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OSCILLATORY AND ASYMPTOTIC PROPERTIES OF THIRD AND FOURTH ORDER LINEAR DIFFERENTIAL EQUATIONS

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This paper is a continuation of [3] and [4] and deals with oscillatory and asymptotic properties of solutions of linear differential equations of the form

$$(1) \qquad (r(t)\,(r(t)\,y'(t))')' + p(t)\,y(t) = 0\,,$$

$$(2) \qquad (r(t)\,(r(t)\,(r(t)\,y'(t))')')' + p(t)\,y(t) = 0\,,$$

where $r, p \in C([t_0, \infty))$, $r(t) > 0$. The technique used in the paper is based on the notion of a v-derivative of a function.

In the sequel we will restrict our attention to those solutions of the equations considered which exist on some ray $[T, \infty)$ and are non-trivial in any neighborhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros, and nonoscillatory otherwise.

1. THE v-DERIVATIVE AND THE v-TRANSFORMATION

Throughout this section suppose that f, g, v, φ are real-valued functions of one real variable. Let the interval $(-\infty, \infty)$ be denoted by $\mathbb{R}$. Now we recall the following definitions, remarks and theorems from [3].

Definition 1.1. Let functions f and v be defined on some neighborhood $O(t)$ of a point $t \in \mathbb{R}$ and let the conditions $x \in O(t)$, $x \neq t$ imply $v(x) \neq v(t)$. If the limit

$$\lim_{x \to t} \frac{f(x) - f(t)}{v(x) - v(t)}$$

is finite, then it is called the *v-derivative of the function f at the point t* and denoted by $f'_v(t)$ or $\mathrm{d}f(t)/\mathrm{d}v$.

Remark 1.1. It follows from Definition 1.1 that $f'_f(t) = 1$ for every t such that f is defined on $O(t)$ and the conditions $x \in O(t)$, $x \neq t$ imply $f(x) \neq f(t)$.

Theorem 1.1. *Let the following conditions be satisfied:*

(i) *a function v is continuous at a point t,*

(ii) *a function g has a v-derivative at the point t,*

(iii) *a function f has the ordinary derivative at the point $g(t)$.*

Then the composite function $f(g)$ has the v-derivative at the point t and

$$(f(g))'_v(t) = f'(g(t))\, g'_v(t)\,.$$

Theorem 1.2. *Let there exist $v'(t) \neq 0$ on an interval I. Then for $t \in I$ the v-derivative $f'_v(t)$ exists if and only if the derivative $f'(t)$ exists. At the same time,*

$$f'_v(t) = \frac{f'(t)}{v'(t)}\,.$$

In this paper we shall need the following simple form of Definition 1.2 in [3].

Definition 1.2. Let $n > 1$ be a natural number. Let functions v and $f^{(n-1)}_{v^{n-1}}$ be defined on some neighborhood $O(t)$ of a point $t \in \mathbb{R}$. If the limit

$$\lim_{x \to t} \frac{f^{(n-1)}_{v^{n-1}}(x) - f^{(n-1)}_{v^{n-1}}(t)}{v(x) - v(t)}$$

is finite, then it is called the *n-th v-derivative of the function f at the point t* and denoted by

$$f^{(n)}_{v^n}(t) \quad \text{or} \quad \frac{\mathrm{d}^n f(t)}{\mathrm{d}v^n}\,.$$

We simplify also the notion of a v-transformation of a differential equation presented in [3] to fit the needs of this paper. Thus suppose that the following conditions are satisfied:

a) $n \geqq 1$ is integer,

b) I and I_1 are intervals in $\mathbb{R}$,

c) $v \in C(I_1)$, v is a strictly monotonous function, $v: I_1 \to I$,

d) φ is the inverse function to v,

e) $p: I \to \mathbb{R}$.

Consider the differential equation

$$y^{(n)}(t) + p(t)\, y(t) = 0\,, \quad t \in I\,. \tag{3}$$

If the independent variable t is replaced by the function $v(t)$ in the coefficient $p(t)$ of the equation (3) and $y^{(n)}(t)$ is replaced by $y^{(n)}_{v^n}(t)$ in the sense that $v(t)$ replaces even the independent variable as an argument of the function with respect to which the derivatives of the unknown function are calculated, then the equation (3) is transformed into the equation

$$y^{(n)}_{v^n}(t) + p(v(t))\, y(t) = 0\,, \quad t \in I_1\,. \tag{4}$$

In the sequel we shall call the abovementioned process of obtaining (4) from (3) a v-transformation of the differential equation (3).

