Czechoslovak Mathematical Journal

Bohdan Zelinka

Spanning trees of locally finite graphs

Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 2, 193–197

Persistent URL: http://dml.cz/dmlcz/102294

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of the Czechoslovak Academy of Sciences
V. *39* (*114*), PRAHA 14. 6. 1989, No 2

SPANNING TREES OF LOCALLY FINITE GRAPHS

BOHDAN ZELINKA, Liberec

(Received April 24, 1986)

We shall consider infinite undirected graphs without loops and multiple edges. A graph G will be called *locally finite*, if each vertex of G has a finite degree.

If R is a subset of the vertex set $V(G)$ of a graph G, then by $G - R$ we shall denote the graph obtained from G by deleting all vertices of the set R.

For locally finite graphs, R. Halin [1] introduced the concept of an end of a graph. Before giving the definition, we define some auxiliary concepts.

A rest of a one-way infinite path P is a one-way infinite path, all of whose vertices and edges belong to P. Two one-way infinite paths P_1, P_2 of a locally finite fraph G are called *equivalent*, if there exists a one-way infinite path P_0 in G (which may coincide with P_1 or with P_2) with the property that every rest of P_0 has common vertices with both P_1 and P_2. This relation defined on the set of all one-way infinite paths of G in this way is really an equivalence relation [1]. Its equivalence classes are called the *ends of* G.

An end $\mathfrak{E}$ of C is called *free*, if there exists a finite subset R of the vertex set $V(G)$ of G such that in the graph $G - R$ there exists a connected component which contains paths from $\mathfrak{E}$, but no one-way infinite paths from any other ends of G. (We say that R separates $\mathfrak{E}$ from the other ends.)

If a locally finite graph G has finitely many ends, then all of them are free [1]. Obviously an infinite locally finite graph G contains at least one end, because it contains at least one infinite path.

We shall study ends of spanning trees of a graph G. The following propositions are easy to prove.

Proposition 1. *Let T be an infinite locally finite tree. Then two one-way infinite paths of T belong to the same end of T if and only if their intersection is a rest of both of them.*

Proposition 2. *Let G be an infinite locally finite graph, let T be its spanning tree.*

If two one-way infinite paths in T belong to the same end of T, then they belong to the same end of G, but not vice versa.

Now we prove a lemma.

Lemma 1. *Let G be a connected infinite locally finite graph, let T be its spanning tree, let $\mathfrak{E}$ be a free end of G. Then T contains at least one path from $\mathfrak{E}$.*

Proof. As $\mathfrak{E}$ is free, there exists a finite set R of vertices of G which separates $\mathfrak{E}$ from the other ends. Let G_0 be the connected component of $G - R$ containing paths from $\mathfrak{E}$, let H_0 be the subgraph of G induced by the union of R and the vertex set $V(G_0)$ of G_0. Let T_0 be the subgraph of T induced by the vertex set $V(H_0)$ of H_0; T_0 is a forest. Each connected component of T_0 contains at least one vertex of R; otherwise it would be also a connected component of T and T would not be a tree. This implies that the number of connected components of T_0 is at most $|R|$. As T_0 is infinite and has a finite number of connected components, at least one connected component of T_0 is infinite. As T_0 is locally finite, this connected component contains a one-way infinite path. This path is also in G; as it is in H_0, it belongs to $\mathfrak{E}$.

Now we shall define a concept which will be useful in the sequel.

Let A be a non-empty finite subset of $V(G)$, let $\mathfrak{E}$ be an end of G. We say that a subset R of $V(G)$ *separates A from* $\mathfrak{E}$, if each path from $\mathfrak{E}$ with the initial vertex in A contains a vertex of R and $A \cap R = \emptyset$. Evidently, each non-empty finite subset A of $V(G)$ is separated from each end $\mathfrak{E}$ of G by a finite set; for example, we may choose R as the set of all vertices of G which do not belong to A and are adjacent to at least one vertex of A. The cardinality of R is less than or equal to the sum of degrees of vertices of A. As A is finite and G is locally finite, this sum is finite and so is the cardinality of R. Hence we may define $c(A, \mathfrak{E})$ as the minimum cardinality of a set separating A from $\mathfrak{E}$; it is a positive integer. Now take the supremum of $c(a, \mathfrak{E})$ for all non-empty finite subsets A of $V(G)$. This supremum will be called the *degree* of $\mathfrak{E}$ and denoted by $d(\mathfrak{E})$. It is either a positive integer, or $\aleph_0$.

