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Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 2, 262–279

Persistent URL: `http://dml.cz/dmlcz/102301`

PRODUCTS OF SEQUENTIAL CONVERGENCE PROPERTIES

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(Received April 28, 1987)

1. INTRODUCTION

A topological space X is said to be *strongly Fréchet* [15] (=countably bi-sequential in the sense of E. Michael [8]) if for any decreasing sequence $\{A_n: n \in N\}$ of subsets of X such that $x \in \bar{A}_n$, it follows that there exists a point x_n in A_n such that $\{x_n: n \in N\}$ converges to x. If we put $A_n = A$ for every $n \in N$, then the resulting space is said to be *Fréchet*. In general Fréchet spaces behave very badly with respect to product operations. In fact P. Simon [14] showed that the product of two compact Fréchet spaces need not be Fréchet. Nevertheless it is known that there exist some subclasses of Fréchet spaces which behave nicely with respect to product operations; bi-sequential spaces [8], W-spaces [5] and $\langle\alpha_3\text{-}FU\rangle$-spaces [1, 2, 10, 11] form such nice subclasses. The purpose of this paper is to present such nice classes. Before we mention the purpose in detail we would like to note an example obtained by G. Gruenhage [6]: Assuming a Martin's axiom there exists a space X such that X^n is strongly Fréchet for every $n \in N$, but X^ω is not Fréchet. In this paper we shall study the following problems:

Problem 1.1. *Let $\prod_{i=1}^{n} X_i$ be Fréchet for each $n \in N$. Under what additional conditions for X_i, is $\prod_{i=1}^{\infty} X_i$ Fréchet?*

Problem 1.2. *Let X and Y be regular countably compact Fréchet spaces. Under what additional conditions for X, is $X \times Y$ Fréchet?*

Since W-spaces and bi-sequential spaces are $\langle\alpha_3\text{-}FU\rangle$-spaces, so fas as the author knows, the class $\langle\alpha_3\text{-}FU\rangle$ is the widest class which satisfies the problems 1.1 and 1.2 at the same time ([2], [10]). But recently the author showed that there exists a non-$\langle\alpha_3\text{-}FU\rangle$-space X which gives an answer to the problem 1.2 ([12]). Therefore it is expected to find out new classes which contain $\langle\alpha_3\text{-}FU\rangle$-spaces and the above example X, and which give answers to problems 1.1 and 1.2 at the same time.

We shall introduce new classes $\langle\alpha_3\text{-}FU\rangle = \langle\alpha^0\text{-}St\rangle \subset \langle\alpha^1\text{-}St\rangle \subset \langle\alpha^2\text{-}St\rangle \subset \ldots \ldots \subset \langle\alpha^k\text{-}St\rangle \subset \ldots \subset \langle\alpha^\omega\text{-}St\rangle$ and show that problems 1.1 and 1.2 are true for the class $\langle\alpha^k\text{-}St\rangle$, $k \in N$. We also show that the class $\langle\alpha^\omega\text{-}St\rangle$ satisfies the problem 1.2. We construct, under the continuum hypothesis (CH), an $\langle\alpha^n\text{-}St\rangle$-space which is not an $\langle\alpha^m\text{-}St\rangle$-space for $m < n$. In fact the example obtained in [12] is an $\langle\alpha^1\text{-}St\rangle$-space which is not an $\langle\alpha_3\text{-}FU\rangle$-space.

In this paper all spaces are assumed to be Hausdorff topological spaces.

2. DEFINITIONS AND PRODUCTS OF $\langle\alpha^k\rangle$, $\langle\alpha^\omega\rangle$-SPACES

Let X be a space. In this paper a sequence in X means a map $f: M \to X$, where M is a countable set. If M is infinite, then the sequence is said to be *infinite*. As usual we also denote the sequence f as $\{x_n: n \in M\}$. But we strictly distinguish the sequence f from the image of f. If M is infinite, then the sequence $\{x_n: n \in M\}$ is infinite but the set $\{x_n: n \in M\}$ may be finite. We denote by $|A|$ the cardinality of a set A. For a collection $\mathscr{A} = \{A_\alpha: \alpha \in \Omega\}$ of subsets of a set X and a map $g: X \to Y$, we put $g(\mathscr{A}) = \{g(A_\alpha): \alpha \in \Omega\}$.

A collection $\mathscr{A}$ of convergent sequences in X is said to be a *sheaf* in X if all members of $\mathscr{A}$ converge to the same point of X, which is said to be the *vertex* of the sheaf $\mathscr{A}$. In this paper all sheaves are assumed to be countable. We consider the following properties of X which were introduced by A. V. Arhangel'skii [1, 2]:

Let $\mathscr{A} = \{A_n: n \in N\}$ be a sheaf in X with vertex $x \in X$. Then there exists a sequence B converging to x such that:

(α_3) $|\{n \in N: A_n \cap B$ is an infinite subsequence of A_n and $B\}| = \omega$, where ω is the first infinite cardinal number.

(α_4) $|\{n \in N: A_n \cap B \neq \emptyset\}| = \omega$.

We say B satisfies (α_i) with respect to $\mathscr{A}$ if B satisfies the property (α_i) for $i = 3, 4$. The class of spaces satisfying the property (α_i) for every sheaf $\mathscr{A}$ and vertex $x \in X$ is denoted by $\langle\alpha_i\rangle$. We denote by $\langle\alpha_i\text{-}FU\rangle$ the intersection of the class $\langle\alpha_i\rangle$ and the class of Fréchet spaces. For a class $\mathscr{C}$ of spaces an element of $\mathscr{C}$ is said to be a *$\mathscr{C}$-space*. A. V. Arhangel'skii showed that the class $\langle\alpha_4\text{-}FU\rangle$ is precisely the class of strongly Fréchet spaces, and showed that every bi-sequential space is an $\langle\alpha_3\text{-}FU\rangle$-space. For further properties and applications of $\langle\alpha_i\rangle$-spaces $(i = 1, 2, 3, 4)$, see [1], [2], [4], [10], [11] and [12]. We introduce new properties (α^k) and (α^ω) as follows.

Let $\mathscr{A} = \{A_n: n \in N\}$ be a sheaf in X with vertex $x \in X$. A sheaf $\mathscr{B}$ with the same vertex x is said to be a *cross-sheaf* of $\mathscr{A}$ if $|\{n \in N: A_n \cap B \neq \emptyset\}| = \omega$ for each $B \in \mathscr{B}$ and $\bigcup\mathscr{B} \subset \bigcup\mathscr{A}$. A sheaf $\mathscr{A}' = \{\mathscr{A}'_m: m \in M\}$ is said to be a *subsheaf* of $\mathscr{A}$ if M is an infinite subset of N and for each $m \in M$, A'_m is an infinite subsequence of A_m. A sheaf $\mathscr{A}$ is said to be 0-*nice* if it is convergent, $\mathscr{A}$ is said to be $(k + 1)$-*nice* if each cross-sheaf of $\mathscr{A}$ has a k-nice subsheaf.

(α^k) For each sheaf $\mathscr{A}$ in X, there exists a k-nice subsheaf of $\mathscr{A}$.

We denote by $\langle\alpha^k\rangle$ $(\langle\alpha^\omega\rangle)$ the class of all spaces satisfying (α^k) $((\alpha^n)$ for an $n \in N$, respectively) for every sheaf $\mathscr{A}$ and vertex $x \in X$. Clearly $\langle \alpha^0 \rangle = \langle \alpha_3 \rangle$. If $0 \leqq k \leqq \leqq n \leqq \omega$, then every $\langle \alpha^k \rangle$-space is an $\langle \alpha^n \rangle$-space.

Now we study products of $\langle \alpha^k \rangle$-spaces.

Lemma 2.1. *Let $\mathscr{C}$ be a sheaf in $Z = X \times Y$ with vertex z. If each subsheaf of $\pi_X(\mathscr{C})$ and $\pi_Y(\mathscr{C})$ has a k-nice subsheaf, then there exists a subsheaf $\mathscr{C}'$ of $\mathscr{C}$ such that $\pi_X(\mathscr{C}')$ and $\pi_Y(\mathscr{C}')$ are k-nice subsheaves of $\pi_X(\mathscr{C})$ and $\pi_Y(\mathscr{C})$, respectively, where π_X and π_Y are the projections from Z to X and Y, respectively.*

Proof. By the assumption, there exist $L \subset N$ and an infinite subsequence C''_n of C_n for every $n \in L$ such that $\{\pi_X(C''_n): n \in L\}$ is a k-nice subsheaf of $\pi_X(\mathscr{C})$. Put $\mathscr{C}'' = \{C''_n: n \in L\}$. Since $\pi_Y(\mathscr{C}'')$ has a k-nice subsheaf, there exists $M \subset L$ and an infinite subsequence C'_n of C''_n for every $n \in M$ such that $\{\pi_Y(C'_n): n \in M\}$ is a k-nice subsheaf of $\{\pi_Y(C''_n): n \in L\}$. $\mathscr{C}' = \{C'_n: n \in M\}$ is a desired subsheaf of $\mathscr{C}$. The proof is completed.

