Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 2, 225–231

Persistent URL: http://dml.cz/dmlcz/102297

Czechoslovak Mathematical Journal, 39 (114) 1989, Praha

TESTS FOR DISCONJUGACY AND STRICT DISCONJUGACY OF LINEAR DIFFERENTIAL EQUATIONS WITH DELAYS

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(Received September 29, 1986)

In [2], the author has introduced the notions of disconjugate and strictly disconjugate linear differential equation with delay. They are generalizations of similar notions for an ordinary linear differential equation without delay (see Definition 2 and Definition 4 below), and are closely related to the existence and uniqueness of the solution of multipoint boundary value problems for linear differential equations with delay (see [2]). The purpose of this paper is to give tests for disconjugacy and strict disconjugacy of linear differential equations with delays. The paper is divided in two parts: in the first part we give one criterion for strict disconjugacy and in the second part one criterion for disconjugacy of an ordinary linear differential equation with delays. In the sequel we will follow the paper [3] by A. Ju. Levin in which he deals with the case of linear differential equations without delay.

I

Let us consider the n-th order linear differential equation with delays

$$x^{(n)}(t) + \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}(t)\, x^{(n-i)}(t - \Delta_j(t)) = 0 \tag{E_n}$$

with continuous coefficients $a_{ij}(t)$ and delays $\Delta_j(t) \geqq 0$ on an interval I ($i = 1, 2, \ldots, n;\ j = 1, \ldots, m$).

For $t_0 \in I$ let us denote by $B'(E_n, t_0)$ the set of all solutions of (E_n) with constant initial functions

$$\phi_0(t), \phi_1(t), \ldots, \phi_{n-1}(t)$$

which are defined on the initial set E_{t_0}, and by $B(E_n, t_0)$ the set of all $x(t) \in B'(E_n, t_0)$ with $\phi_0(t) = 0,\ t \in E_{t_0}$.

Let $x(t) \in B'(E_n, t_0)$, $x(t) \not\equiv 0$. The n-th consecutive zero (including multiplicity) of $x(t)$ to the right of t_0 will be denoted by $\eta(x, t_0)$.

Definition 1. (A. Haščák [2].) Let $a \in I$. By the *first adjoint point* to the point a (with respect to (E_n)) we mean the point

$$\alpha_1(a) = \inf\{\eta(x, a):\ x \in B'(E_n, a),\ x(t) \not\equiv 0\}\,.$$

Definition 2. (A. Haščák [2].) The equation (E_n) is said to be *strictly disconjugate on an interval J* iff for each $a \in J$ the following implication holds:

$$a \in J \Rightarrow \alpha_1(a) \notin J\,.$$

Lemma 1. (A. Ju. Levin [3], see also [1] p. 84.) *Suppose* $y \in C^n$ *on the interval* $I = \langle a, b \rangle$ *and*

$$|y^{(n)}(t)| \leqq \mu \quad \text{for} \quad a \leqq t \leqq b\,,$$

$$y(a_0) = y'(a_1) = \ldots = y^{(n-1)}(a_{n-1}) = 0\,, \tag{1}$$

where

$$a \leqq a_0 \leqq a_1 \leqq \ldots \leqq a_{n-1} \leqq b \tag{2}$$

or

$$a \leqq a_{n-1} \leqq \ldots \leqq a_1 \leqq a_0 \leqq b\,. \tag{2'}$$

Then

$$|y^{(n-k)}(t)| \leqq \frac{\mu(b-a)^k}{k\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]!} \tag{3}$$

for $a \leqq t \leqq b$; $k = 1, 2, \ldots, n$.

Lemma 2. *Suppose* $x \in B'(E_n, t_0)$, $t_0 \in \langle a, b)$ *and*

$$|x^{(n)}(t)| \leqq \mu \quad \text{for} \quad t_0 \leqq t \leqq b\,,$$

$$x(a_0) = x'(a_1) = \ldots = x^{(n-1)}(a_{n-1}) = 0 \tag{4}$$

where

$$t_0 \leqq a_0 \leqq a_1 \leqq \ldots \leqq a_{n-1} \leqq b \tag{5}$$

or

$$t_0 \leqq a_{n-1} \leqq \ldots \leqq a_1 \leqq a_0 \leqq b\,. \tag{5'}$$

Then

$$|x^{(n-k)}(t)| \leqq \frac{\mu(b-a)^k}{k\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]!} \tag{6}$$

or $t_0 \leqq t \leqq b$, $k = 1, 2, \ldots, n$, *and*

$$|x^{(n-k)}(t - \Delta_j(t))| \leqq \frac{\mu(b-a)^k}{k\left[\frac{k-1}{2}\right]!\left[\frac{k}{2}\right]!} \tag{6'}$$

for $t_0 \leqq t \leqq b$, $k = 1, 2, \ldots, n$ *and* $j = 1, \ldots, m$.