Lemma 2. *Let G be a connected infinite locally finite graph. Let $\mathfrak{E}$ be a free end of G, let $d(\mathfrak{E})$ be finite. Let T be a spanning tree of G. Then the number of ends of T which are included in $\mathfrak{E}$ is at most $d(\mathfrak{E})$.*

Proof. Let k be the number of ends of T which are contained in $\mathfrak{E}$. From the definition of the end and from the fact that T is a tree it follows that T contains k one-way infinite paths $P_1, \ldots, P_k$, where $k = d(\mathfrak{E})$, which are pairwise vertex-disjoint and have the property that the initial vertex of any P_i $(i = 1, \ldots, k)$ separates all other vertices of P_i from all vertices of the paths P_j for $i \neq j$. All paths $P_1, \ldots, P_k$ belong to the end $\mathfrak{E}$ of G, but to pairwise distinct ends of T. Let A be the set of initial vertices of the paths $P_1, \ldots, P_k$. Any set R separating A from $\mathfrak{E}$ in G must have at least k vertices, otherwise there would be a path among $P_1, \ldots, P_k$ which would contain a vertex of R. Thus $c(A, \mathfrak{E}) \leqq k$ and also $d(\mathfrak{E}) \leqq k$.

Lemma 3. *Let G be a connected infinite locally finite graph. Let $\mathfrak{E}$ be a free end* *of G, let $d(\mathfrak{E})$ be finite. Let k be an integer, $1 \leqq k \leqq d(\mathfrak{E})$. Then there exists a spanning tree T of G which has exactly k ends contained in $\mathfrak{E}$.*

Proof. First, consider $k = \mathrm{d}(\mathfrak{E})$. As $\mathfrak{E}$ is free, there exists a finite set $R_0 \subset V(G)$ separating $\mathfrak{E}$ from all other ends. Let H be the subgraph of G induced by the union of R_0 and the vertex of the connected component of $G - R_0$ containing paths from $\mathfrak{E}$. The graph H evidently has exactly one end $\mathfrak{E}_0$ which is a subset of $\mathfrak{E}$. Take a set R_1 of the least cardinality which separates R_0 from $\mathfrak{E}_0$. We have $|R_1| \leqq d(\mathfrak{E})$. If $|R_1| < d(\mathfrak{E})$, we find a set R_2 separating R_1 from $\mathfrak{E}$; if $|R_2| < d(\mathfrak{E})$, we continue by finding R_3 separating R_2 from $\mathfrak{E}$, etc. If we can proceed to infinity in this way, then every subset of $V(G)$ can be separated from $\mathfrak{E}$ by less than $d(\mathfrak{E})$ vertices, which is a contradiction. Thus there exists a positive integer m such that R_m is separated from $\mathfrak{E}$ by a set S_0 such that $|S_0| = d(\mathfrak{E})$ and by no set of a lesser cardinality. Now we can construct an infinite sequence $(S_i)_{i=0}^{\infty}$ recurrently. If S_j is constructed for some j, then we can find a set S_{j+1} such that $|S_{j+1}| = d(\mathfrak{E})$ and S_{j+1} separates S_j from $\mathfrak{E}_0$. Now for each positive integer i let F_i be the subgraph of G induced by the union of $S_i \cup S_{i+1}$ and the vertex set of the connected component of $G - (S_i \cup S_{i+1})$ which contains paths from S_i to S_{i+1}. The sets S_i, S_{i+1} in F_i are separated by not less than $d(\mathfrak{E})$ vertices. Thus according to Menger's Theorem there exist $k = d(\mathfrak{E})$ pairwise vertex-disjoint paths from S_i to S_{i+1}. If the vertices of S_i are denoted by $a_1^{(i)}, \ldots, a_k^{(i)}$, then we denote these paths by $P_1^{(i)}, \ldots, P_k^{(i)}$ in such a way that $a_j^{(i)}$ is a terminal vertex of $P_j^{(i)}$ for $j = 1, \ldots, k$. Then the terminal vertex of $P_j^{(i)}$ in S_{i+1} will be denoted by $a_j^{(i+1)}$. We proceed in this way for all i's. The union of $P_j^{(i)}$ for all i's is a one-way infinite path P_j for each $j = 1, \ldots, k$. Thus we have constructed k pairwise vertex-disjoint one-way infinite paths $P_1, \ldots, P_k$. Now let H' be the connected component of $G - (R_0 \cup S_0)$ which contains paths from R_0 to S_0; it is a finite graph, because it contains no infinite path. Let H'' be the graph by the union of R_0, S_0 and the vertex set of H'. Let T_0' be a spanning tree of H''. If we add the paths $P_1, \ldots, P_k$ to it, we obtain a spanning tree T_0 of H. We can construct a spanning tree T of G having T_0 as a subtree and this is the required spanning tree for $k = d(\mathfrak{E})$.