Lemma 2.2. *Let $\mathscr{C}$ be a sheaf in $Z = X \times Y$ with vertex $z = (x, y)$ and $\mathscr{C}'$ be a subsheaf of $\mathscr{C}$ such that $\pi_X(\mathscr{C}')$ and $\pi_Y(\mathscr{C}')$ are k-nice subsheaves of $\pi_X(\mathscr{C})$ and $\pi_Y(\mathscr{C})$, respectively. Then $\mathscr{C}'$ has a k-nice subsheaf.*

Proof. We show the assertion by induction on k. For $k = 0$, put

$$\mathscr{C}' = \{C'_m: m \in M\}\,, \quad C'_m = \{(x^m_n, y^m_n): n \in N\}$$

Without loss of generality we can assume $x^m_n \neq x$ and $y^m_n \neq y$ for every $(m, n) \in \in M \times N$. Put

$$D_m = C'_m - \pi_X^{-1}(\{x^i_j: i, j \leqq m - 1\}) \cup \pi_Y^{-1}(\{y^i_j: i, j \leqq m - 1\})\,.$$

Then $C'_m - D_m$ is a finite set. Put

$$\mathscr{D} = \{D_m: m \in M\}\,.$$

We show $\mathscr{D}$ is a 0-nice subsheaf of $\mathscr{C}'$, i.e. $\bigcup \mathscr{D}$ a is convergent sequence.

Let $U \times V$ be an open neighborhood of (x, y) in $X \times Y$. Put

$$S = \{(x^m_n, y^m_n) \in \bigcup\mathscr{D}: x^m_n \notin U\}\,,$$
$$T = \{(x^m_n, y^m_n) \in \bigcup\mathscr{D}: y^m_n \notin V\}\,.$$

We show $S \cup T$ is a finite set. Assume S is infinite. Then, since $\pi_X(S)$ is finite, there exist $m_0, m_i \in M$, $n_0, n_i \in N$ $(i \in N)$ such that $(x^{m_0}_{n_0}, y^{m_i}_{n_i}) \in D_{n_i}$ for $i \in N$. But this is impossible since $D_{n_i} \cap \pi_X^{-1}(x^{m_0}_{n_0}) = \emptyset$ for $n_i > n_0$. Similarly T is finite.

Assume the assertion has been proved for k $(\geqq 0)$. We show the case $k + 1$.

Let $\mathscr{B}$ be a cross-sheaf of $\mathscr{C}'$. Then $\pi_X(\mathscr{B})$ and $\pi_Y(\mathscr{B})$ are cross-sheaves of $\pi_X(\mathscr{C}')$ and $\pi_Y(\mathscr{C}')$, respectively. Since $\pi_X(\mathscr{C}')$ and $\pi_Y(\mathscr{C}')$ are $(k + 1)$-nice subsheaves of $\pi_X(\mathscr{C})$ and $\pi_Y(\mathscr{C})$, respectively, $\pi_X(\mathscr{B})$ and $\pi_Y(\mathscr{B})$ have k-nice subsheaves. By Lemma 2.1 and by the inductive assumption, $\mathscr{B}$ has a k-nice subsheaf. This shows the assertion is true for the $k + 1$. The proof is completed.

By Lemma 2.1 and 2.2 we have the following theorem.

Theorem 2.3. *Let X and Y be $\langle\alpha^k\rangle$-spaces ($\langle\alpha^\omega\rangle$-spaces). Then $Z = X \times Y$ is an $\langle\alpha^k\rangle$-space, ($\langle\alpha^\omega\rangle$-space, respectively).*

Lemma 2.4. *Let $\mathscr{A}$ be a sheaf in $X = \prod_{i=1}^{\infty} X_i$ with vertex $x \in X$. If each subsheaf of $\pi_n(\mathscr{A})$ has a k-nice subsheaf for every $n \in N$, then there exists a subsheaf $\mathscr{A}'$ of $\mathscr{A}$ such that $\pi_n(\mathscr{A}')$ is a k-nice subsheaf of $\pi_n(\mathscr{A})$ for every $n \in N$, where $\pi_n: X \to \to \prod_{i=1}^{n} X_i$ is the projection for every $n \in N$.*

Proof. We first construct a subsheaf $\mathscr{A}_n$ of $\pi_n(\mathscr{A})$ and it's k-nice subsheaf $\mathscr{A}'_n$ such that $\pi_n^m(\mathscr{A}'_m)$ is a k-nice subsheaf of $\mathscr{A}'_n$ for $m \geqq n$, where $\pi_n^m: \prod_{i=1}^{m} X_i \to \prod_{i=1}^{n} X_i$ is the projection. Put $\mathscr{A}_1 = \{\pi_1(A_n): n \in N\}$ and $M_0 = N$. Let $\mathscr{A}'_1 = \{A_n^1: n \in M_1\}$ be a k-nice subsheaf of $\mathscr{A}_1$ such that A_n^1 is an infinite subsequence of $\pi_1(A_n)$ for every $n \in M_1$. Inductively we construct $\mathscr{A}_j$ and $\mathscr{A}'_j$ as follows: Put

$$\mathscr{A}_j = \{(\pi_{j-1}^j)^{-1}(A_n^{j-1}) \cap \pi_j(A_n): n \in M_{j-1}\} \quad \text{for} \quad j \geqq 2 .$$

Then $\mathscr{A}_j$ is a subsheaf of $\pi_j(\mathscr{A})$. Let $\mathscr{A}'_j = \{A_i^j: i \in M_j\}$ be a k-nice subsheaf of $\mathscr{A}_j$, where $M_j \subset M_{j-1}$ and A_n^j is an infinite subsequence of $(\pi_{j-1}^j)^{-1}(A_n^{j-1}) \cap \pi_j(A_n)$ for $n \in M_j$. Note that M_j is infinite by the definition of a subsheaf. Choose $n_j \in M_j$ such that $n_j < n_{j+1}$ for each $j \in N$.

Now we define

$$\mathscr{A}' = \{\pi_{n_j}^{-1}(A_{n_j}^j) \cap A_{n_j}: j \in N\} .$$

Then $\mathscr{A}'$ is a subsheaf of $\mathscr{A}$ and $\pi_n(\mathscr{A}')$ is a subsheaf of $\mathscr{A}'_n$. Clearly a subsheaf of a k-nice sheaf is k-nice. Therefore $\pi_n(\mathscr{A}')$ is k-nice for every $n \in N$. The proof is completed.

Lemma 2.5. *Let $\mathscr{A} = \{A_n: n \in N\}$ be a sheaf in $X = \prod_{i=1}^{\infty} X_i$ with vertex x such that each subsheaf of $\pi_n(\mathscr{A})$ has a k-nice subsheaf for every $n \in N$. Then $\mathscr{A}$ has a k-nice subsheaf.*

Proof. We prove the assertion by induction on k. Let $k = 0$. By Lemma 2.4, choose a subsheaf $\mathscr{A}'$ of $\mathscr{A}$ such that $\pi_n(\mathscr{A}')$ is 0-nice for every $n \in N$. To avoid the complexity of the notations we put $\mathscr{A}' = \mathscr{A}$. Put

$$A_n = \{x_m^n: m \in N\} ,$$

$$A'_n = A_n - \bigcup_{m=1}^{n} \pi_m^{-1}(\{\pi_m(x_j^i): i, j \leqq m,\ \pi_m(x_j^i) \neq \pi_m(x)\}) .$$

Then $\mathscr{A}' = \{A'_n: n \in N\}$ is a subsheaf of $\mathscr{A}$. We show $\mathscr{A}'$ is 0-nice. Let $V = \prod_{i=1}^{n} U_i \times \times \prod_{i=n+1}^{\infty} X_i$ be an open neighborhood of x in X. Since $\bigcup \pi_n(\mathscr{A}')$ is a convergent sequence, $\bigcup \pi_n(\mathscr{A}') - \prod_{i=1}^{n} U_i$ is a finite set. Put

$$\pi_n(\{x_{j_1}^{i_1}, \ldots, x_{j_m}^{i_m}\}) = \bigcup \pi_n(\mathscr{A}') - \prod_{i=1}^{n} U_i\,,$$

$$t = \max\{n, i_1, \ldots, i_m, j_1, \ldots, j_m\}\,.$$

Then

$$A'_s \cap \pi_n^{-1}(\pi_n(\{x_{j_1}^{i_1}, \ldots, x_{j_m}^{i_m}\}) = \emptyset \quad \text{for} \quad s \geqq t$$

and

$$|A'_s \cap \pi_n^{-1}(\pi_n(\{x_{j_1}^{i_1}, \ldots, x_{j_m}^{i_m}\}))| < \omega \quad \text{for} \quad s < t$$

because $\pi_n(A'_s)$ is a sequence converging to $\pi_n(x)$ and $\pi_n(x_{j_r}^{i_r}) \neq \pi_n(x)$ for every $r = 1, 2, \ldots, m$. These show $(X - V) \cap (\bigcup \mathscr{A}')$ is a finite set. The proof of the case $k = 0$ is completed.

Assume the assertion has been proved for the case $k \geqq 0$. We show the case $k + 1$.

Let $\mathscr{A}$ be a sheaf with vertex x such that $\pi_n(\mathscr{A})$ is $(k + 1)$-nice for every $n \in N$. Then, by Lemma 2.4, there exists a subsheaf $\mathscr{A}'$ of $\mathscr{A}$ such that $\pi_n(\mathscr{A}')$ is a $(k + 1)$-nice subsheaf of $\pi_n(\mathscr{A})$ for every $n \in N$. We show $\mathscr{A}'$ is $(k + 1)$-nice.

