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PRINCIPAL TOLERANCES ON LATTICES

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By a *tolerance* on an algebra A we mean a reflexive and symmetrical binary relation on A which is compatible with operations of A, i.e. it is a subalgebra of the direct product $A \times A$. Thus each congruence on A is a tolerance but not vice versa. It is known that the set of all tolerances on A forms an algebraic lattice $LT(A)$ with respect to set inclusion, see e.g. [3], [4]. Hence, for each two elements a, b of A there exists a tolerance on A which is the least tolerance containing the pair $\langle a, b\rangle$; we denote it by $T(a, b)$ and call the *principal tolerance* (generated by $\langle a, b\rangle$), see [3]. A tolerance T on an algebra A is *finitely generated* if there exists a finite set $F = \{a_1, \ldots, a_n\}$ of elements of A such that T is the least tolerance on A containing all pairs $\langle a_i, a_j\rangle$ for all $a_i, a_j \in F$; denote it by $T(a_1, \ldots, a_n)$. The aim of this note is to describe finitely generated tolerances and joins and meets of principal tolerances on lattices.

1. FINITELY GENERATED TOLERANCES

First, we try to characterize varieties of algebras whose every finitely generated tolerance is principal. For varieties of *idempotent algebras* (i.e. algebras satisfying $f(a, \ldots, a) = a$ for every n-ary operation f and each element a of A), the answer is the following:

Theorem 1. *Let $\mathscr{V}$ be a variety of idempotent algebras. The following conditions are equivalent:*

(1) *every finitely generated tolerance on each $A \in \mathscr{V}$ is principal;*
(2) *for every integer $n \geqq 2$ there exist n-ary polynomials p, q such that*

$$\langle x_i, x_j\rangle \in T(p(x_1, \ldots, x_n), q(x_1, \ldots, x_n)) \quad \textit{for all} \quad i, j \in \{1, \ldots, n\}\,.$$

Proof. (1) $\Rightarrow$ (2): Let $F \in \mathscr{V}$ be a free algebra with free generators $x_1, \ldots, x_n$ and $T(x_1, \ldots, x_n) \in LT(F)$. By (1), there exist elements c, d of F such that

$$T(x_1, \ldots, x_n) = T(c, d)\,.$$

Since F is freely generated by $x_1, \ldots, x_n$, then

$$c = p(x_1, \ldots, x_n), \quad d = q(x_1, \ldots, x_n)$$

for some n-ary polynomials p, q. Since $\langle x_i, x_j \rangle \in T(x_1, \ldots, x_n)$, we have (2).

(2) $\Rightarrow$ (1): Suppose $A \in \mathscr{V}$, $a_1, \ldots, a_n \in A$ and $T(a_1, \ldots, a_n) \in LT(A)$. By (2), there exist n-ary polynomials p, q with

$$\langle a_i, a_j \rangle \in T(p(a_1, \ldots, a_n), q(a_1, \ldots, a_n))$$

for $i, j \in \{1, \ldots, n\}$. Hence

$$T(a_1, \ldots, a_n) \subseteq T(p(a_1, \ldots, a_n), q(a_1, \ldots, a_n)).$$

Conversely, we have

$$\langle a_1, a_1 \rangle \in T(a_1, \ldots, a_n), \ldots, \langle a_1, a_n \rangle \in T(a_1, \ldots, a_n),$$

thus

$$\langle q(a_1, \ldots, a_1), q(a_1, \ldots, a_n) \rangle \in T(a_1, \ldots, a_n).$$

Since A is idempotent, we have

$$\langle a_1, q(a_1, \ldots, a_n) \rangle \in T(a_1, \ldots, a_n).$$

Analogously, we obtain

$$\langle a_2, q(a_1, \ldots, a_n) \rangle \in T(a_1, \ldots, a_n),$$
$$\vdots$$
$$\langle a_n, q(a_1, \ldots, a_n) \rangle \in T(a_1, \ldots, a_n),$$

hence

$$\langle p(a_1, \ldots, a_n), p(q(a_1, \ldots, a_n), \ldots, q(a_1, \ldots, a_n)) \rangle \in T(a_1, \ldots, a_n).$$

The idempotency of A implies

$$\langle p(a_1, \ldots, a_n), q(a_1, \ldots, a_n) \rangle \in T(a_1, \ldots, a_n),$$

whence

$$T(p(a_1, \ldots, a_n), q(a_1, \ldots, a_n)) \subseteq T(a_1, \ldots, a_n)$$

which completes the proof.

Corollary 1. *For every lattice L, each finitely generated tolerance on L is principal.*

Proof. Put

$$p(x_1, \ldots, x_n) = x_1 \wedge \ldots \wedge x_n,$$
$$q(x_1, \ldots, x_n) = x_1 \vee \ldots \vee x_n.$$

By Lemma 2 in [4], $\langle x_i, x_j \rangle \in T(p(x_1, \ldots, x_n), q(x_1, \ldots, x_n))$ for each $i, j \in \{1, \ldots, n\}$. Since lattices are idempotent algebras, the assertion follows directly from Theorem 1.

