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LEXICOGRAPHIC FACTORS OF A LINEARLY ORDERED GROUP

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This paper is a continuation of the article [5]. In [5], we investigated the lattice $L(G)$, whose elements are certain equivalence classes of lexicographic decompositions of a linearly ordered group G. These equivalence classes are defined in a natural way by applying the well-known Malcev's theorem [6] on isomorphic refinements of lexicographic decompositions of G.

Suppose that $G \neq \{0\}$. Let $F_1(G)$ be the set of all nonzero lexicographic factors of G. Put $F_2(G) = F_1(G) \cup \{\{0\}\}$. We can define a quasiorder on the set $F_2(G)$ in a similar way as we did for the set of all lexicographic decompositions of G in [5]. Let $F(G)$ be the corresponding partially ordered set.

In more detail, the quasiorder $\leqq$ on $F_2(G)$ is defined as follows. For each $A \in F_1(G)$ we set $\{0\} \leqq A$. Let $A, B \in F_1(G)$. We put $A \leqq B$ if $B^l \subseteq A^l \subset A^1 \subseteq B^1$. Here, A^1 denotes the l-ideal of G generated by A, and $A^l = \{g \in G\colon |g| < |a|$ for each nonzero element a of $A\}$.

In this paper it will be proved that $F(G)$ is a lattice which is modular if and only if card $F(G) \leqq 4$. Moreover, $F(G)$ satisfies the upper covering condition; hence if it is finite, then it is semimodular. Each closed dual ideal of $F(G)$ distinct from $F(G)$ is a completely distributive lattice. Some further results on $F(G)$ are also established.

1. PRELIMINARIES

We shall apply the same notation as in [2] with the distinction that the group operation will be denoted additively. The commutativity of this operation will not be assumed. Throughout the paper, G denotes a nonzero linearly ordered group. For the more specific notions and notation concerning lexicographic decompositions of G cf. [5], § 1 and § 2.

By considering a lexicographic decomposition

$$\alpha\colon G \to \Gamma_{i \in I} G_i\,,$$

we always assume that all linearly ordered groups G_i are nonzero. For $g \in G$ and $i \in I$ let g_i be the component of g in G_i.

For each $i \in I$ we put

$$G_i^0 = \{g \in G\colon g_j = 0 \text{ for each } j \in I \setminus \{i\}\}\,,$$
$$G_i^1 = \{g \in G\colon g_j = 0 \text{ for each } j \in I \text{ with } j < i\}\,,$$
$$G_i^l = \{g \in G\colon g_j = 0 \text{ for each } j \in I \text{ with } j \geqq i\}\,,$$
$$G_i^u = \{g \in G\colon g_j = 0 \text{ for each } j \in I \text{ with } j \leqq i\}\,.$$

G_i^0, G_i^1, G_i^l and G_i^u are subgroups of G; they are linearly ordered by the induced linear order. G_i^0 is said to be a *lexicographic factor of* G.

Let us have another lexicographic decomposition

$$\beta\colon G \to \Gamma_{j \in J} B_j\,;$$

for the lexicographic decomposition β we adopt notation analogous to that introduced for α.

1.1. Lemma. *Let $i \in I$ and $j \in J$. Suppose that $G_i^1 = G_j^1$ and $G_i^l = G_j^l$. Then the lexicographic factors G_i^0 and G_j^0 are isomorphic.*

Proof. This is a consequence of Malcev's theorem [6] (cf. also [5], Thm. 1.1; for a more general result cf. Fuchs [2], Chap. II, Thm. 9, and the author [3], [4]).

Let us remark that under the assumptions as in 1.1 we need not have $G_i^0 = G_j^0$.

1.2. Lemma. *Let $i \in I$. Then G_i^1 is the l-ideal of G generated by the set G_i^0, and G_i^l is the set of all $g \in G$ such that $|g| < |g_0|$ for each $g_0 \in G_i^0$ with $g_0 \neq 0$.*

Proof. This is an immediate consequence of the definitions of G_i^0, G_i^1 and G_i^l.

1.3. Lemma. *Let $i \in I$ and $j \in J$. If $G_i^0 = G_j^0$, then $G_i^1 = G_j^1$ and $G_i^l = G_j^l$.*

Proof. This follows from 1.2.