Let $\mathscr{B}$ be a cross-sheaf of $\mathscr{A}'$. Then $\pi_n(\mathscr{B})$ is a cross-sheaf of $\pi_n(\mathscr{A}')$. Since $\pi_n(\mathscr{A}')$ is $(k + 1)$-nice, $\pi_n(\mathscr{B})$ has a k-nice subsheaf. By Lemma 2.4, $\mathscr{B}$ has a subsheaf $\mathscr{B}'$ such that $\pi_n(\mathscr{B}')$ is k-nice for every $n \in N$. Therefore, by the inductive assumption $\mathscr{B}'$ has a k-nice subsheaf. This shows $\mathscr{A}'$ is $(k + 1)$-nice. The proof is completed.

Theorem 2.6. *Let X_i be an $\langle\alpha^k\rangle$-space for every $i \in N$. Then $\prod_{i=1}^{\infty} X_i$ is an $\langle\alpha^k\rangle$-space.*

Proof. Let $\mathscr{A}$ be a sheaf in X with vertex x. By Theorem 2.3, $\prod_{i=1}^{n} X_i$ is an $\langle\alpha^k\rangle$-space for every $n \in N$. Therefore each subsheaf of $\pi_n(\mathscr{A})$ has a k-nice subsheaf. By Lemma 2.4 and 2.5, we can choose a k-nice subsheaf $\mathscr{A}'$ of $\mathscr{A}$. The proof is completed.

In § 3 we shall show that the countable product of $\langle\alpha^\omega\rangle$-spaces need not be an $\langle\alpha^\omega\rangle$-space under (CH).

3. $\langle\alpha^k\text{-}St\rangle$-SPACES AND FINITE PRODUCTS

According to E. Michael [8], the product $X \times C$ is Fréchet if and only if X is strongly Fréchet, where $C = \{0\} \cup \{1/n\colon n \in N\}$ with the usual topology. Hence if we expect the Fréchetness of products, it is natural to require that each factor space is strongly Fréchet. We denote by $\langle\alpha^k\text{-}St\rangle$ ($\langle\alpha^\omega\text{-}St\rangle$), the class of all strongly Fréchet spaces with the property (α^k) ((α^ω), respectively).

Lemma 3.1. *Let X be a strongly Fréchet space and Y be a regular countably compact Fréchet space. Let $C_n = \{(x_m^n, y_m^n)\colon m \in N\}$, $n \in N$, be convergent sequences in $Z = X \times Y$ satisfying the following conditions:*

(1) *Each C_n converges to a point (x, y^n) for $n \in N$.*

(2) $\{y^n: n \in N\}$ *is a sequence in* Y *converging to a point* y, *where* $y^n \neq y$ *for every* $n \in N$.

Put $A_n = \{x^n_m: m \in N\}$. *Then*

(3) $\mathscr{A} = \{A_n: n \in N\}$ *is a k-nice sheaf with vertex* x.

Then there exists a sequence $\{(x^{n_j}_{m_j}, y^{n_j}_{m_j}) \in C_{n_j}: j \in N\}$ *converging to the point* (x, y).

Proof. We prove the assertion by induction on k. Let $k = 0$. Then $\mathscr{A}$ is a 0-nice sheaf, therefore $\bigcup\mathscr{A}$ is a sequence converging to x. Put $A = \bigcup\{C_n: n \in N\}$. Then $A \subset (\bigcup\mathscr{A} \cup \{x\}) \times Y$ and $(x, y) \in \bar{A}$. Since a regular countably compact Fréchet space is strongly Fréchet [13], $(\bigcup\mathscr{A} \cup \{x\}) \times Y$ is Fréchet by Michael's theorem (see introduction of this section). Therefore there exists a sequence B in A converging to (x, y). If $B \cap C_n$ is an infinite set for some $n \in N$, then B converges to the point (x, y^n); this contradicts $y^n \neq y$. Therefore $B \cap C_n$ is a finite set for every $n \in N$. Choose $n_1 < n_2 < \ldots$ such that $B \cap C_{n_j} \neq \emptyset$ and choose $(x^{n_j}_{m_j}, y^{n_j}_{m_j}) \in B \cap C_{n_j}$. Then $\{(x^{n_j}_{m_j}, y^{n_j}_{m_j}): j \in N\}$ converges to the point (x, y). The proof for the case $k = 0$ is completed.

Assume the assertion has been proved for k $(\geqq 0)$. We show the case $k + 1$.

Put

$$\mathscr{B} = \{B \subset \bigcup\{C_n: n \in N\}: |B \cap C_n| < \omega \quad \text{for every} \quad n \in N, \quad \pi_X(B)$$
$$\text{converges to the point } x \text{ and } \pi_Y(B) \text{ converges to a point } y_B \in Y\}.$$

We show

(*) $y \in \mathrm{cl}\,\{y_B: B \in \mathscr{B}\}$.

Let V be an open neighborhood of y in Y and W be an open set such that $y \in W \subset \subset \overline{W} \subset V$. Since $\{y^n: n \in N\}$ converges to y, there exist $n_0, n_m \in N$ such that

$$\{y^m_i: i \geqq n_m,\ m \geqq n_0\} \subset W.$$

Since $\{x^m_i: i \geqq n_m\}$ converges to x for each $m \in N$, by using the strongly Fréchetness of X, there exist $x^{m_j}_{i_j} \in A_{m_j}$, $m_0 < m_j < m_{j+1}$, $n_{m_j} < i_j$ such that $\{x^{m_j}_{i_j}: j \in N\}$ converges to the point x. Since Y is countably compact Fréchet, there exist a convergent subsequence $\{y^{m_{j(r)}}_{i_{j(r)}}: r \in N\}$ of $\{y^{m_j}_{i_j}: j \in N\}$ and its limit point $y(W) \in \overline{W} \subset V$. Put $B = \{(x^{m_{j(r)}}_{i_{j(r)}}, y^{m_{j(r)}}_{i_{j(r)}}): r \in N\}$. Then clearly $B \in \mathscr{B}$ and $y_B = y(W) \in \overline{W} \subset V$. The proof of (*) is completed.

Since Y is Fréchet, there exists $B_n \in \mathscr{B}$ such that $\{y_{B_n}: n \in N\}$ converges to the point y. Without loss of generality we can assume $y_{B_n} \neq y$. Put $B'_n = B_n - \bigcup\{C_i: i \leqq n\}$. Then B'_n is an infinite subsequence of B_n. Clearly $\{\pi_X(B'_n): n \in N\}$ is a cross-sheaf of $\{\pi_X(C_n): n \in N\}$. Since $\{\pi_X(C_n): n \in N\}$ is $(k + 1)$-nice, $\{\pi_X(B'_n): n \in N\}$ has a k-nice subsheaf. Therefore we can choose an infinite subset M of N and an infinite subsequence B''_m of B'_m for each $m \in M$ such that $\{\pi_X(B''_m): m \in M\}$ is a k-nice subsheaf of $\{\pi_X(B_n): n \in N\}$. By the inductive assumption for k, there exists a sequence $A = \{(x^{n_i}_{m_i}, y^{n_i}_{m_i}) \in B''_{n_i}: i \in N\}$ converging to the point x. By the definition of B''_{n_i}, $A \cap C_n$ is a finite set for every $n \in N$. Hence, by taking a subsequence of A, we can assume each $(x^{n_i}_{m_i}, y^{n_i}_{m_i})$ is contained in different C_n. The proof is completed.

Since each $\langle\alpha_3\text{-}FU\rangle$-space is an $\langle\alpha^{\omega}\text{-}St\rangle$-space, the following theorem gives a generalization of Theorem 5.16 of [2].

Theorem 3.2. *Let X be an $\langle\alpha^{\omega}\text{-}St\rangle$-space and Y be a regular countably compact Fréchet space. Then $Z = X \times Y$ is Fréchet.*

Proof. Let $A \subset Z$, $z = (x, y) \in \bar{A}$. If $z \in \operatorname{cl}(A \cap \{x\} \times Y)$, then the arguments are completed trivially. Therefore without loss of generality we can assume $A \cap \{x\} \times \times Y = \emptyset$. Let $\mathscr{V}_y$ be a neighborhood base of y in Y. Let $V \in \mathscr{V}_y$. We first show that there exist a convergent sequence $A(V)$ in $A \cap (X \times V)$ and its limit point $(x, y(V))$ such that $y(V) \in \bar{V}$. Clearly $x \in \operatorname{cl}(\pi_X(A \cap X \times V))$, so by using the Fréchetness of X, we can choose $x_n \in \pi_X(A \cap X \times V)$ such that $\lim_{n\to\infty} x_n = x$. Choose $(x_n, y_n) \in \pi_X^{-1}(x_n) \cap \cap A \cap X \times V$. Then, since Y is countably compact Fréchet, there exist a convergent subsequence $\{y_{n_k}: k \in N\}$ of $\{y_n: n \in N\}$ and its limit point $y(V)$. Put $A(V) = \{(x_{n_k}, y_{n_k}): k \in N\}$. Clearly $A(V)$ converges to $(x, y(V))$ and $y(V) \in \bar{V}$. By using the regularity of Y, it is easy to see $y \in \operatorname{cl}\{y(V): V \in \mathscr{V}\}$. Now the Fréchetness of Y implies that there exists $\{V_n: n \in N\}$ such that $\lim_{n\to\infty} y(V_n) = y$. Let $C_n = A(V_n)$ be a convergent sequence constructed for V_n in the above arguments. Put

$$C_n = \{(x_k^n, y_k^n): k \in N\}\,, \quad A_n = \{x_k^n: k \in N\}\,,$$
$$\mathscr{A} = \{A_n: n \in N\}\,, \quad y(V_n) = y^n\,.$$

Note that $\mathscr{A}$ has a k-nice subsheaf in X with vertex x for some k. By Lemma 3.1, there exists a sequence in A converging the point (x, y). The proof is completed.