It is useful to note that a φ-transformation of (4) leads again to (3).

Now we introduce the following result which is a special case of Theorem 2.1 proved in [3].

Theorem 1.3. *Let the conditions* a) – e) *be satisfied. The function* $u(t)$ *is a solution of the equation* (3) *on* I *if and only if the function* $u(v(t))$ *is a solution of the equation* (4) *on* I_1.

2. DEFINITIONS AND COMPARISON THEOREMS

Without mentioning it again, the following notation will be used throughout this paper:

$$R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)}, \quad t \geqq t_0,$$

ϕ is the inverse function to R.

Consider the n-th order differential equation

$$(5) \qquad (r(t) \ldots (r(t)(r(t)\, y'(t))')' \ldots)' + p(t)\, y(t) = 0, \quad t \geqq t_0.$$

Definition 2.1. The equation (5) is said to *have the property* (A) if for n even, every solution of (5) is oscillatory, and for n odd, every solution $y(t)$ of (5) is either oscillatory or

$$(6) \qquad \frac{\mathrm{d}^i y(t)}{\mathrm{d}R^i} \to 0 \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \ldots, n-1).$$

Definition 2.2. The equation (5) is said to *have the property* (B) if for n even, every solution $y(t)$ of (5) is oscillatory or satisfies the condition (6) or satisfies the condition

$$(7) \qquad \left|\frac{\mathrm{d}^i y(t)}{\mathrm{d}R^i}\right| \to \infty \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \ldots, n-1),$$

and for n odd, every solution $y(t)$ of (5) is either oscillatory or satisfies the condition (7).

It is easy to see that in the case $r(t) \equiv 1$ the conditions (6) and (7) assume the form

$$(6_1) \qquad y^{(i)}(t) \to 0 \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \ldots, n-1)$$

and

$$(7_1) \qquad |y^{(i)}(t)| \to \infty \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \ldots, n-1).$$

Now we introduce the following comparison theorems of V. A. Kondratev [2] and T. A. Čanturija [1].

Theorem A. *Let* $n \geqq 3$. *Let functions* p *and* q *be integrable on every finite and closed subinterval of the interval* $[0, \infty)$.

A$_1$) *If* $p(t) \geqq q(t) \geqq 0$ *for* $t \in [0, \infty)$ *and the equation*

$$(8) \qquad u^{(n)}(t) + q(t)\, u(t) = 0$$

has the property (A), *then the equation*

$$(9) \qquad u^{(n)}(t) + p(t)\, u(t) = 0$$

has the property (A).

A_2) *If* $p(t) \leqq q(t) \leqq 0$ *for* $t \in [0, \infty)$ *and the equation* (8) *has the property* (B), *then the equation* (9) *has the property* (B).

Remark 2.1. It is known (see e.g. [2]) that if $p(t) \geqq 0$ ($p(t) \leqq 0$) and the equation

$$y^{(n)}(t) + p(t)\, y(t) = 0$$

has the property (A) (the property (B)) then it has solutions of every type mentioned in Definition 2.1 (in Definition 2.2).

3. TWO LEMMAS

Lemma 3.1. *The Euler equation*

$$(10) \qquad t^3 y''' + \alpha y = 0\,, \quad t \geqq t_0 > 0$$

a) *has the property* (A) *if* $\alpha > 2/3\,\sqrt{3}$,
b) *has the property* (B) *if* $\alpha < -2/3\,\sqrt{3}$,
c) *has not the property* (A) *if* $0 \leqq \alpha \leqq 2/3\,\sqrt{3}$,
d) *has not the property* (B) *if* $-2/3\,\sqrt{3} \leqq \alpha \leqq 0$.