For the lexicographic factors G_i^0 and G_j^0 we put $G_i^0 \leqq G_j^0$ if the relation

$$G_j^l \subseteq G_i^l \subset G_i^1 \subseteq G_j^1$$

is valid. We also set $\{0\} \leqq G_i^0$ for each lexicographic factor G_i^0. For each G_i^0 we denote by $(G_i^0)^*$ the set of all lexicographic factors G_k^0 such that

$$G_i^0 \leqq G_k^0 \quad \text{and} \quad G_k^0 \leqq G_i^0$$

is valid.

From 1.1 we infer:

1.4. Lemma. *Let $G_k^0 \in (G_i^0)^*$. Then G_k^0 is isomorphic to G_i^0.*

If G_i^0 and G_j^0 are isomorphic lexicographic factors, then the relation $(G_i^0)^* = (G_j^0)^*$ need not hold.

1.5. Lemma. *Let $F_2(G)$ be as in the introduction. Then the relation $\leqq$ is a quasi-order on $F_2(G)$.*

Proof. This is an immediate consequence of the definition of the relation $\leqq$.

Let $F(G)$ be the partially ordered set corresponding to the quasiordered set $F_2(G)$

(in the sense of [1], Chap. II, § 1). From the correspondence between $F_2(G)$ and $F(G)$ we obtain

1.6. Lemma. *$F(G)$ is the set whose elements are either of the form $(G_i^0)^*$ (where G_i^0 runs over the set of all lexicographic factors of G) or of the form $\{\{0\}\}$. Next, $\{\{0\}\}$ is the least element of $F(G)$. For lexicographic factors G_i^0 and G_j^0 we have $(G_i^0)^* \leqq (G_j^0)^*$ if and only if $G_i^0 \leqq G_j^0$.*

Let α and β be as above. Let I_1 be a nonempty subset of I. Next, let I_1^0 be a convex subset of I generated by I_0. Put

$$G(I_1^0) = \{g \in G\colon g_j = 0 \text{ for each } j \in I \setminus I_1^0\}\,.$$

The following lemma is an immediate consequence of the definition of $G(I_1^0)$.

1.7. Lemma. *$G(I_1^0)$ is a lexicographic factor of G. If H is a lexicographic factor of G such that $(G_i^0)^* \leqq H^*$ for each $i \in I_1$, then $(G(I_1^0))^* \leqq H^*$.*

In view of Malcev's theorem [6] (cf. also [5], Thm. 1.1) there are lexicographic decompositions $f(\alpha, \beta)$ and $f(\beta, \alpha)$ of G such that $f(\alpha, \beta)$ is a refinement of α, $f(\beta, \alpha)$ is a refinement of β. The factors in $f(\alpha, \beta)$ are denoted as G_{ij} for certain $(i, j) \in \in I \times J$. (The fact that we do not consider (i, j) as running over the whole set $I \times J$ is a consequence of the assumption that only nonzero lexicographic factors are taken into account. In [5], Thm. 1.1, some lexicographic factors under consideration can be zero.) Similarly, the factors in $f(\beta, \alpha)$ are G_{ji}, where (j, i) is an element of $J \times I$ such that G_{ij} is a factor of $f(\alpha, \beta)$. For each such (j, i) we have $(G_{ji}^0)^* = (G_{ij}^0)^*$. For each fixed $i \in I$ there is $J(i) \subseteq J$ such that G_i is a lexicographic product of linearly ordered groups G_{ij} $(j \in J(i))$. Analogously, for each fixed $j \in J$ there is $I(j) \subseteq I$ such that G_j is a lexicographic product of linearly ordered groups G_{ji} $(i \in I(j))$.

From the above mentioned properties of $f(\alpha, \beta)$ and $f(\beta, \alpha)$ and from 1.7 we obtain the following assertion:

1.8. Lemma. *Let $i \in I$ and $j \in J$. Suppose that G_{ij} is a lexicographic factor in $f(\alpha, \beta)$. Let H be a lexicographic factor in G with $H^* \leqq (G_i^0)^*$ and $H^* \leqq (G_j^0)^*$. Then $H^* \leqq (G_{ij}^0)^*$.*

Let i_0 be a fixed element of I and let A be a lexicographic factor of G. Suppose that there exists a lexicographic product decomposition

$$\gamma\colon G \to \Gamma_{i \in I}\, C_i$$

such that $C_{i_0}^0 = A$, and for each $i \in I \setminus \{i_0\}$ we have $C_i^0 = G_i^0$. Then we say that $G_{i_0}^0$ can be replaced by A.

The following result will be stated without proof.