Proof. Applying Lemma 1 to $x(t)$ on the interval $\langle t_0, b \rangle$ we get (6). From this inequality we get that (6') is valid for

$$t_0 \leqq t - \Delta_j(t) \leqq b\,.$$

Further, since $x \in B'(E_n, t_0)$, we have

$$x(t - \Delta_j(t)) = x(t_0) \quad \text{for} \quad t - \Delta_j(t) < t_0$$

and thus (6') is valid also for $t - \Delta_j(t) \leqq b$, i.e. for $t \in \langle t_0, b \rangle$.

Example 1. The function $x(t) = (t - 1)^2$, $t \in \langle 0, 1 \rangle$ is a solution of the equation

$$x''(t) - 2\,x(t - 1) = 0\,, \quad t \in \langle 0, 1 \rangle\,, \tag{7}$$

with the initial functions $\phi_0(t) = 1$, $\phi_1(t) = -2$ for $t \in \langle -1, 0 \rangle$. Thus $\mu = 2$. By (6') we get

$$1 = x(t - 1) \leqq \frac{2(1 - 0)^2}{2\left[\frac{2 - 1}{2}\right]!\left[\frac{2}{2}\right]!} = 1\,,$$

i.e. equality in (6') may be really attained.

Theorem 1. *Suppose*

$$|a_{ij}(t)| \leqq A_{ij} \tag{8}$$

for all t in a compact interval $I = \langle a, b \rangle$ $(i = 1, \ldots, n,\ j = 1, \ldots, m)$. Then the equation (E_n) is strictly disconjugate on I if

$$\chi(b - a) < 1 \tag{9}$$

where

$$\chi(h) = \sum_{i=1}^{n} \frac{\sum_{j=1}^{m} A_{ij}}{i\left[\frac{i - 1}{2}\right]!\left[\frac{i}{2}\right]!}\, h^i\,. \tag{10}$$

Proof. Suppose on the contrary that (E_n) is not strictly disconjugate on I. Then there is a point $t_0 \in \langle a, b)$ such that (E_n) has a nontrivial solution $x \in B'(E_n, t_0)$ with at least n zeros (including multiplicity) in $\langle t_0, b \rangle$. From this fact and Rolle's Theorem there are points $a_0, a_1, \ldots, a_{n-1}$ such that the inequalities (5) (or (5')) and the equalities (4) are fulfilled. The interval $\langle t_0, a_{n-1} \rangle$ is nondegenerate, since $x(t)$ cannot have a zero of multiplicity n at t_0 (if $x^{(k)}(t_0) = 0$, $k = 0, 1, \ldots, n - 1$ then $x(t)$ is a solution of (E_n) with the initial functions $\phi_k(t) = 0$, $t \in E_{t_0}$, $k = 0, 1, \ldots \ldots, n - 1$, and thus $x(t)$ is the trivial solution). Let us denote $a_{n-1} = c$. Applying Lemma 2 to the interval $\langle t_0, c \rangle$ we obtain

$$\max_{t_0 \leqq t \leqq c} |x^{(n-k)}(t - \Delta_j(t))| \leqq \frac{\mu(b - a)^k}{k\left[\frac{k - 1}{2}\right]!\left[\frac{k}{2}\right]!}\,, \tag{11}$$

$$k = 1, 2, \ldots, n\,; \quad j = 1, \ldots, m$$

where

$$\mu = \max_{t_0 \leqq t \leqq c} |x^{(n)}(t)|\,.$$

However, for some $\tau \in \langle t_0, c\rangle$ we have

(12) $$\max_{t_0 \leqq t \leqq c} |x^{(n)}(t)| = |x^{(n)}(\tau)| = \mu .$$

From (12), (E_n) and (11) we get

$$\mu = |x^{(n)}(\tau)| = \Big|\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}(\tau)\, x^{(n-i)}(\tau - \Delta_j(\tau))\Big| \leqq$$

$$\leqq \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} |x^{(n-i)}(\tau - \Delta_j(\tau))| \leqq \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} \frac{\mu(b-a)^i}{i\left[\dfrac{i-1}{2}\right]!\left[\dfrac{i}{2}\right]!} =$$

$$= \mu \sum_{i=1}^{n} \frac{\sum_{j=1}^{m} A_{ij}}{i\left[\dfrac{i-1}{2}\right]!\left[\dfrac{i}{2}\right]!} (b-a)^i .$$