Now suppose $1 \leqq d(\mathfrak{E}) < k$. We proceed by induction. Suppose that there exists a spanning tree T_1 of G having $k+1$ ends contained in $\mathfrak{E}$. Then T_1 contains $k+1$ pairwise vertex-disjoint one-way infinite paths $P_1, \ldots, P_{k+1}$ belonging to $\mathfrak{E}$. There exists a subgraph G^* of G such that the subgraph T_1^* of T_1 induced by $V(G_0)$ consists of the rests $P_1^*, \ldots, P_{k+1}^*$ of $P_1, \ldots, P_{k+1}$. As P_1^*, P_{k+1}^* belong to the same end of G, there exists a one-way infinite path Q in G having infinitely many common vertices with both P_1^* and P_{k+1}^*. We traverse Q starting at its initial vertex. Whenever we enter a vertex v of P_{k+1}^* by an edge e not belonging to P_{k+1}, we add e to T_1^* (previously e was not in T_1^*). Simultaneously we delete the edge of P_{k+1}^* ending at v (when traversing P_{k+1}^* from its initial vertex). As Q has infinitely many common vertices with P_{k+1}^*, in this way we delete infinitely many edges of P_{k+1}^*, thus none of its rests is in the resulting graph. Evidently neither a new one-way infinite path, nor a circuit is obtained; thus we have constructed the required tree.

These lemmas imply a theorem.

Theorem 1. *Let G be a connected infinite locally finite graph, let $\mathfrak{E}$ be its free end, let $d(\mathfrak{E})$ be finite. Let k be an integer. Then the following two assertions are equivalent:*

(i) $1 \leqq k \leqq d(\mathfrak{E})$.

(ii) *There exists a spanning tree of G having exactly k ends included in $\mathfrak{E}$.*

Corollary 1. *Let G be a connected infinite locally finite graph with finitely many ends $\mathfrak{E}_1, \ldots, \mathfrak{E}_m$ of finite degrees. Let k be an integer. Then the following two assertions are equivalent:*

(i) $m \leqq k \leqq \sum_{i=1}^{m} d(\mathfrak{E}_i)$.

(ii) *There exists a spanning tree of G having exactly k ends.*

Now we shall consider the case when $d(\mathfrak{E})$ is infinite.

Theorem 2. *Let G be a connected infinite locally finite graph, let $\mathfrak{E}$ be its free end, let $d(\mathfrak{E}) = \aleph_0$. Then there exists a spanning tree T of G having infinitely many ends belonging to $\mathfrak{E}$.*

Proof. The construction is similar to that from the proof of Lemma 3. However, here the cardinalities of the sets S_i are not equal; they form a non-decreasing sequence tending to infinity. If $|S_i| = |S_{i+1}| = k$, we construct the paths $P_1^{(i)}, \ldots, P_k^{(i)}$ in the same way as in the proof of Lemma 3. If $k = |S_i| < |S_{i+1}| = l$, we construct again $P_1^{(i)}, \ldots, P_k^{(i)}$ and denote their terminal vertices in S_{i+1} by $a_1^{(i+1)}, \ldots, a_k^{(i+1)}$. The remaining vertices in S_{i+1} will be denoted arbitrarily by $a_{k+1}^{(i+1)}, \ldots, a_l^{(i+1)}$. Now for each positive integer j the path P_j is the union of paths $P_j^{(i)}$ for all i's for which such a path exists. Thus we have infinitely many pairwise vertex-disjoint one-way infinite paths $P_1, P_2, \ldots$. From G we delete all edges which join a non-initial vertex of one of these paths with a vertex of another one; then we construct a spanning tree of the graph thus obtained. This tree is the required tree T.

Now we propose two conjectures.

Conjecture 1. *Let G be a connected infinite locally finite graph, let $\mathfrak{E}$ be its free end, let $d(\mathfrak{E}) = \aleph_0$. Then for each positive integer k there exists a spanning tree T of G having exactly k ends included in $\mathfrak{E}$.*

Conjecture 2. *The assertion of Lemma 1 holds even without the assumption that $\mathfrak{E}$ is free.*

We present a partial result concerning these conjectures.

Theorem 3. *There exists a connected infinite locally finite graph G with one end $\mathfrak{E}$ such that $d(\mathfrak{E}) = \aleph_0$ and with property that for each positive integer k there exists a spanning tree T_k of G having exactly k ends.*

Proof. We will construct the graph G. Its vertex set is the set of all ordered pairs

(i, j) of positive integers. Two vertices (i_1, j_1), (i_2, j_2) are adjacent if and only if either $i_1 = i_2$ and $|j_1 - j_2| = 1$, or $j_1 = j_2$ and $|i_1 - i_2| = 1$. Let N denote the set of all positive integers. Let P_0 be the one-way infinite path with the vertex set $\{(i, 0) \mid i \in N\}$. For each $k \in N$ let P_k be the one-way infinite path with the vertex set $\{(k, j) \mid j \in N\}$. Further, for positive integers i, k let $Q_i^{(k)}$ be the finite path which is the union of the path with the vertex set $\{(k + i, j) \mid j \leqq i + 1\}$ and the path with the vertex set $\{(j, i + 1) \mid k \leqq j \leqq k + i\}$. Now the tree T_k is the union of the paths $P_0, P_1, \ldots, P_{k-1}, Q_1^{(k)}, Q_2^{(k)}, \ldots$.

Reference

[1] *Halin, R.:* Über unendliche Wege in Graphen; Math. Annalen *157* (1964), 125—137.

Author's address: 461 17 Liberec 1, Studentská 1292, Czechoslovakia (katedra tváření a plastů VŠST).