It is easy to see that an $\langle\alpha_3\text{-}FU\rangle$-space is an $\langle\alpha_4\text{-}FU\rangle$-space (hence strongly Fréchet space) by the definition. But as is shown by the following example, Fréchet space with the property (α^k) need not be strongly Fréchet.

Example 3.3. Let S_ω be the sequential fan, i.e. $S_\omega = \{\infty\} \cup N \times N$, where each point of $N \times N$ is an isolated point, and the collection $\{\{\infty\} \cup \{(m, n): n > f(m)\}: f$ is a function from N into $N\}$ forms a neighborhood base of the point ∞. Then as is well-known S_ω is a Fréchet space with the property (α^k), but it is not strongly Fréchet.

Clearly an $\langle\alpha_3\text{-}FU\rangle$-space is $\langle\alpha^k\text{-}St\rangle$, but the converse need not be true under (CH). In fact we construct an $\langle\alpha^{k+1}\rangle$-space X_{k+1} which is not an $\langle\alpha^k\rangle$-space for each $k = 0, 1, \ldots$ under (CH).

We denote by βN the Stone-Čech compactification of N. For a subset A of N, $A^* = \operatorname{Cl}_{\beta N} A$. In the arguments below of this section, $\operatorname{Int} A$ and $\operatorname{Bdy} A$ denote respectively the interior of A in N^* and the boundary of A in N^* for a set $A \subset N^*$.

We recall the following well-known facts we shall use later.

Fact 1. *Two disjoint cozero sets in N^* have disjoint closures.*

Fact 2. *Let Z be a non-empty zero set in N^*. Then* $\operatorname{Int} Z \neq \emptyset$.

Lemma 3.4 (CH). *Let Z be a zero set in N^* with non-empty boundary* $\operatorname{Bdy} Z$

in N^*. Let H be a non-empty closed subset of Bdy Z *and U be a clopen subset of N^* with $H \subset U$. Then there exists a regular closed subset E_1 in N^* such that:*

(i-1) $E_1 \subset Z \cap U$,

(ii-1) $H = \text{Bdy}\, E_1$,

(iii-1) $H \subset \text{Bdy}\,(U - E_1)$.

Proof. Without loss of generality we can assume $U = N^*$. We construct F_1 by transfinite induction. Note that the cardinality of the set of all zero sets in N^* equals the cardinality of the continuum. Let $\{Z_\alpha: \alpha < \omega_1\}$ be the family of all zero sets in Z such that $H \cap Z_\alpha \neq \emptyset$ for $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal. Let $\{K_\alpha: \alpha < \omega_1\}$ be a family of zero sets in Z such that $K_\alpha \subset K_\beta$ for $\alpha > \beta$, $K_\alpha = \bigcap\{K_\beta: \beta < \alpha\}$ if α is a limit ordinal and $H = \bigcap\{K_\alpha: \alpha < \omega_1\}$.

Let O_1 and V_1 be non-empty disjoint clopen subsets of N^* contained in $Z_1 \cap K_1$. Inductively we assume that we have defined, for each $\beta < \alpha$, non-empty clopen subsets O_β and V_β of N^* such that:

(1) $\bigcup\{O_\gamma: \gamma < \beta\} \subset O_\beta$, $\bigcup\{V_\gamma: \gamma < \beta\} \subset V_\beta$,

(2) $(O_\beta - \bigcup\{O_\gamma: \gamma < \beta\}) \cup (V_\beta - \bigcup\{V_\gamma: \gamma < \beta\}) \subset K_\beta$,

(3) $O_\beta \cap Z_\beta \neq \emptyset$, $V_\beta \cap Z_\beta \neq \emptyset$,

(4) $O_\gamma \cap V_\delta = \emptyset$ for $\gamma, \delta < \alpha$.

We define O_α and V_α. We first define O'_α and V'_α as follows: Put $\hat{O}_\alpha = \bigcup\{O_\beta: \beta < \alpha\}$. If α is isolated, we put $O'_\alpha = \hat{O}_\alpha (= O_{\alpha-1})$ and $V'_\alpha = \hat{V}_\alpha (= V_{\alpha-1})$. Assume α is a limit ordinal. Then the relation

$$\bigcup\{(N^* - K_\beta) - O_\beta: \beta < \alpha\} = (N^* - \bigcap\{K_\beta: \beta < \alpha\}) - \hat{O}_\alpha$$

shows that $(N^* - \bigcap\{K_\beta: \beta < \alpha\}) - \hat{O}_\alpha$ is a cozero set in N^*. Thus it follows that $\hat{O}_\alpha$ and $(N^* - \bigcap\{K_\beta: \beta < \alpha\}) - \hat{O}_\alpha$ are disjoint cozero sets in N^*. By Fact 1 we can choose disjoint clopen subsets O'_α and V'_α which contain $\hat{O}_\alpha$ and $(N^* - \bigcap\{K_\beta: \beta < \alpha\}) - \hat{O}_\alpha$, respectively.

Choose non-empty disjoint clopen subsets S_α and T_α such that

$$S_\alpha \cup T_\alpha \subset K_\alpha \cap \text{Int}\, Z - O'_\alpha \cup V'_\alpha\,.$$

Let $O_\alpha = O'_\alpha \cup S_\alpha$ and $V_\alpha = V'_\alpha \cup T_\alpha$. We have chosen O_α and V_α $(\alpha < \omega_1)$ satisfying the conditions (1)–(4).

Put

$$E_1 = \text{Cl}(\bigcup\{O_\alpha: \alpha < \omega_1\})\,.$$

Then clearly E_1 is a regular closed set in N^* which satisfies (i-1). We show (ii-1) and (iii-1). Let U be any clopen subset of N^* with $U \cap H \neq \emptyset$. Then $U \cap Z \neq \emptyset$ by Fact 2. Since $U \cap Z$ is a zero set in N^* which has non-empty intersection with H, there exists $\alpha < \omega_1$ such that $Z_\alpha = U \cap Z$. Then $\emptyset \neq Z_\alpha \cap E_1 \subset U \cap E_1$ and $\emptyset \neq Z_\alpha \cap (N^* - E_1) \subset U \cap (N^* - E_1)$ by (3). Therefore $H \subset \text{Bdy}\, E_1$ and $H \subset \text{Bdy}\,(N^* - E_1)$. We show $\text{Bdy}\, E_1 \subset H$. By (2) $\text{Bdy}\, E_1 \subset K_\alpha$ for every $\alpha < \omega_1$, therefore $\text{Bdy}\, E_1 \subset \bigcap\{K_\alpha: \alpha < \omega_1\} = H$. The proof of Lemma 3.4 is completed.

We call a regular closed set E_1 constructed by the above method a *1-nice set with respect to* (Z, H, U). We shall define a k-nice set ($k \geqq 1$) with respect to (Z, H, U) by induction on k. The definition depends heavily on the above construction. Assume a k-nice set E_k with respect to (Z', H', U') has been defined for a zero set Z', a non-empty closed subset $H' \subset \text{Bdy } Z'$ and a clopen set $U'(\supset H')$.

$(k + 1)$-construction: Let Z, H, U be the same as the ones in Lemma 3.4. We shall define a $(k + 1)$-nice set E_{k+1} with respect to (Z, H, U) as follows: The notation used below is the same as in the proof of Lemma 3.4. We first define a regular closed set $L_\alpha \subset N^*$ by transifinite induction on α. Put $L_1 = O_1$. Assume L_β, $\beta < \alpha < \omega_1$ has been defined satisfying

(5) $L_\beta \subseteq U_\beta$ for $\beta < \alpha$,

(6) $L_\gamma \subset L_\beta$ for $\gamma < \beta < \alpha$,

(7) $S_\beta \subset L_\beta$ for $\beta < \alpha$,

(8) $L_\beta \cap O_\gamma \doteq L_\gamma$ for $\gamma < \beta < \alpha$,

(9) $\text{Cl}\, \hat{L}_\beta - \hat{L}_\beta \subset N^* - \hat{O}_\beta$, where $\hat{L}_\beta = \bigcup\{L_\gamma : \gamma < \beta\}$.

If α is isolated, then put $L_\alpha = L_{\alpha-1} \cup S_\alpha$. Let α be a limit ordinal. Put

(10) $H_\alpha = \text{Cl}\, \hat{L}_\alpha \cap (N^* - \hat{O}_\alpha)$.

Then

(11) $H_\alpha = \text{Cl}\, \hat{L}_\alpha - \hat{L}_\alpha$.

We show (11). Since $\hat{L}_\alpha \subset \hat{O}_\alpha$ by (5), $H_\alpha \cap \hat{L}_\alpha \subset (N^* - \hat{O}_\alpha) \cap \hat{L}_\alpha = \emptyset$. This shows $H_\alpha \subset \text{Cl}\, \hat{L}_\alpha - \hat{L}_\alpha$. We show the converse implication. Let $p \in \text{Cl}\, \hat{L}_\alpha - \hat{L}_\alpha$. It is enough to show $p \in N^* - \hat{O}_\alpha$. If $p \in \hat{O}_\alpha$, then $p \in O_\beta$ for some $\beta < \alpha$. By (8) $O_\beta - \hat{L}_\alpha = O_\beta - L_\beta$, therefore $O_\beta - L_\beta$ is an open neighborhood of p which is disjoint from $\hat{L}_\alpha$. This is a contradiction. The proof of the equality (11) is completed. By (10) and (11), H_α is a non-empty closed subset of the boundary of a zero set $N^* - \hat{O}_\alpha$; we can choose a k-nice set L'_α with respect to $(N^* - O_\alpha, H_\alpha, O'_\alpha)$ by the inductive assumption for k. Put $L_\alpha = L'_\alpha \cup \hat{L}_\alpha \cup S_\alpha$. We have defined L_α ($\alpha < \omega_1$) satisfying (5)–(9).