Proof. Let y be a solution of (10). Since

$$(11) \qquad \frac{\mathrm{d}y}{\mathrm{d}\ln t} = ty'\,, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}(\ln t)^2} = t^2 y'' + ty'\,, \quad \frac{\mathrm{d}^3 y}{\mathrm{d}(\ln t)^3} = t^3 y''' + 3t^2 y'' + ty'$$

the equation (10) may be written in the form

$$(10_1) \qquad \frac{\mathrm{d}^3 y}{\mathrm{d}(\ln t)^3} - 3\,\frac{\mathrm{d}^2 y}{\mathrm{d}(\ln t)^2} + 2\,\frac{\mathrm{d}y}{\mathrm{d}\ln t} + \alpha y = 0\,.$$

The v-transformation of the equation (10_1) with $v(t) = \exp t$ yields the equation

$$(12) \qquad y''' - 3y'' + 2y' + \alpha y = 0\,,$$

the characteristic equation of which is

$$(13) \qquad k^3 - 3k^2 + 2k + \alpha = 0\,.$$

Solving it we find

$$k_1 = 1 + a + b\,,$$

$$k_2 = 1 - \frac{a + b}{2} + \frac{a - b}{2}\,\mathrm{i}\,\sqrt{3}\,,$$

$$k_3 = 1 - \frac{a + b}{2} - \frac{a - b}{2}\,\mathrm{i}\,\sqrt{3}\,,$$

where

$$a = \sqrt[3]{\left(-\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha^2}{4} - \frac{1}{27}\right)}\right)},$$

$$b = \sqrt[3]{\left(-\frac{\alpha}{2} - \sqrt{\left(\frac{\alpha^2}{4} - \frac{1}{27}\right)}\right)}.$$

Now we put

$$D = \frac{\alpha^2}{4} - \frac{1}{27}$$

and suppose that $D > 0$, i.e. $\alpha > 2/3\sqrt{3}$ or $\alpha < -2/3\sqrt{3}$. Then k_1 is a real root and k_2, k_3 are complex conjugate roots of (13).

In a simple way we find out that for $\alpha > 2/3\sqrt{3}$,

$$k_1 = 1 + a + b < 0$$

and

$$h = 1 - \frac{a+b}{2} > 1\,.$$

We know that the fundamental system of solutions of (12) consists of the functions

$$y_1(t) = \exp(k_1 t)\,,$$
$$y_2(t) = \exp(ht)\cos\left(\frac{a-b}{2}\,t\sqrt{3}\right),$$
$$y_3(t) = \exp(ht)\sin\left(\frac{a-b}{2}\,t\sqrt{3}\right).$$

Hence, according to Theorem 1.3, the fundamental system of solutions of (10) consists of the functions

$$u_1(t) = y_1(\ln t) = t^{k_1}\,, \tag{14}$$
$$u_2(t) = y_2(\ln t) = t^h\cos\left(\frac{a-b}{2}\sqrt{3}\ln t\right),$$
$$u_3(t) = y_3(\ln t) = t^h\sin\left(\frac{a-b}{2}\sqrt{3}\ln t\right).$$

From this we see that for $\alpha > 2/3\sqrt{3}$ the equation (10) has the property (A).

Similarly as before we find out that for $\alpha < -2/3\sqrt{3}$,

$$k_1 = 1 + a + b > 2$$

and

$$h = 1 - \frac{a+b}{2} < 0{\cdot}5\,.$$

Now we see from (14) that for $\alpha < -2/3\sqrt{3}$ the equation (10) has the property (B).

It is easy to see that for

$$\alpha \in \left\{0, \frac{2}{3\sqrt{3}}, \frac{-2}{3\sqrt{3}}\right\}$$

the roots k_1, k_2, k_3 of (13) are real. It means that (12) and also (10) has no oscillatory solution, i.e. the equation (10) has neither the property (A) nor the property (B).

In the case $D < 0$ and $0 < \alpha < 2/3\sqrt{3}$ we have

$$k_1 = 1 + \frac{\sqrt{3}}{3}\cos\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right) + \sin\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right),$$

$$k_2 = 1 - \frac{2\sqrt{3}}{3}\cos\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right),$$

$$k_3 = 1 + \frac{\sqrt{3}}{3}\cos\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right) - \sin\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right).$$

Similarly as before, since k_1, k_2, k_3 are real roots, the equation (10) has no oscillatory solution, i.e. it has not the property (A).