1.9. Proposition. *Let α be as above and let A be a lexicographic factor of G. Let $i_0 \in I$. Then the following conditions are equivalent:*

(i) *$G_{i_0}^0$ can be replaced by A,*

(ii) *$G_{i_0}^0 \in A^*$.*

In this section it will be shown that $F(G)$ is a lattice; the lattice operations in $F(G)$ will be constructively described.

Let G_i^0 and G_j^0 be lexicographic factors of G; let α and β be as in Section 1. It is obvious that G_i^1, G_i^l and G_j^l are convex subgroups of G; moreover, any two convex subgroups of G are comparable. Hence some of the following five possibilities must occur:

(1) $G_j^l \subseteq G_i^l \subset G_i^1 \subseteq G_j^1$;

(2) $G_i^l \subseteq G_j^l \subset G_j^1 \subseteq G_i^1$;

(3) $G_i^l \subseteq G_j^l \subset G_i^1 \subseteq G_j^1$;

(4) $G_j^l \subseteq G_i^l \subset G_j^1 \subseteq G_i^1$;

(5) either $G_j^0 \subseteq G_i^l$ or $G_i^0 \subseteq G_j^l$.

If (1) is valid, then $G_i^0 \leqq G_j^0$, whence $(G_i^0)^* \leqq (G_j^0)^*$ and thus $\inf\{(G_i^0)^*, (G_j^0)^*\} = (G_i^0)^*$. Similarly, in the case (2) we have $\inf\{(G_i^0)^*, (G_j^0)^*\} = (G_j^0)^*$.

Suppose that (5) holds and that for some lexicographic factor G_k^0 the relations $(G_k^0)^* \leqq (G_i^0)^*$ and $(G_k^0)^* \leqq (G_j^0)^*$ are valid. Hence we have $G_k^0 \subseteq G_i^0$ and $G_k^0 \subseteq G_j^0$. Therefore

$$G_i^l \subseteq G_k^l \subset G_k^1 \subseteq G_i^1$$

and

$$G_j^l \subseteq G_k^l \subset G_k^1 \subseteq G_j^1 .$$

In view of (5), we have a contradiction. Thus $\inf\{(G_i^0)^*, (G_j^0)^*\} = 0^-$.

Suppose that (3) is valid. Consider the isomorphic refinements $f(\alpha, \beta)$ and $f(\beta, \alpha)$ of α and β, respectively.

We adopt the same notation as in Section 1. According to the construction of G_{ij} (cf. [5]) we have $j \in J(i)$, whence G_{ij}^0 is a lexicographic factor of G_i^0 and G_{ji}^0 is a lexicographic factor of G_j^0.

If G_k^0 is any lexicographic factor of G with $(G_k^0)^* \leqq (G_i^0)^*$ and $(G_k^0)^* \leqq (G_j^0)^*$, then in view of 1.9 we have $(G_k^0)^* \leqq (G_{ij}^0)^* = (G_{ji}^0)^*$. Therefore $\inf\{(G_i^0)^*, (G_j^0)^*\} = (G_{ij}^0)^*$. The case (4) is analogous. Summarizing, we obtain

2.1. Lemma. *The partially ordered set $F(G)$ is a $\wedge$-semilattice.*

By quoting this lemma, we shall apply also the construction of $(G_i^0)^* \wedge (G_j^0)^*$ as described above.

Now, let us investigate the existence of $\sup\{(G_i^0)^*, (G_j^0)^*\}$ in $F(G)$. The cases (1) and (2) are clear. Suppose that some of the cases (3)–(5) is valid. Let G_k^0 be a lexicographic factor of G such that $(G_i^0)^* \leqq (G_k^0)^*$ and $(G_j^0)^* = (G_k^0)^*$. Hence for each $t \in J(i)$ and each $s \in I(j)$ we have

$$(G_{it}^0)^* \leqq (G_k^0)^* \quad \text{and} \quad (G_{js}^0)^* \leqq (G_k^0)^* .$$

Let H^0 be the lexicographic factor of G generated by the set $\{G_{it}^0\}_{t \in J(i)} \cup \{G_{sj}^0\}_{s \in I(j)}$.

Then $(G_i^0)^* \leqq H^*$ and $(G_j^0)^* \leqq H^*$. According to Lemma 1.8 we have $(H^0)^* \leqq \leqq (G_k^0)^*$. Moreover, in view of the definition of H^0, the relations $(G_i^0)^* \leqq (H^0)^*$ and $(G_j^0)^* \leqq (H^0)^*$ are valid. Thus $\sup\{(G_i^0)^*, (G_j^0)^*\} = (H^0)^*$. We obtain that $F(G)$ is a join-semilattice. Hence in view of 2.1 we have

2.2. Theorem. *The partially ordered set $F(G)$ is a lattice.*

3. THE UPPER COVERING CONDITION

A lattice L is said to satisfy the upper covering conditions if, whenever x and y are elements of L which are covered by $x \vee y$, then $x \wedge y$ is covered by both x and y.