Put

$$E_{k+1} = \text{Cl} \bigcup\{L_\alpha : \alpha < \omega_1\}.$$

It is easy to prove that E_{k+1} is a regular closed set. We call a regular closed set E_{k+1} *$(k + 1)$-nice with respect to* (Z, H, U). For the sake of the convenience, we call a set E_0 *0-nice with respect to* (Z, H, U) if $E_0 = Z \cap U$.

Remark 3.5. *Let W be a clopen set with $W \cap H \neq \emptyset$. Then clearly $W \cap E_{k+1}$ is a $(k + 1)$-nice set with respect to $(W \cap Z, W \cap H, W \cap U)$.*

The properties of a k-nice set E_k are summarized by the following lemma.

Lemma 3.6 (CH). *Let Z, H, and U be the same as in Lemma 3.4. Then there exists a regular closed subset E_k in N^* such that:*

(i-k) $E_k \subset Z \cap U$,

(ii-k) $H = Bdy\, Z \cap \text{Bdy}\, E_k$,

(iii-k) $H \subset \text{Bdy}\,(U - E_k)$,

(iv-k) *Let J be a zero set in N^* such that $J \cap E_k \neq \emptyset$. Then* $\text{Int}\, J \cap E_k \neq \emptyset$.

(v-k) *Let J be a zero set in N^* such that $J \cap H \neq \emptyset$. Then there exist non-empty clopen subsets $P_i \subset Z \cap E_k$ and a clopen subset W of N^* such that $W \cap E_k - P$ is a $(k$-$1)$-nice set with respect to $(W - P$, Bdy P, $W)$, where $P = \bigcup\{P_i\colon i \in N\}$.*

(vi-k) *Let $\{P_i\colon i \in N\}$ be a collection of non-empty clopen subsets of N^* such that $P_i \subset E_k$ for every $n \in N$. Then there exist an infinite subset $M \subset N$, non-empty clopen set $Q_m \subset P_m$ for every $m \in M$ and a clopen set W of N^* such that $W \cap E_k - Q$ is a i-nice set $(0 \leqq i \leqq k - 1)$ with respect to $(W - Q$, Bdy Q, $W)$, where $Q = \bigcup\{Q_m\colon m \in M\}$.*

Proof. It is easy to prove (i-k) – (iii-k) by the construction and the arguments in the case $k = 1$. Since the boundary of E_k is the boundary of some i-nice set $(1 \leqq i \leqq k$-$1)$ by the inductive construction, it is easy to prove (iv-k) by the construction. We prove (v-k). Since $J \cap H \neq \emptyset$, by (5) and (7), we can choose $\alpha_1 < \alpha_2 < \ldots$ such that $P_{\alpha_i} \subset J \cap E_k$. Put $P_i = P_{\alpha_i}$ and $\alpha = \sup\{\alpha_i\colon i \in N\}$. Then the relation

$$(N^* - K_\alpha) - P = \bigcup\{(N^* - K_{\alpha_n}) - \bigcup\{P_{\alpha_i}\colon i = 1, 2, \ldots n\}\colon n \in N\}$$

shows that $(N^* - K_\alpha) - P$ is a cozero set in N^*. Let $W \subset O_\alpha$ be a clopen set which contains P and disjoint from $(N^* - K_\alpha) - P$. Then

$$W \cap \text{Bdy}\, E_k = W \cap \text{Bdy}\, L_\alpha = \text{Bdy}\, P$$

and $W \cap \hat{L}_\alpha = P$. By Remark 3.5, $W \cap E_k - P$ is a $(k$-$1)$-nice subset with respect to $(W - P$, Bdy P, $W)$. The proof of (v-k) is completed. The proof of (vi-k) can be done using a similar method as in the proof of (v-k) and the inductive hypothesis but it is routine. We left it to the reader. The proof of Lemma 3.6 is completed.

Let F be a closed subset of N^*. Put $X = N \cup \{F\}$ and topologize X as follows: each point of N is isolated and the family $\{U \cup \{F\}\colon U \subset N$ and $F \subset U^*\}$ is a neighborhood base of the point F in X. The following facts are well known.

Fact 3. *Let $A \subset N$. Then A converges to the point F in X if and only if $A^* \subset F$.*

Fact 4. ([7, Theorem 1]). *$X = N \cup \{F\}$ is strongly Fréchet if and only if F is regular closed in N^* and, for each zero set K in N^* such that $K \cap F \neq \emptyset$,* $\text{Int}\, K \cap F \neq \emptyset$.

Now we construct an $\langle\alpha^{k+1}$-$St\rangle$-space X_{k+1} which is not an $\langle\alpha^k$-$St\rangle$-space.

Example 3.7 (CH). Let E_{k+1} be a $(k + 1)$-nice set with respect to $(Z$, Bdy Z, $N^*)$. Put $F_{k+1} = (N^* - Z) \cup E_{k+1}$. Then $X_{k+1} = N \cup \{F_{k+1}\}$ is an $\langle\alpha^{k+1}$-$St\rangle$-space which is not an $\langle\alpha^k$-$St\rangle$-space for $k = 0, 1, \ldots$.

Proof. It is easy to see that the zero set Z can be expressed in the form $Z = N^* - \bigcup\{T_n^*\colon n \in N\}$, where $\{T_n\colon n \in N\}$ is a family of pairwise disjoint infinite subsets of N and $\bigcup\{T_n\colon n \in N\} = N$. Clearly $\mathscr{T} = \{T_n\colon n \in N\}$ is a sheaf in X with vertex F_{k+1} by Fact 3. We call such a sheaf $\mathscr{T}$ a principal sheaf in X_{k+1}.

Assertion 1. *The space X_1 does not satisfy* (α^0).

Proof. Let $\mathscr{T} = \{T_n : n \in N\}$ be a principal sheaf in X_1. Let $\mathscr{T}' = \{T'_m : n \in M\}$ be any sub-sheaf of $\mathscr{T}$. Put $T = \bigcup \mathscr{T}'$. Then $T^* \cap \operatorname{Bdy} Z \neq \emptyset$. Therefore there exists an infinite subset A of T such that $A^* \subset Z$ and $A^* \cap E_1 = \emptyset$ by (iii-1) By Fact 3 T is not a convergent sequence.

Assertion 2. *The space X_1 is an $\langle \alpha^1\text{-}St \rangle$-space.*

Proof. By fact 4 X_1 is strongly Fréchet. We show X_1 satisfies (α^1).

Let $\mathscr{A} = \{A_n : n \in N\}$ be a sheaf with vertex x. If $x \neq F_1$, then the arguments are completed trivially. So we assume $x = F_1$. By Fact 3, $A_n^* \subset F_1$. Note that the sets $A_n^* \cap (N^* - Z)$ and $A_n^* \cap E_1$ are clopen subsets of N^* for each $n \in N$. Indeed, if $A_n^* \cap (N^* - Z)$ is not clopen in N^*, then $\operatorname{Bdy} H \cap \operatorname{Cl}(A_n^* \cap (N^* - Z)) \neq \emptyset$, hence $A_n^* \cap \operatorname{Bdy} Z \neq \emptyset$. By (iii-1), $A_n^* \cap (Z - E_1) \neq \emptyset$, which contradicts $A_n^* \subset F_1$. Similarly we can show $A_n^* \cap E_1$ is a clopen subset in N^*. Since $A_n^* \cap (N^* - Z)$ and $A_n^* \cap E_1$ are clopen subsets in N^* we can choose $B_n \subset A_n$ and $C_n \subset A_n$ such that $B_n^* = A_n^* \cap (N^* - Z)$ and $C_n^* = A_n^* \cap E_1$.

Case I. $|\{n \in N : C_n^* \neq \emptyset\}| = \omega$.

Put $M = \{n \in N : C_n^* \neq \emptyset\}$. By (vi-1) in Lemma 3.6 there exist an infinite subset L of M and an infinite subset W of N such that $W^* \cap E_1 - P$ is a 0-nice set with respect to $(W^* - P, \operatorname{Bdy} P, W^*)$, where $P = \bigcup \{C_n^* : n \in N\}$. Therefore by the definition of a 0-nice set with respect to $(W^* - P, \operatorname{Bdy} P, W^*)$, $W^* \cap E_1 - P = W^* \cap \cap (W^* - P) = W^* - P$. Since $W^* \cap E_1 \subset E_1$ and $P \subset E_1$, $W^* \subset E_1$.

Let $\mathscr{A}' = \{A_m \cap W : m \in M\}$. Since $A_m \cap W$ contains $A_m \cap C_m$ except finitely many elements of N for each $m \in M$, it follows that $\mathscr{A}'$ is a subsheaf of $\mathscr{A}$. Let $\mathscr{D} = \{D_i : i \in N\}$ be a cross-sheaf of $\mathscr{A}'$. Then $D_i \subset W$ for any $i \in N$. The implication $W^* \subset E_1 \subset F_1$ shows W converges to F_1 in X_1 by Fact 1. Since $\bigcup \mathscr{D} \subset W$, $\mathscr{D}$ is 0-nice. This implies $\mathscr{A}'$ is 1-nice.

Case II. $|\{n \in N : C_n^* \neq \emptyset\}| < \omega$.