2. JOINS AND MEETS OF PRINCIPAL TOLERANCES

An algebra A is *congruence principal* if for each $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ there exist $a, b \in A$ such that

$$\Theta(a_1, b_1) \wedge \ldots \wedge \Theta(a_n, b_n) = \Theta(a, b)$$

in *Con A*. Varieties of such algebras were investigated in [2], [5], [6], [7]. A similar concept was introduced also for tolerances, see [3]: An algebra A is *tolerance principal* if for every $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ thre exist elements a, b of A such that

$$T(a_1, b_1) \wedge \ldots \wedge T(a_n, b_n) = T(a, b)$$

in $LT(A)$. Varieties of tolerance principal algebras were characterized in [3]. This concept can be modified for algebras with a constant element: An algebra A with a constant 0 is 0-*tolerance principal* if for every $a_1, \ldots, a_n \in A$ there exists an element $a \in A$ such that

$$T(0, a_1) \vee \ldots \vee T(0, a_n) = T(0, a)$$

in $LT(A)$. It was proved in [3] that every lattice L with 0 is 0-*tolerance principal.* On the other hand, it is an easy excecise to show that tolerance principality is an exceptional property on lattices.

The dual property is the so called *intersection property*: An algebra A has the *congruence intersection property* if every meet of a finite number of principal congruences is principal, i.e., if for each $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ there exist $a, b \in A$ such that

$$\Theta(a_1, b_1) \wedge \ldots \wedge \Theta(a_n, b_n) = \Theta(a, b) .$$

An algebra A with a constant 0 has the 0-*congruence intersection property* if for each $a_1, \ldots, a_n \in A$ there exists $a \in A$ with

$$\Theta(0, a_1) \wedge \ldots \wedge \Theta(0, a_n) = \Theta(0, a) .$$

We will investigate lattice having analogous properties for tolerances:

An algebra A has the *tolerance intersection property* if for each $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ there exist $a, b \in A$ with

$$T(a_1, b_1) \wedge \ldots \wedge T(a_n, b_n) = T(a, b) .$$

An algebra A with 0 has the 0-*tolerance intersection property* if for each $a_1, \ldots, a_n \in A$ there exists $a \in A$ such that

$$T(0, a_1) \wedge \ldots \wedge T(0, a_n) = T(0, a) .$$

The starting point is the result of K. A. Baker [1]:

Proposition 1 ([1], Theorems 2.8, 2.9). *Let $\mathscr{V}$ be a congruence distributive variety. The following conditions are equivalent*:

(1) *algebras of $\mathscr{V}$ have the congruence intersection property*;
(2) *there exist 4-ary polynomials d_0, d_1 such that $d_0(x, y, u, v) = d_1(x, y, u, v)$ iff $x = y$ or $u = v$ hold on any SI member of $\mathscr{V}$.*

Theorem 2. *Every distributive lattice has the tolerance intersection property.*

Proof. **(A)** Let $\mathscr{V}$ be the variety of all distributive lattices. Clearly $\mathscr{V}$ is congruence distributive. Put

$$d_0(x, y, u, v) = (x \vee u) \wedge (x \vee v) \wedge (u \vee v),$$
$$d_1(x, y, u, v) = (y \vee u) \wedge (y \vee v) \wedge (u \vee v).$$

In any lattice L, $x = y$ or $u = v$ imply

$$d_0(x, y, u, v) = d_1(x, y, u, v).$$

Conversely, the only subdirectly irreducible distributive lattices are the one element lattice and the two element chain. It is easy to show that in the two element chain, the implication

$$d_0(x, y, u, v) = d_1(x, y, u, v) \Rightarrow x = y \quad \text{or} \quad u = v$$

holds. For one element lattice, this is trivial. Thus the distributive lattices have the congruence intersection property.

(B) By [4], we have $\Theta(a, b) = T(a, b)$ on every distributive lattice L and each $a, b \in L$. Since the operation meet is the same in *Con* L as in $LT(L)$, **(A)** implies that L has also the tolerance intersection property.

In a way similar to that of [1], we can prove:

Proposition 2. *Let $\mathscr{V}$ be a variety with a nullary operation* 0 *having distributive congruences. The following conditions are equivalent*:

(1) *algebras of $\mathscr{V}$ have the* 0*-congruence intersection property*;

(2) *there exists a binary polynomial $b(x, y)$ such that $b(x, y) = 0$ if and only if $x = 0$ or $y = 0$ holds on any SI member of $\mathscr{V}$.*

Corollary 2. *Every distributive lattice with the least element* 0 *has the* 0*-tolerance intersection property.*

Proof. Put $b(x, y) = x \wedge y$. The rest of the proof is similar to that of Theorem 2.

Remark. The congruence (or tolerance) intersection property on an algebra A with 0 does not imply the 0-congruence (or 0-tolerance) intersection property. For instance, let $L = \{0, x, y, a, 1\}$ be the non-modular lattice N_5. The only principal congruences on L are $\omega = \Theta(0, 0)$, $\Theta(0, a)$, $\Theta(0, x)$, $\Theta(x, y)$ and $\Theta(0, 1) = L \times L$. Clearly

$$\Theta \wedge \omega = \omega \quad \text{and} \quad \Theta \wedge \Theta(0, 1) = \Theta$$

for each $\Theta \in Con\ L$. Moreover,

$$\Theta(x, y) \wedge \Theta(0, a) = \Theta(x, y),$$
$$\Theta(x, y) \wedge \Theta(0, x) = \Theta(x, y),$$
$$\Theta(0, a) \wedge \Theta(0, x) = \Theta(x, y),$$

thus L has the congruence intersection property. On the other hand, there exists no element $c \in L$ with

$$\Theta(0, a) \wedge \Theta(0, x) = \Theta(0, c),$$

thus L has not the 0-congruence intersection property.

Theorem 3. *Let D be a distributive lattice with the least element 0. The set of all principal tolerances of the form $T(0, x)$ forms a sublattice of the tolerance lattice $LT(D)$.*

Proof. Corollary 2 yields that the set $S = \{T(0, x);\ x \in D\}$ is closed under meet of $LT(D)$. By [3], S is closed also under join of $LT(D)$, whence the assertion follows.

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