Since we assume that $G \neq \{0\}$, the lattice $F(G)$ has at least two elements, namely $\{\{0\}\}$ and $\{G\}$. If G is lexicographically indecomposable (e.g., if G is the additive group of all integers with the natural linear order), then card $F(G) = 2$, hence $F(G)$ is a two-element chain.

If α is as in Section 1 and $I = \{1, 2\}$ (with the natural linear order), then we shall write $G = G_2^0 \circ G_1^0$.

Suppose that card $F(G) > 2$. Then there exists a lexicographic factor A of G with $\{0\} \neq A \neq G$. Hence some of the following possibilities must occur:

(1) there are lexicographic factors A_1, A_2 of G such that $G = A_1 \circ A \circ A_2$;

(2) there is a lexicographic factor A_1 of G such that $G = A_1 \circ A$;

(3) there is a lexicographic factor A_2 of G such that $G = A \circ A_2$.

In the case (1), the classes A_1^*, A^* and A_2^* are incomparable in the lattice $F(G)$. Similarly, in the cases (2) and (3) the classes A_1^* and A^*, or A_2^* and A^*, respectively, are incomparable in the lattice $F(G)$. Hence we have

3.1. Lemma. (i) card $F(G) \geqq 2$. (ii) *If* card $F(G) > 2$, *then* card $F(G) \geqq 4$. (iii) *If* card $F(G) \leqq 4$, *then the lattice $F(G)$ is distributive.*

Now assume that card $F(G) > 4$. Then the condition (1) is satisfied. From the properties of the lattice operations in $F(G)$ we infer that the sublattice of $F(G)$ generated by the elements A_1^*, A^* and A_2^* is nonmodular. Thus in view of 3.1 we get

3.2. Theorem. *The lattice $F(G)$ is modular if and only if* card $F(G) \leqq 4$.

The following lemma is an immediate consequence of Malcev's refinement theorem in [6].

3.3. Lemma. *Let A, B be lexicographic factors of G such that $(A)^* \leqq (B)^*$ and $A \neq B$. Then there is a lexicographic factor A' with $(A')^* = A^*$ and some of the following conditions are valid:*

(1') *there are lexicographic factors A_1 and A_2 of G such that $B = A_1 \circ A' \circ A_2$;*

(2') *there is a lexicographic factor A_1 of G such that $B = A_1 \circ A'$;*

(3') *there is a lexicographic factor A_2 of G such that $B = A' \circ A_2$.*

3.4. Corollary. *Let $A^*, B^* \in F(G)$ such that A^* is covered by B^* and $A^* \neq 0^-$. Then either*

(2″) the condition (2′) is valid and A_1 is lexicographically indecomposable,

or

(3″) the condition (3′) is valid and A_2 is lexicographically indecomposable.

3.5. Corollary. *Let $A^* \in F(G)$, $A^* \neq 0^-$. Let X be a subset of $F(G)$ such that for each pair of distinct elements B_1^*, B_2^* of X we have $B_1^* \wedge B_2^* = A^*$. Then* card $X \leqq 2$.

3.6. Corollary. *Let $A^* \in F(G)$, $A^* \neq 0^-$. Let X be a subset of $F(G)$ such that each element of X covers A^*. Then* card $X \leqq 2$.

Also, from 3.3 we infer

3.7. Lemma. *Let A_i, A_j and B be lexicographic factors of G such that A_i^* is incomparable with A_j^*, and B^* covers both A_i^* and A_j^*.* a) *Suppose that $A_i^* \wedge A_j^* > 0^-$. Then there are lexicographic factors A', A_1 and A_2 of G such that* (i) *either $A_1 \circ A' \in A_i^*$ and $A' \circ A_2 \in A_j^*$, or $A_1 \circ A' \in A_j^*$ and $A' \circ A_2 \in A_i^*$;* (ii) *A_1 and A_2 are lexicographically indecomposable. Moreover, $A_i^* \wedge A_j^* = (A')^*$.* b) *Suppose that $A_i^* \wedge A_j^* = 0^-$. Then there are $A_1 \in A_i^*$ and $A_2 \in A_j^*$ such that* (i) *A_1 and A_2 are lexicographically indecomposable, and* (ii) *either $B = A_1 \circ A_2$ or $B = A_2 \circ A_1$.*

3.8. Corollary. *The lattice $F(G)$ satisfies the upper covering condition.*

Hence we have (cf. [1])

3.9. Theorem. *If the lattice $F(G)$ is finite, then it is semimodular.*

4. INTERVALS OF THE FORM $[A^*, \{G\}]$

Let A be a fixed lexicographic factor of G, $A \neq G$. In this section the interval $[A^*, \{G\}]$ of the lattice $F(G)$ will be investigated.