Finally, in the case $D < 0$ and $-2/3\sqrt{3} < \alpha < 0$ we have

$$k_1 = 1 + \frac{2\sqrt{3}}{3}\cos\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right),$$

$$k_2 = 1 - \frac{\sqrt{3}}{3}\cos\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right) + \sin\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right),$$

$$k_3 = 1 - \frac{\sqrt{3}}{3}\cos\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right) - \sin\left(\frac{1}{3}\operatorname{arctg}\sqrt{\left(\frac{4}{27\alpha^2} - 1\right)}\right).$$

Now again k_1, k_2, k_3 are real roots, thus the equation (10) has no oscillatory solution, i.e. it has not the property (B). The proof is complete.

Lemma 3.2. *The Euler equation*

$$t^4 y^{(4)} + \alpha y = 0\,, \quad t \geqq t_0 > 0 \tag{15}$$

a) *has the property* (A) *if* $\alpha > 1$,
b) *has the property* (B) *if* $\alpha < -9/16$,
c) *has not the property* (A) *if* $0 \leqq \alpha \leqq 1$,
d) *has not the property* (B) *if* $-9/16 \leqq \alpha \leqq 0$.

Proof. Let y be a solution of (15). Since (11) holds and

$$\frac{d^4 y}{d(\ln t)^4} = t^4 y^{(4)} + 6t^3 y''' + 7t^2 y'' + ty'\,,$$

the equation (15) may be written in the form

$$\frac{d^4 y}{d(\ln t)^4} - 6\frac{d^3 y}{d(\ln t)^3} + 11\frac{d^2 y}{d(\ln t)^2} - 6\frac{dy}{d\ln t} + \alpha y = 0\,. \tag{15_1}$$

The v-transformation of the equation (15_1) with $v = \exp t$ yields the equation

$$y^{(4)} - 6y''' + 11y'' - 6y' + \alpha y = 0 \tag{16}$$

the characteristic equation of which is

$$k^4 - 6k^3 + 11k^2 - 6k + \alpha = 0\,. \tag{17}$$

Solving (17) we obtain

$$k_{1,2,3,4} = \tfrac{3}{2} \pm \sqrt{(\tfrac{1}{8}(5 + \sqrt{(9 + 16\alpha)}))} \pm \sqrt{(\tfrac{1}{8}(5 - \sqrt{(9 + 16\alpha)}))}\,. \tag{18}$$

Now from (18) we see that

$$9 + 16\alpha \geqq 0 \quad \text{and} \quad 5 - \sqrt{(9 + 16\alpha)} \geqq 0 \quad \text{for} \quad \alpha \in [-9/16, 1]\,.$$

Then the roots k_1, k_2, k_3, k_4 of (17) are real provided $\alpha \in [-9/16, 1]$, i.e. all solutions of (16) are nonoscillatory. But then, according to Theorem 1.3, all solutions of (15) are nonoscillatory and by Remark 2.1 we obtain the assertions c) and d) of Lemma 3.2.

The case $\alpha > 1$ yields $5 - \sqrt{(9 + 16\alpha)} < 0$, i.e. the roots k_1, k_2, k_3, k_4 are complex numbers of the form

$$k_{1,2,3,4} = \tfrac{3}{2} \pm \sqrt{(\tfrac{1}{8}(5 + \sqrt{(9 + 16\alpha)}))} \pm \mathrm{i}\sqrt{(\tfrac{1}{8}(\sqrt{(9 + 16\alpha)} - 5))}\,.$$

It means that all solutions of (16) are oscillatory. From this, according to Theorem 1.3, we conclude that all solutions of (15) are oscillatory and thus the assertion a) of Lemma 3.2 holds true.

