies $\not{f} = 0$ on U and, by analyticity, on the whole π. Thus $J = 0$ on π, QED.

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