4.1. Lemma. *Suppose that either the condition (2) or the condition (3) from Section 3 is valid. Then the interval $[A^*, \{G\}]$ is a chain.*

Proof. This is a consequence of 3.3.

Now suppose that the condition (1) from Section 4 is valid. Let X be the set of all $B^* \in F(G)$ such that the condition (2′) from Section 3 holds for B (where $A' \in A^*$). Similarly, let Y be the set of all $B^* \in F(G)$ such that the condition (3′) from Section 3 is satisfied (again, $A' \in A^*$).

4.2. Lemma. *Both X and Y are chains.*

Proof. This follows immediately from the definition of X and Y.

Put $X_0 = \{0^-\} \cup X$, $Y_0 = \{0^-\} \cup Y$. Then X_0 and Y_0 are chains as well.

The following assertion is easy to verify.

4.3. Lemma. *Let A and B be as in 3.3. Let $B' \in B^*$. If (1′) is valid, then there are* *lexicographic factors* A_1', A'', A_2' *in* G *such that* $B' = A_1' \circ A'' \circ A_2'$ *and* $A'' \in A^*$, $A_1' \in A_1^*$ *and* $A_2' \in A_2^*$. *In the cases* (2') *and* (3') *analogous results are valid.*

We define a mapping $f: [A^*, \{G\}] \to X_0 \times Y_0$ as follows. Let $B^* \in [A^*, \{G\}]$. (i) If the condition (1') from 3.3 holds, then we put $f(B^*) = (A_1^*, A_2^*)$. (ii) If (2') holds, then we set $f(B^*) = (A_1^*, 0^-)$. (iii) In the case (3') let $f(B^*) = (0^-, A_2^*)$. (iv) If $B^* = A^*$, then we put $f(B^*) = (0^-, 0^-)$.

In view of 4.3, the mapping f is correctly defined. Moreover, f is a bijection and we have

$$B_1^* \leqq B_2^* \Leftrightarrow f(B_1^*) \leqq f(B_2^*) .$$

We have obtained the following result:

4.4. Theorem. *The lattice* $[A^*, \{G\}]$ *is isomorphic to the direct product* $X_0 \times Y_0$ *of linearly ordered sets* X_0 *and* Y_0.

A lattice L is said to be *completely distributive* if the relations (4) and (4') in [1], Chap. V, § 5 are valid in L whenever all the corresponding suprema and infima do exist in L.

Since each chain is completely distributive, 4.4 yields

4.5. Theorem. *Let* A *be a lexicographic factor of* G. *Then the interval* $[A^*, \{G\}]$ *of the lattice* $F(G)$ *is completely distributive.*

4.6. Corollary. *Each closed dual ideal of* $F(G)$ *distinct from* $F(G)$ *is completely distributive.*

The question of a generalization of Theorem 4.5 for lexicographic factors of partially ordered groupoids investigated in [3] remains open.

Theorem 4.5 cannot be generalized for factors in mixed product decompositions of partially ordered groups (as studied in [4]).

5. TRANSPOSED INTERVALS

Let A and B be lexicographic factors of G. The intervals $[A^*, A^* \vee B^*]$ and $[A^* \wedge B^*, B^*]$ are called *transposed* to each other. (Cf. [1].)

For leach lexicographic factor A_1 of G with $A^* \leqq A_1^* \leqq A^* \vee B^*$ we put $f_1(A_1^*) = A_1^* \wedge B^*$.

Similarly, for each lexicographic factor B_1 of G with $A^* \wedge B^* \leqq B_1^* \leqq B^*$ we set $f_2(B_1^*) = B_1^* \vee A^*$.

Let us consider the following conditions for A and B:

(α_1) f_1 is an isomorphism of the interval $[A^*, A^* \vee B^*]$ onto the interval $[A^* \wedge B^*, B^*]$;

(α_2) f_2 is an isomorphism of the interval $[A^* \wedge B^*, B^*]$ onto the interval $[A^*, A^* \vee B^*]$.