The case $\alpha < -9/16$ yields $9 + 16\alpha < 0$, and simple calculation gives

$$\begin{aligned} k_1 &= \tfrac{3}{2} + \tfrac{1}{2}\sqrt{(4\sqrt{(1 - \alpha)} + 5)} > 3\,, \\ k_2 &= \tfrac{3}{2} - \tfrac{1}{2}\sqrt{(4\sqrt{(1 - \alpha)} + 5)} < 0\,, \\ k_{3,4} &= \tfrac{3}{2} \pm \mathrm{i}/2\sqrt{(4\sqrt{(1 - \alpha)} - 5)}\,. \end{aligned}$$

Since now linearly independent solutions of (16) are the functions

$$\begin{aligned} &y_1(t) = \exp(k_1 t)\,, \quad y_2(t) = \exp(k_2 t)\,, \\ &y_3(t) = \exp(\tfrac{3}{2}t)\cos(t/2\sqrt{(4\sqrt{(1 - \alpha)} - 5)})\,, \\ &y_4(t) = \exp(\tfrac{3}{2}t)\sin(t/2\sqrt{(4\sqrt{(1 - \alpha)} - 5)})\,, \end{aligned}$$

then linearly independent solutions of (15) are the functions

$$\begin{aligned} &u_1(t) = y_1(\ln t) = t^{k_1}\,, \quad u_2(t) = y_2(\ln t) = t^{k_2}\,, \\ &u_3(t) = y_3(\ln t) = t^{3/2}\cos(\ln t/2\sqrt{(4\sqrt{(1 - \alpha)} - 5)})\,, \\ &u_4(t) = y_4(\ln t) = t^{3/2}\sin(\ln t/2\sqrt{(4\sqrt{(1 - \alpha)} - 5)})\,. \end{aligned} \tag{19}$$

If we take into account that $k_1 > 3$ and $k_2 < 0$ then from (19) it is easy to see that the assertion b) of Lemma 3.2 holds true. This completes the proof.

Theorem 4.1. *Let $r, p \in C([t_0, \infty))$, $r(t) > 0$ and $R(t) \to \infty$ as $t \to \infty$. Then the equation* (1) *has the property* (A) *if*

$$\liminf_{t\to\infty} R^3(t)\, r(t)\, p(t) > \frac{2}{3\sqrt{3}}$$

and the equation (1) *has the property* (B) *if*

$$\limsup_{t\to\infty} R^3(t)\, r(t)\, p(t) < -\frac{2}{3\sqrt{3}}\,.$$

Proof. We can write the equation (1) in the form

$$(1_1) \qquad \frac{d^3 y(t)}{dR^3} + r(t)\, p(t)\, y(t) = 0\,, \quad t \geqq t_0\,.$$

By the v-transformation of (1_1) with $v = \phi$ we obtain

$$(20) \qquad y'''(t) + r(\phi(t))\, p(\phi(t))\, y(t) = 0\,, \quad t \geqq 0\,.$$

A simple calculation shows that

$$\liminf_{t\to\infty} t^3 r(\phi(t))\, p(\phi(t)) = \liminf_{t\to\infty} R^3(t)\, r(t)\, p(t)\,.$$

Putting $a = \liminf_{t\to\infty} R^3(t)\, r(t)\, p(t) > 2/3\sqrt{3}$ we obtain that for every a_1 with $2/3\sqrt{3} < a_1 < a$ there exists t_1 (>0) such that

$$t^3 r(\phi(t))\, p(\phi(t)) \geqq a_1 \quad \text{if} \quad t \geqq t_1\,.$$

From this inequality, according to Lemma 3.1 and Theorem A we conclude that the equation (20) has the property (A). But then by Theorem 1.3, the equation (1_1) and also the equation (1) has the property (A). Indeed, if $y(t)$ is a solution of (20) then $u(t) = y(R(t))$ is a solution of (1_1) and also of (1). Further (since $R(t) \to \infty$ as $t \to \infty$), if $y(t)$ is an oscillatory solution of (20) then $u(t) = y(R(t))$ is an oscillatory solution of (1). Finally, if $y(t)$ is a solution of (20) with the property (6_1) then for $u(t) = y(R(t))$ we have

$$u(t) = y(R(t)) \to 0 \quad \text{as} \quad t \to \infty\,,$$

$$\frac{du(t)}{dR} = y'(R(t)) \to 0 \quad \text{as} \quad t \to \infty\,,$$

$$\frac{d^2u(t)}{dR^2} = y''(R(t)) \to 0 \quad \text{as} \quad t \to \infty\,.$$

To prove the second part of Theorem 4.1 we can proceed similarly as above. Since now

$$\limsup_{t\to\infty} t^3 r(\phi(t))\, p(\phi(t)) = \limsup_{t\to\infty} R^3(t)\, r(t)\, p(t) < -\frac{2}{3\sqrt{3}}$$

thus putting

$$b = \limsup_{t\to\infty} R^3(t)\, r(t)\, p(t)$$

we obtain that for every b_1 with $b < b_1 < -2/3\sqrt{3}$ there exists t_2 (>0) such that

$$t^3 r(\phi(t))\, p(\phi(t)) \leqq b_1 \quad \text{if} \quad t \geqq t_2 .$$

This inequality together with Lemma 3.1 and Theorem A implies that the equation (20) has the property (B), and by Theorem 1.3 we find out that the equation (1) has the property (B). This completes the proof.