In this case $|\{n \in N : B_n^* \neq \emptyset\}| = \omega$. We prove this case dividing it into two cases.

Case II-i. There exist infinitely many $B_{n_j} (\neq \emptyset)$ and their infinite subsets $\tilde{B}_{n_j}$ such that $\operatorname{Bdy} Z \cap \operatorname{Cl} \bigcup \{\tilde{B}_{n_j}^* : j \in N\} = \emptyset$.

Choose a subset C of N such that

$$\operatorname{Cl} \bigcup \{\tilde{B}_{n_j}^* : j \in N\} \subset C^* \subset N^* - Z\,.$$

Put $\mathscr{A}' = \{\tilde{B}_{n_j} \cap U : j \in N\}$.

If $\tilde{B}_{n_j}^* \neq \emptyset$, then $A_{n_j} \cap C$ contains $A_{n_j} \cap \tilde{B}_{n_j}$ except for finitely many elements of N. Therefore $\mathscr{A}'$ is a subsheaf of $\mathscr{A}$. Let $\mathscr{D} = \{D_j : j \in N\}$ be a cross-sheaf of $\mathscr{A}'$. Then $D_j \subset C$ for any $j \in N$. The implication $C^* \subset N^* - Z \subset F$ shows C converges to F_1 by Fact 3. Hence $\mathscr{D}$ is 0-nice. Therefore $\mathscr{A}'$ is 1-nice.

Case II-ii. For each infinite subset B_{n_j}, and its infinite subset $\tilde{B}_{n_j}$, $\mathrm{Bdy}\, Z \cap \cap \mathrm{Cl} \bigcup \{\tilde{B}^*_{n_j}: n \in N\} \neq \emptyset$.

We first show $(N^* - Z) - \bigcup\{B^*_n: n \in N\}$ is a cozero set in N^*. Let $\mathcal{T} = \{T_i: i \in N\}$ be a principal sheaf in X_1. Since $N^* - Z = \bigcup\{T^*_i: i \in N\}$, it is enough to show $T^*_i - \bigcup\{B^*_n: n \in N\}$ is a cozero set in N^*. In fact, put $M_i = \{n \in N: T^*_i \cap B^*_n \neq \emptyset\}$, then by our assumption, M_i is a finite set. Therefore $T^*_i - \bigcup\{B^*_n: n \in N\}$ is a cozero set in N^*.

Since $(N^* - Z) - \bigcup\{B^*_n: n \in N\}$ and $\bigcup\{B^*_n: n \in N\}$ are disjoint cozero sets in N^*, it follows by Fact 1 that these have disjoint closures. Let C be a subset of N such that

$$\bigcup\{B^*_n: n \in N\} \subset C^* \subset (N^* - Z) - \bigcup\{B^*_n: n \in N\}.$$

Put $\mathcal{A}' = \{B_n \cap U: B_n \text{ is an infinite subset of } N\}$. Then $\mathcal{A}'$ is a subsheaf of $\mathcal{A}$. Let $\mathcal{D} = \{D_n: n \in N\}$ be a cross-sheaf of $\mathcal{A}'$. Then $D^*_n \subset C^*$ and $B^*_m \cap D^*_n = \emptyset$ for each $m \in N$. These show $D^*_n \cap (N^* - Z) = \emptyset$ for each $n \in N$. Therefore $D^*_n \subset E_1$. Since $\bigcup\{D^*_n: n \in N\}$ and $N^* - Z$ are disjoint cozero sets in N^*, they have disjoint closures in N^*. Choose a subset D of N such that

$$\bigcup\{D^*_n: n \in N\} \subset D^* \subset E_1.$$

Then D converges to F_1. Therefore $\mathcal{D}$ is 0-nice which implies $\mathcal{A}'$ is 1-nice.

To prove that X_{k+1} is not an $\langle\alpha^k\rangle$-space, it is enough to show the following assertion.

Assertion 3. *Assume a principal sheaf of X_i does not have any i-nice subsheaf for $i \leq k$. We show that a principal sheaf $\mathcal{T} = \{T_n: n \in N\}$ in X_{k+1} doesn't have a $(k+1)$-nice subsheaf.*

Proof. Let $\mathcal{T}' = \{T'_m: m \in M\}$ be an arbitrary subsheaf of $\mathcal{T}$. Then $Z' = N^* - - \bigcup(T'^*_m: n \in N\}$ is a zero set in N^* such that $Z' \cap \mathrm{Bdy}\, Z \neq \emptyset$. By (v−k+1) in Lemma 3.6, we can choose non-empty clopen sets $P_i \subset Z \cap E_{k+1}$ and a clopen set W in N^* such that $W \cap E_{k+1} - P$ is a k-nice set with respect to $(W - P, \mathrm{Bdy}\, P, W)$, where $P = \bigcup\{P_i: i \in N\}$. Let $\mathcal{S} = \{S_i: i \in N\}$ be a disjoint family of subsets of N such that $S^*_i = P_{\alpha_i}$ and $W = (\bigcup\mathcal{S})^*$. Then $\mathcal{S}$ is a principal sheaf of a subspace $Y = \bigcup\mathcal{S} \cup \{F_{k+1}\} = \bigcup\mathcal{S} \cup \{P \cap F_{k+1}\} = \bigcup\mathcal{S} \cup \{(N^* - P) \cup (W \cap E_{k+1} - P)\}$. We can consider Y as X_k. Therefore the principal sheaf $\mathcal{S}$ of Y does not have a k-nice subsheaf by the inductive assumption. Since $\mathcal{S}$ is a cross-sheaf of $\mathcal{T}'$ and $\mathcal{T}'$ is an arbitrary subsheaf of $\mathcal{T}$, $\mathcal{T}$ does not have a (k+1)-nice subsheaf. The proof of Assertion 3 is completed.

The proof that X_{k+1} is an $\langle\alpha^{k+1}\text{-}St\rangle$-space can be done by similar arguments of Assertion 2 using (vi-k+1), but it is routine. We left it to the reader.

Example 3.8. Let X_k be an $\langle\alpha^k\rangle$-space constructed in the above example 3.7. Then $\prod_{i=1}^{k} X_i$ is an $\langle\alpha^k\rangle$-space (hence $\langle\alpha^\omega\rangle$-space) for every $k \in N$ by Theorem 2.3, but $X = \prod_{i=1}^{\infty} X_i$ is not an $\langle\alpha^\omega\rangle$-space.

Proof. Let $\mathcal{T}_k = \{T_n^k: n \in N\}$ be a principal sheaf in X_k. Put $T_n^k = \{x_n^k(m): m \in N\}$ and

$$T(n) = \{(x_n^1(m), x_n^2(m), \ldots, x_n^n(m), F_{n+1}, F_{n+2}, \ldots): m \in N\} .$$

Then $T(n)$ is a convergent sequence in X with limit point $(F_1, F_2, \ldots)$. Put $\mathcal{A} = \{T(n): n \in N\}$. Then $\mathcal{A}$ is a sheaf in X with vertex $(F_1, F_2, \ldots)$. We show that $\mathcal{A}$ does't have a k-nice subsheaf for every $k \in N$. Let $\mathcal{A}'$ be a k-nice subsheaf of $\mathcal{A}$. Then $\pi_n(\mathcal{A}')$ is k-nice for every $n \in N$, where $\pi_n: X \to X_n$ is the projection. Since $\pi_n(\mathcal{A}')$ is a subsheaf of the principal sheaf $\mathcal{T}_n$, it is not n-nice, therefore it is not k-nice for $n \geqq k$. This is a contradiction. The proof is completed.

4. INVERSE LIMITS AND INFINITE PRODUCTS

The following lemma is well known but we give its proof for the sake of completeness.

Lemma 4.1. *A Fréchet space is strongly Fréchet if and only if it contains no subspace which is homeomorphic to S_ω* (see Example 3.2).

Proof. Assume X is Fréchet but not strongly Fréchet (hence X is not an $\langle\alpha_4\text{-}FU\rangle$-space). Let $\mathcal{A} = \{A_n: n \in N\}$ be a sheaf with vertex x which does not satisfy (α_4). Without loss of generality we assume $A_n \cap A_m = \emptyset$ for $n \neq m$. Then it is easy to show $\{x\} \cup \bigcup\{A_n: n \in N\}$ is homeomorphic to S_ω. The proof is completed.

Let $\{X_m: \pi_n^m\}$ be an inverse limiting system of a sequence of spaces X_m with the onto projections $\pi_n^m: X_m \to X_n$ $(m \geqq n)$. Let $X = \varprojlim \{X_m: \pi_n^m\}$ be the inverse limit of this system and $\pi_m: X \to X_m$ the projections.

The following theorem has been proved by the author and Y. Tanaka. We present it here wich the kind permission of Y. Tanaka.

Theorem 4.2. *Let $X = \varprojlim \{X_m; \pi_n^m\}$ and let X_m be strongly Fréchet for every $n \in N$. Then the following conditions are equivalent.*

(i) *X is a Fréchet space.*

(ii) *X is a strongly Fréchet space.*

Proof. Obviously (ii) $\to$ (i). We shall show (i) $\to$ (ii). Assume X is Fréchet but not strongly Fréchet. Then by Lemma 4.1 X contains a copy of $S_\omega = \{\infty\} \cup N \times N$. Put $T_n = \{n\} \times N$. In the arguments below we assume S_ω is a subspace of X. Put

$\mathcal{B}_n = \{B: B \text{ is open in } X_n \text{ and } \pi_n(\infty) \in B\}$,

$\mathcal{B}'_n = \{S_\omega \cap \pi_n^{-1}(B): B \in \mathcal{B}_n\}$,

$\mathcal{B}' = \bigcup\{\mathcal{B}'_n: n \in N\}$.