(For analogous mappings in general lattices cf. [1], Chap. I, Theorem 13.)

The lexicographic factors A and B will be said to be *adjacent* if there are lexicographic factors C and D of G such that $C^* = A^*$, $D^* = B^*$, and either

(i) $C \circ D$ is a lexicographic factor of G,

or

(ii) $D \circ C$ is a lexicographic factor of G.

It is easy to verify that A and B are adjacent if and only if either

(i_1) $A^1 = B^l$

or

(ii_1) $B^1 = A^l$.

5.1. Lemma. *If $A^* \wedge B^* > 0^-$, then both (α_1) and (α_2) are valid.*

Proof. This is a consequence of 4.5 (in view of [1], Chap. I, Theorem 13).

5.2. Lemma. *If both A^* and B^* are covered by $A^* \vee B^*$, then (α_1) and (α_2) hold.*

Proof. This follows from 3.8.

It will be shown below that the condition from Lemma 5.2 is not necessary for (α_1) and (α_2) to be valid. (Cf. Corollary 5.5 below.)

5.3. Lemma. *Let A and B be adjacent. Let A_1 and B_1 be lexicographic factors of G such that $A^* < A_1^* < A^* \vee B^*$ and $0^- = A^* \wedge B^* < B_1^* < B^*$.*

(a) *Let (i) be valid. Then there are lexicographic factors D_i ($i = 1, 2$) of D such that $D = D_1 \circ D_2$ and $A_1^* = (C \circ D_1)^*$.*

(b) *Let (ii) be valid. Then there are D_i ($i = 1, 2$) as in (a) with the distinction that instead of $A_1^* = (C \circ D)^*$ we now have $A_1^* = (D_2 \circ C)^*$.*

Proof. This is implied by the Mal'cev Theorem (loc. cit.).

5.4. Corollary. *Let A and B be adjacent. Suppose that B is lexicographically indecomposable. Then the conditions (α_1) and (α_2) hold.*

Proof. The assertion follows from the fact that in view of 5.3 both the intervals $[A^*, A^* \vee B^*]$ and $[A^* \wedge B^*, B^*]$ are prime.

5.5. Remark. Under the assumptions as in 5.4, A need not be lexicographically indecomposable; this implies that $A^* \vee B^*$ need not cover B^*.

If A^* and B^* are comparable, then the conditions (α_1) and (α_2) obviously hold. Thus, when investigating these conditions it suffices to assume that A^* and B^* are incomparable; also, in view of 5.1 it suffices to assume that $A^* \wedge B^* = 0^-$.

5.6. Lemma. *Assume that $A^* \wedge B^* = 0^-$. Suppose that f_1 is a monomorphism. Then A and B are adjacent.*

Proof. By way of contradiction, assume that A and B fail to be adjacent. Then we have either

(a) $A^1 \subset B^l$,

or

(b) $B^1 \subset A^l$.

Suppose that (a) is valid. Then there is a lexicographic factor C of G such that $A \circ C$ is a lexicographic factor of G as well, and $(A \circ C)^* \wedge B^* = 0^-$, $(A \circ C)^* < A^* \vee B^*$. Thus $A^* \neq (A \circ C)^*$ and $f_1(A^*) = f_1((A \circ C)^*)$, which is a contradiction.

The case (b) is analogous.

5.7. Lemma. *Let A and B be adjacent. Suppose that f_1 is an epimorphism. Then B is lexicographically indecomposable.*

Proof. By way of contradiction, assume that we have $B = C_1 \circ D_1$. We have either (i_1) or (ii_1). Let (i_1) be valid. Then $0^- = A^* \wedge B^* < D_1^* < B^*$ and if A_1 is a direct factor of G with $A^* \leqq A_1^* \leqq A^* \vee B^*$, then $f_1(A_1^*) \neq D_1^*$, which is a contradiction. The case (ii_1) is analogous.

The proof of the following lemmas uses the same idea which was applied in the proofs of 5.6 and 5.7.

5.8. Lemma. *Let (α_2) be valid. Then A and B are adjacent and B is lexicographically indecomposable.*

From 5.4, 5.6, 5.7 and 5.8 we obtain

5.9. Theorem. *Let A and B be lexicographic factors of G. Then (α_1) is equivalent to the following condition.*

(β_1) *either $A^* \wedge B^* > 0^-$, or A is adjacent to B and B is lexicographically indecomposable. Moreover, (α_1) and (α_2) are equivalent.*

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