For other related results concerning the property (A) of retarded differential equations the reader is referred to the paper [5].

Theorem 4.2. *Let* $r, p \in C([t_0, \infty))$, $r(t) > 0$ *and* $R(t) \to \infty$ *as* $t \to \infty$.

a) *Let* $p(t) \geqq 0$ *and*

$$\limsup_{t\to\infty} R^3(t)\, r(t)\, p(t) < \frac{2}{3\sqrt{3}} .$$

Then the equation (1) *has not the property* (A).

b) *Let* $p(t) \leqq 0$ *and*

$$\liminf_{t\to\infty} R^3(t)\, r(t)\, p(t) > \frac{-2}{3\sqrt{3}} .$$

Then the equation (1) *has not the property* (B).

Proof. As we already know the v-transformation of (1) with $v = \phi$ yields (20). Since now

$$\limsup_{t\to\infty} t^3 r(\phi(t))\, p(\phi(t)) = \limsup_{t\to\infty} R^3(t)\, r(t)\, p(t) < \frac{2}{3\sqrt{3}}$$

thus putting

$$a = \limsup_{t\to\infty} R^3(t)\, r(t)\, p(t)$$

we obtain that for every a_1 with $a < a_1 < 2/3\sqrt{3}$ there exists t_1 (>0) such that

$$t^3 r(\phi(t))\, p(\phi(t)) \leqq a_1 \quad \text{if} \quad t \geqq t_1 .$$

From this inequality, according to Lemma 3.1 and Theorem A we see that the equation (20) has not the property (A). Now by Theorem 1.3 we conclude that the equation (1) has not the property (A), and part a) of Theorem 4.2 is proved. Part b) of Theorem 4.2 can be proved similarly. Therefore the proof is complete.

If we use Lemma 3.2 instead of Lemma 3.1, we can prove the following two theorems in the same way as above.

Theorem 4.3. *Let* $r, p \in C([t_0, \infty))$, $r(t) > 0$ *and* $R(t) \to \infty$ *as* $t \to \infty$. *Then the equation* (2) *has the property* (A) *if*

$$\liminf_{t\to\infty} R^4(t)\, r(t)\, p(t) > 1$$

and the equation (2) *has the property* (B) *if*

$$\limsup_{t \to \infty} R^4(t)\, r(t)\, p(t) < -9/16 .$$

Theorem 4.4. *Let $r, p \in C([t_0, \infty))$, $r(t) > 0$ and $R(t) \to \infty$ as $t \to \infty$.*

a) *Let $p(t) \geqq 0$ and*

$$\limsup_{t \to \infty} R^4(t)\, r(t)\, p(t) < 1 .$$

Then the equation (2) *has not the property* (A).

b) *Let $p(t) \leqq 0$ and*

$$\liminf_{t \to \infty} R^4(t)\, r(t)\, p(t) > -9/16 .$$

Then the equation (2) *has not the property* (B).

Note that the previous results can be generalized e.g. to equations

$$(r_2(t)\,(r_1(t)\, y'(t))')' + p(t)\, y(t) = 0 ,$$
$$(r_3(t)\,(r_2(t)\,(r_1(t)\, y'(t))')')' + p(t)\, y(t) = 0$$

by suitable comparison theorems. On the other hand, in the case $r(t) \equiv 1$ Theorems 4.1, 4.2, 4.3 and 4.4 give results for the equations

$$y'''(t) + p(t)\, y(t) = 0 ,$$
$$y^{(4)}(t) + p(t)\, y(t) = 0 ,$$

which we can regard as an extension of Hille's results (see [6], p. 194) to the third and fourth order differential equations.

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