Then clearly $\mathcal{B}'_n \subset \mathcal{B}'_{n+1}$ for each $n \in N$ and $\mathcal{B}'$ is a neighborhood base of $\{\infty\}$ in S_ω.

Assertion. *For each $n \in N$, there exists an open neighborhood W_n of ∞ in S_ω such that*

$(*)$ *For any $B' \in \mathcal{B}'_n$, $|\{i \in N: B' \cap T_i - W_n \neq \emptyset\}| = \omega$.* We shall prove the above assertion in two steps.

Case I. For each $n \in N$, there exists a finite set $M_n \subset N$ such that

$$|\pi_n(T_j) - \bigcup\{\pi_n(T_i): i \in M_n\}| < \omega \quad \text{for each} \quad j \in N .$$

In this case there exist $i_0 \in M_n$, an infinite subset N_n of N and an infinite subsequence T_k' of T_k for each $k \in N_n$ such that

$$\pi_n(T_k') \subset \pi_n(T_{i_0}) \quad \text{for each} \quad k \in N_n .$$

Case I-i. $\pi_n(T_{i_0})$ is a finite set.

Since $\pi_n(T_k')$ is a convergent sequence with limit point $\pi_n(\infty)$, there exists an infinite subset $T_k'' \subset T_k'$ with $\pi_n(\infty) = \pi_n(T_k'')$ for $k \in N_n$. Choose $x_{m_k}^k \in T_k''$. Then $\pi_n(x_{m_k}^k) = \pi_n(\infty)$. Put

$$W_n = \{\infty\} \cup \bigcup\{\{x_m^k: m > n_k\}: k \in N_n\} \cup \{T_i: i \in N - N_n\} .$$

We show W_n satisfies $(*)$. Let $B' \in \mathscr{B}_n'$. Then $B' = S_\omega \cap \pi_n^{-1}(B)$ for some $B \in \mathscr{B}_n$. We show $B' \cap T_k - W_n \neq \emptyset$ for $k \in N_n$. The relation $\pi_n(\infty) \in B$ implies $T_k'' \subset B'$, hence $x_{n_k}^k \in B'$. However $x_{m_k}^k \notin W_n$. This shows $B' \cap T_k - W_n \neq \emptyset$. The proof of Case I-i is completed.

Case I-ii. $\pi_n(T_{i_0})$ is an infinite set.

Put

$$\pi_n(T_{i_0}) - \{\pi_n(\infty)\} = \{y_k: k \in N\} \quad \text{and} \quad \lim_{k \to \infty} y_k = \pi_n(\infty) .$$

Put

$$W_n = \{\infty\} \cup \{T_k - \pi_n^{-1}(\{y_j: j \leqq k\}): k \in N_n\} \cup \{T_k: k \in N - N_n\} .$$

We show W_n satisfies $(*)$. Let $B' \in \mathscr{B}_n'$, $B' = S_\omega \cap \pi_n^{-1}(B)$ for some $B \in \mathscr{B}_n$. Then $\pi_n(\infty) \in B$ and $\lim_{k \to \infty} y_k = \pi_n(\infty)$ imply that there exists $k_0 \in N$ such that $y_k \in B$ for $k \geqq k_0$. For $k \geqq k_0$ and $k \in N_n$, $S_\omega \cap \pi_n^{-1}(y_k) \subset B'$ but $\pi_n^{-1}(y_k) \cap T_k = \emptyset$. This shows $B' \cap T_k - W_n = \emptyset$. The proof of Assertion is completed in this case.

Case II. There exists $n \in N$ such that: for each finite subset M of N, there exists $j \in N$ such that

$$|\pi_n(T_j) - \bigcup\{\pi_n(T_i): i \in M\}| = \omega .$$

By induction on n it is easy to choose an infinite subset N_n of N and an infinite subset T_k' of T_k for each $k \in N_n$ such that

(2) $$\pi_n|T_k': T_k' \to \pi_n(T_k') \quad \text{is one-to-one} ,$$

(3) $$\pi_n(T_k') \cap \pi_n(T_m') = \emptyset \quad \text{for} \quad k, m \in N_n \quad \text{and} \quad k \neq m .$$

Since X_n is strongly Fréchet (hence $\langle \alpha_4\text{-}FU \rangle$), there exists a convergent sequence K in X_n which satisfies (α_4) with respect to $\{\pi_n(T_k'): k \in N_n\}$. Put $K = \{\pi_n(x_{m_i}^{n_i}): i \in N\}$. By taking a subsequence of K we can assume $n_i \neq n_j$ if $i \neq j$. Put $L = \{n_i: i \in N\}$ and put

$$W_n = \{\infty\} \cup \bigcup\{\{x_m^{n_i}: m > m_i\}: i \in N\} \cup \bigcup\{T_j: j \in N - L\} ,$$

it is a neighborhood of ∞ in S_ω. We show W_n satisfies $(*)$.

Let $B' \in \mathscr{B}_n$, $B' = S_\omega \cap \pi_n^{-1}(B)$ for some $B \in \mathscr{B}_n$. Since $\{\pi_n(x_{m_i}^{n_i}): m \geqq n_i\}$ converges to $\pi_n(\infty)$, B contains $\{\pi_n(x_{m_i}^{n_i}): m \geqq n_i\}$ except for finitely many elements. Therefore B' contains $\{x_{m_i}^{n_i}: m \geqq n\}$ except for finitely many elements. On the other hand $W_n \cap \{x_{m_i}^{n_i}: i \in N\} = \emptyset$. These show that $B' - W_n \neq \emptyset$. The proof of the Assertion is completed.

Let $\{W_n: n \in N\}$ be a sequence of open neighborhoods of ∞ in S_ω the existence of which is guranteed by the Assertion. We can assume $W_n \supset W_{n+1}$ for each $n \in N$. Put

$$T_n' = T_n \cap W_n ,$$
$$W = \{\infty\} \cup \{T_n': n \in N\}.$$

Then W is a neighborhood of ∞ in S_ω. We shall show $B' - W \neq \emptyset$ for each $B' \in \mathscr{B}'$. This is a contradiction because $\mathscr{B}'$ is a neighborhood base of ∞ in S_ω. Note that

$$W - W_n \subset \bigcup\{T_i: i = 1, 2, \ldots, n\} .$$

Therefore

$$B' - W \supset B' - (W_n - \bigcup\{T_i: i = 1, 2, \ldots, n\}) .$$

By Assertion, there exists T_k, $k > n$ such that

$$B' \cap T_k - W_n \neq \emptyset .$$

This shows

$$B' - W \supset B' \cap T_k - (W_n \cup \{T_i: i = 1, 2, \ldots, n\}) \cap T_k \supset$$
$$\supset B' \cap T_k - W_n \cap T_k \neq \emptyset .$$

The proof is completed.

Corollary 4.3. *Let $|X_i| \geqq 2$ for infinitely many $n \in N$. Then the following* (i) *and* (ii) *are equivalent.*

(i) $\prod_{i=1}^{\infty} X_i$ *is Fréchet.*

(ii) $\prod_{i=1}^{\infty} X_i$ *is strongly Fréchet.*

Proof. The implication (ii) $\rightarrow$ (i) is trivial. We show (i) $\rightarrow$ (ii). It is enough to show that $\prod_{i=1}^{n} X_i$ is strongly Fréchet for each $n \in N$. In fact $\prod_{i=n+1}^{\infty} X_n$ is a non-discrete Fréchet space. Choose an infinite set $\{x_i: i \in N\}$ and a point x_0 in $\prod_{i=n+1}^{\infty} X_i$ such that $\lim_{i \to \infty} x_i = x_0$. Then $\prod_{i=1}^{n} X_i \times (\{x_0\} \cup \{x_i: i \in N\})$ is Fréchet. By E. Michael's theorem (see § 3, introduction), $\prod_{i=1}^{n} X_i$ is strongly Fréchet. The proof is completed.

Lemma 4.4. *Let $X = \varprojlim \{X_m: \pi_n^m\}$ and each X_n be an $\langle \alpha^k\text{-}St \rangle$-space. Let $A \subset X$ and $x \in \bar{A} - A$. Then there exists a k-nice sheaf $\mathscr{A}_n$ in X_n with vertex $\pi_n(x)$ satisfying*:

(i) $\bigcup \mathscr{A}_n \subset \pi_n(A)$,

(ii) $\pi_n^m(\mathscr{A}_m)$ *is a subsheaf of $\mathscr{A}_n$ for $m > n$.*

Proof. Since $\pi_n(x) \in \mathrm{Cl}\, \pi_n(A)$, the Fréchetness of X_n implies that there exists an infinite sequence $A_n \subset \pi_n(A)$ converging to $\pi_n(x)$. Without loss of generality, we can choose the sequence satisfying:

$(*)$ $\pi_i^n(A_n) = \pi_i(x)$ or otherwise $\pi_i^n(A_n)$ is an infinite set which does not contain $\pi_i(x)$ for each $i \leqq n$.

Put $\mathscr{A}^1 = \{\pi_1^n(A_n): n \in N\}$. Then $\mathscr{A}^1$ is a sheaf with vertex $\pi_1(x)$. As X_1 is an $\langle\alpha^k\text{-}St\rangle$-space, choose a k-nice subsheaf $\mathscr{A}_1 = \{A_m^1: m \in N_1\}$ of $\mathscr{A}^1$, where A_m^1 is an infinite subsequence of $\pi_1^m(A_m)$ for $m \in N_1$.