Evidently $\mu > 0$, since otherwise $x(t)$ would coincide on $\langle t_0, c\rangle$ with a polynomial of degree $m < n$ and $x^{(m)}(t)$ would not vanish on $\langle t_0, c\rangle$ (but $x^{(m)}(a_m) = 0$). Hence $\chi(b - a) \geqq 1$, which is a contradiction with (9).

Corollary 1. *Equation (7) is not strictly disconjugate on $\langle 0, 1\rangle$ and $\chi(1 - 0) = 1$. Thus in general the sign of strict inequality in (9) cannot be substituted by the sign $\leqq$.*

II

Definition 3. (A. Haščák [2].) Let $a \in I$. By the *adjoint point to the point a* (with respect to (E_n)) we mean the point

$$\alpha(a) = \inf\{\eta(x, a)\colon\ x \in B(E_n, a),\ x(t) \not\equiv 0\} .$$

Definition 4. (A. Haščák [2].) The equation (E_n) is said to be *disconjugate on an interval J* iff for each $a \in J$ the following implication holds:

$$a \in J \Rightarrow \alpha(a) \notin J .$$

Lemma 3. *Suppose $x \in B(E_n, t_0)$, $t_0 \in \langle a, b)$ (and $x(t) \not\equiv 0$, $t \in \langle t_0 b\rangle$) has at least n zeros on $I = \langle t_0, b\rangle$. Then*

i) *there are points*

$$a_0, a_1, \ldots, a_{2n-2}$$

such that

(13) $$t_0 \leqq a_0 \leqq a_1 \leqq \ldots \leqq a_{2n-2} \leqq b ,$$

(14) $$0 = x(a_0) = x'(a_1) = \ldots = x^{(n-2)}(a_{n-2}) = x^{(n-1)}(a_{n-1}) = \\ = x^{(n-2)}(a_n) = \ldots = x'(a_{2n-3}) = x(a_{2n-2})$$

and

ii) *the subintervals $\langle t_0, a_{n-1}\rangle$, $\langle a_{n-1}, b\rangle$ are nondegenerate.*

Proof. Let $a_1^{(0)}, \dots, a_n^{(0)}$ be n zeros of $x(t)$ such that

$$t_0 \leqq a_1^{(0)} \leqq \dots \leqq a_n^{(0)} \leqq b\,.$$

Since $x \in B(E_n, t_0)$, $x(t) \not\equiv 0$ we have that t_0 is a zero of $x(t)$ of multiplicity less than n (since otherwise $x(t) \equiv 0$ would hold). Hence $x(t)$ has at least two distinct zeros in $\langle t_0, b\rangle$, i.e.

$$t_0 = a_1^{(0)} < a_n^{(0)} \leqq b\,.$$

By Rolle's Theorem there are $(n-1)$ zeros $a_1^{(1)}, \dots, a_{n-1}^{(1)}$ of $x'(t)$ such that $a_i^{(0)} \leqq \leqq a_i^{(1)} \leqq a_{i+1}^{(0)}$, $i = 1, \dots, n-1$ and

$$t_0 \leqq a_1^{(1)} < a_{n-1}^{(1)} \leqq b\,.$$

Repeating this process we eventually obtain a zero $a_1^{(n-1)}$ of $x^{(n-1)}(t)$ between two zeros $a_1^{(n-2)}, a_2^{(n-2)}$ of $x^{(n-2)}(t)$. The points

$$a_1^{(0)}, a_1^{(1)}, \dots, a_1^{(n-2)}, a_1^{(n-1)}, a_2^{(n-2)}, \dots, a_{n-1}^{(1)}, a_n^{(0)}$$

satisfy (13), (14), and since

$$t_0 \leqq a_1^{(n-2)} < a_2^{(n-2)} \leqq b$$

we have

$$t_0 \neq a_1^{(n-1)} \neq b\,.$$

The proof of the lemma is complete.