Assume a k-nice sheaf $\mathscr{A}_i = \{A_m^i: m \in N_i\}$ in X_i has been defined for $i \leqq n$. We define a k-nice sheaf $\mathscr{A}_{n+1}$ in X_{n+1} as follows:

Put

$$\mathscr{A}^{n+1} = \{(\pi_n^{n+1})^{-1}(A_m^n) \cap \pi_{n+1}^m(A_m): m \geqq n+1,\ m \in N_n\} .$$

Then $\mathscr{A}^{n+1}$ is a sheaf in X_{n+1} with vertex $\pi_n(x)$. Choose a k-nice subsheaf $\mathscr{A}_{n+1} = \{A_m^{n+1}: m \in N_{n+1}\}$ of $\mathscr{A}^{n+1}$, where A_m^{n+1} is an infinite subsequence of $(\pi_n^{n+1})^{-1}(A_m^n) \cap \pi_{n+1}^m(A_m)$ for every $m \in N_{n+1}$. We have defined a $(k+1)$-nice sheaf $\mathscr{A}_n$ with vertex $\pi_n(x)$ for every $n \in N$ satisfying the conditions (i) and (ii). The proof is completed.

Lemma 4.5. *Let $X = \varprojlim \{X_m: \pi_n^m\}$ and X_n be a strongly Fréchet space for every $n \in N$. Let $x \in X$ and $\mathscr{A}_n$ be a k-nice sheaf in X_n with vertex $\pi_n(x)$ such that $\pi_n^m(\mathscr{A}_m)$ is a subsheaf of $\mathscr{A}_n$ for every $n \leqq m$. Then there exists a convergent sequence K_n in X_n with limit point $\pi_n(x)$ satisfying*:

(1) $K_n \subset \bigcup \mathscr{A}_n$,

(2) $\pi_n^m(K_m) \subset K_n$ *for* $n \leqq m$.

Proof. We show the assertion by induction on k. For $k = 0$, the assertion is trivial since a 0-nice sheaf is nothing but a convergent sequence. Assume the assertion has been proved for k $(\geqq 0)$. We show the case $k+1$.

Since X_n is a strongly Fréchet space, there exists a convergent sequence B_n which satisfies $\langle\alpha_4\rangle$ with respect to $\mathscr{A}_n$. Put

$$\mathscr{B}_1 = \{\pi_1^n(B_n): n \in N\} .$$

Then $\mathscr{B}_1$ is a cross-sheaf of a $(k+1)$-nice sheaf $\mathscr{A}_1$. Choose a k-nice subsheaf $\mathscr{B}_1' = \{B_m^1: m \in N_1\}$ of $\mathscr{B}_1$, where $N_1 \subset N$ and B_m^1 is an infinite subsequence of $\pi_1^m(B_m)$ for $m \in N_1$. Assume that a k-nice sheaf $\mathscr{B}_i' = \{B_m^i: m \in N_i\}$ which is a cross-sheaf of $\mathscr{A}_i$ has been defined for $i \leqq n$ such that $\pi_j^i(\mathscr{B}_i')$ is a subsheaf of $\mathscr{B}_j'$. We construct $\mathscr{B}_{n+1}'$ as follows: Put

$$\mathscr{B}_{n+1} = \{(\pi_n^{n+1})^{-1}(B_m^n) \cap \pi_{n+1}^m(B_m): m \geqq n+1 \text{ and } m \in N_n\} .$$

Then $\mathscr{B}_{n+1}$ is a cross-sheaf of $\mathscr{A}_{n+1}$. Choose a k-nice subsheaf $\mathscr{B}_{n+1}' = \{B_m^{n+1}: m \in N_{n+1}\}$ of $\mathscr{B}_{n+1}$, where $N_{n+1} \subset N_n$ and B_m^{n+1} is an infinite subsequence of $(\pi_n^{n+1})^{-1}(B_m^n) \cap \pi_{n+1}^m(B_m)$ for $m \in N_{n+1}$. Since $\pi_j^i(\mathscr{B}_i')$ is a k-nice subsheaf of $\mathscr{B}_j'$, by the inductive assumption for k, there exists $K_n \subset \bigcup \mathscr{B}_n' \subset \bigcup \mathscr{A}_n$ satisfying $\pi_n^m(K_m) \subset K_n$ for $n \leqq m$. The proof is completed.

Lemma 4.6. *Let $X = \varprojlim \{X_m, \pi_n^m\}$. Let $A \subset X$ and $x \in \bar{A} - A$. If there exists a convergent sequence K_n in X_n with limit point $\pi_n(x)$ satisfying:*

(1) $K_n \subset \pi_n(A)$,

(2) $\pi_n^m(K_m) \subset K_n$ *for* $n \leqq m$,

then there exists a sequence $\{a_n: n \in N\}$ in A converging to the point x.

Proof. Put $K_n = \{b_m^n: m \in N\}$ and choose $a_1 \in \pi_1^{-1}(b_1^1) \cap A$. Assume $\{a_1, a_2, \ldots, a_n\}$ has been chosen. Let

$$K'_{n+1} = K_{n+1} - \bigcup\{(\pi_j^i)^{-1}(b_j^i): n \geqq i \geqq j,\ b_j^i \neq \pi_i(x)\}.$$

Since $(\pi_j^i)^{-1}(b_j^i) \cap K_i$ is a finite set for $b_j^i \neq \pi_i(x)$, we have $K'_{n+1} \neq \emptyset$. Choose $b \in K'_{n+1}$ and $a_{n+1} \in \pi_n^{-1}(b) \cap A$. We have chosen a_n for every $n \in N$. We show $\lim_{n\to\infty} a_n = x$. Let W be any neighborhood of x in X. Then $x \in \pi_n^{-1}(U) \subset W$ for some $n \in N$, where U is an open neighborhood of $\pi_n(x)$ in X_n. We show $\{a_n: n \in N\}$ is contained in $\pi_n^{-1}(U)$ except for finitely many elements. For $n < m$, $\pi_n(a_m) = \pi_n^m(\pi_m(a_m)) \in \pi_n^m(K_m) \subset K_n$ and K_n converges to $\pi_n(x)$. Since U contains K_n except for finitely many elements, we have that U contains $\{\pi_n(a_m): m \in N\}$ except for finitely many elements, say $\pi_n(a_{i_k}) \notin U$ for $k = 1, 2, \ldots, j$. We show $\pi_n^{-1}(a_{i_k}) \cap \{a_n: n \in N\}$ is a finite set for $k = 1, 2, \ldots, j$. Put $\pi_n(a_{i_k}) = b_k^m$. Then $\pi_n(a_i) \neq b_k^m$ for $i > k$ by the definition of K'_i and a_i. Therefore $\{a_n: n \in N\}$ is contained in $\pi_n^{-1}(U)$ except for finitely many elements. The proof is completed.

The following theorem follows from Theorem 4.2, Lemma 4.4, 4.5 and 4.6.

Theorem 4.7. *Let $X = \varprojlim \{X_m: \pi_n^m\}$. If X_n is an $\langle\alpha^k\text{-}St\rangle$-space for every $n \in N$, then X is an $\langle\alpha^k\text{-}St\rangle$-space.*

Corollary 4.8. *If $\prod_{i=1}^n X_i$ is an $\langle\alpha^k\text{-}St\rangle$-space for every $n \in N$, then*

$$X = \prod_{i=1}^\infty X_i \text{ is an } \langle\alpha^k\text{-}St\rangle\text{-space}.$$

Corollary 4.9. *Let X_i be a regular countably compact Fréchet space with the property (α^k) for every $i \in N$. Then $X = \prod_{i=1}^\infty X_i$ is countably compact Fréchet.*

Proof. According to [9, Theorem 4.5], X is countably compact. We show $\prod_{i=1}^n X_i$ is $\langle\alpha^k\text{-}St\rangle$ for every $n \in N$. Since a regular countably compact Fréchet space is strongly Fréchet [13, Corollary 5.2], we have X_i is an $\langle\alpha^k\text{-}St\rangle$-space for every $i \in N$. By Theorem 2.3, $\prod_{i=1}^n X_i$ is an $\langle\alpha^k\text{-}St\rangle$-space for each $n \in N$. Therefore X is Fréchet. The proof is completed.

Problems 4.10. *Let $\mathscr{P}$ be a class of spaces. Let $\mathscr{F}(\mathscr{P}) = \{X: X \times Y$ is Fréchet for any $Y \in \mathscr{P}\}$. We use the following notations:*

C = *the class of compact Fréchet spaces,*

CC = *the class of countably compact Fréchet spaces,*
B = *the class of bi-sequential spaces* (see [8] for the definition),
St = *the class of strongly Fréchet spaces.*

(1) *Is* $\mathscr{F}(C) = \mathscr{F}(CC)$?
(2) *Give inner characterizations of classes* $\mathscr{F}(C)$, $\mathscr{F}(CC)$ *and* $\mathscr{F}(St)$.
(3) *Is there a "naive" example of a* $\mathscr{F}(CC)$*-space which is not bi-sequential?*
(4) *Is there an example of a* $\mathscr{F}(CC)$*-space which is not an* $\langle\alpha^\omega\text{-}FU\rangle$*-space?*
(5) *Is* $B = \mathscr{F}(St)$?
(6) *Let* X_i *be an* $\langle\alpha^\omega\text{-}St\rangle$*-space for every* $i \in N$. *Then is* $\prod_{i=1}^{\infty} X_i$ *Fréchet?*

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