Corollary 2. i) *of Lemma* 3 *is in fact a consequence of Lemma* 2 *of* [3].

Theorem 2. *Suppose that the estimates* (8) *are valid. Then the equation* (E_n) *is disconjugate on* $I = \langle a, b\rangle$ *if*

$$\chi\left(\frac{b-a}{2}\right) < 1\,. \tag{15}$$

Proof. Suppose on the contrary that (E_n) is not disconjugate on I. Then there is a point $t_0 \in \langle a, b)$ such that (E_n) has a nontrivial solution $x \in B(E_n, t_0)$ with at least n zeros (including multiplicity) in $\langle t_0, b\rangle$. By Lemma 3 there are points $a_0, a_1, \dots \dots, a_{2n-2}$ such that (13), (14) hold and the subintervals $\langle t_0, a_{n-1}\rangle$, $\langle a_{n-1}, b\rangle$ are not degenerate. Let us denote $a_{n-1} = c$. One of these two subintervals, say $\langle t_0, c\rangle$, has length at most $\frac{1}{2}(b-a)$. Applying Lemma 2 to this interval we obtain

$$\max_{t_0 \leqq t \leqq c} \left|x^{(n-k)}(t - \Delta_j(t))\right| \leqq \frac{\mu(b-a)^k}{2^k \left[\dfrac{k-1}{2}\right]! \left[\dfrac{k}{2}\right]!\; k}\,,$$

$$k = 1, 2, \dots, n\,; \quad j = 1, \dots, m\,.$$

where

$$\mu = \max_{t_0 \leqq t \leqq c} \left|x^{(n)}(t)\right|\,.$$

However, for some $\tau \in \langle t_0, c\rangle$ we have

$$\mu = |x^{(n)}(\tau)| = |\sum_{k=1}^{n} \sum_{j=1}^{m} a_{kj}(\tau)\, x^{(n-k)}(\tau - \Delta_j(\tau))| \leqq$$

$$\leqq \sum_{k=1}^{n} \sum_{j=1}^{m} A_{kj} |x^{(n-k)}(\tau - \Delta_j(\tau))| \leqq$$

$$\leqq \sum_{k=1}^{n} \sum_{j=1}^{m} A_{kj} \frac{\mu(b-a)^k}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} = \mu \sum_{k=1}^{n} \frac{\sum_{j=1}^{m} A_{kj}}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} (b-a)^k .$$

Evidently $\mu > 0$, since otherwise $x(t)$ would coincide on $\langle t_0, c\rangle$ with a polynomial of degree $m < n$ and $x^{(m)}(t)$ would not vanish on $\langle t_0, c\rangle$ (but $x^{(m)}(a_m) = 0$). Hence

$$\chi\left(\frac{b-a}{2}\right) \geqq 1 ,$$

but this is a contradiction with (15).

Corollary 3. *In general the sign of strict inequality in* (15) *cannot be replaced by the sign* $\leqq$ *as the following example shows.*

Example 2. The function $x(t) = -(t-1)^2 + 1$, $t \in \langle 0, 2\rangle$ is a solution of the equation

$$x''(t) + x'(t-2) = 0 , \quad t \in \langle 0, 2\rangle \tag{16}$$

with the initial functions $\phi_0(t) \equiv 0$, $\phi_1(t) \equiv 2$ on $\langle -2, 0\rangle$. Since $x(0) = x(2) = 0$ the equation (16) is not disconjugate on $\langle 0, 2\rangle$. Nonetheless,

$$\chi\left(\frac{2-0}{2}\right) = 1 .$$

Corollary 4. *Lemma* 2 *may be proved directly* (*analogously to Lemma* 1 (see [1] p. 84)). *By examining this proof we can find out when the equality in* (6) *and* (6′) *can hold. This enables us to conclude when Theorem* 1 (Theorem 2) *is valid provided the sign of strict inequality in* (9) ((15)) *is substituted by the sign* $\leqq$.

Corollary 5. *Theorem* 2 *is valid also in the case when* $\Delta_j(t) = 0$, $t \in \langle a, b\rangle$; $j = 1, \ldots, m$. *In this special case it is reduced almost to the well known criterion of disconjugacy of the linear differential equation without delay of A. Ju. Levin* (see [3], or [1] p. 86): in *Levin's criterion in* (15) *the sign* $\leqq$ *occurs.*

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