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ON THE PROBLEM OF INVARIANCE UNDER HOLOMORPHIC FUNCTIONS FOR A SET OF CONTINUITY POINTS OF THE SPECTRUM FUNCTION

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INTRODUCTION

In a paper to appear ([B2]) I have introduced a subset $\Sigma_0(X)$ of the set $\Sigma(X)$ of all the continuity points of the spectrum function in the algebra of linear and continuous operators on a complex Banach space X. In the case of a separable Hilbert space, the set $\Sigma(X)$ has been recently characterized by Conway and Morrel ([CM]) and it can be proved that $\Sigma_0(X)$ coincides with $\Sigma(X)$. It is not known yet if this fact is true in any Banach space, but the authors of [AFHV], who give another necessary and sufficient condition for membership in $\Sigma(X)$, in the case of a separable Hilbert space (see [AFHV], Th. 14.15), suspect that such a condition (which, in any Banach space X, is equivalent to membership in $\Sigma_0(X)$, see [B2], 1.15 and 3.1) characterizes $\Sigma(X)$ for any Banach space X (see [AFHV], page 313). The set $\Sigma_0(X)$ is therefore related to $\Sigma(X)$ in a very interesting way.

In [B2] I have also studied algebraic and topological properties of $\Sigma_0(X)$, without treating the problem of invariance under holomorphic functions, that is instead the subject of this paper, in which a systematic study of the conditions that characterize such a property is made.[1])

In Section 1 I give two preliminary topological results, in a much more general ambit than what is needed for the successive proofs.

In Section 2 I give equivalent conditions (Theorem 2.5) and sufficient ones (Corollaries 2.6, 2.7 and 2.8) for membership of $f(A)$ (where $A \in \Sigma_0(X)$, X is a complex nonzero Banach space and f is a complex-valued function, holomorphic on a neighborhood of the spectrum of A) in $\Sigma_0(X)$. Some of the preliminary results in this section before Theorem 2.5 are slight extensions of analogous results in [BHOP].

In the last part of Section 2 I give equivalent conditions (Theorem 2.9) for membership of $f(A)$ in $\Sigma_0(X)$ for any $A \in \Sigma_0(X)$ whose spectrum is contained in the domain

[1]) As far I know, the problem of invariance of $\Sigma(X)$ under holomorphic functions has not been studied yet.

of f, where f is a complex-valued holomorphic function and X is a complex infinite-dimensional Hilbert space (in Corollary 2.10 the particular case of f being a power is treated).

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0.

If X is a Banach space, for any $x \in X$ and for any $\varepsilon > 0$ let $B_X(x, \varepsilon)$ denote the set of all points of X whose distance from x is smaller than ε. We shall denote with X^* the space of linear and continuous functionals on X, with $L_c(X)$ the space of linear and continuous operators on X, with I_X the identity operator on X and with $L_{cc}(X)$ the ideal of compact operators on X.

By a *projection* on X we shall mean an operator $P \in L_c(X)$ such that $P^2 = P$. Obviously, $I_X - P$ is a projection too, $\operatorname{Im} P = \operatorname{Ker}(I_X - P)$ (the symbol Im will be used to denote the range of a function), so that $\operatorname{Im} P$ is closed, and $X = \operatorname{Im} P \oplus \oplus \operatorname{Ker} P$ (where the symbol $\oplus$ means algebraic direct sum).

If A is a linear and continuous operator on a complex nonzero Banach space X, let A^* denote the adjoint of A (if X is a Hilbert space, we shall denote the Hilbert adjoint of A with $A^{(*)}$) and let $\sigma(A)$ denote the spectrum of A. We recall that $\sigma(A)$ is a compact nonempty subset of the complex plane, so that, if $\varrho(A)$ denotes $\boldsymbol{C} \setminus \sigma(A)$, it results that $\varrho(A)$ is open and nonempty. We also recall that the resolvent function: $\lambda \in \varrho(A) \to (\lambda I_X - A)^{-1} \in L_c(X)$ is analytic in $\varrho(A)$. Let $\sigma_e(A)$ denote the essential spectrum of A, that is the spectrum of the class of A in the quotient algebra $L_c(X)/L_{cc}(X)$.

If f is a complex-valued function, holomorphic in an open neighborhood Δ of $\sigma(A)$, the operator $f(A) \in L_c(X)$ is defined in the following way:

$$f(A) = (1/(2\pi i)) \int_{+\partial D} f(\lambda)(\lambda I_X - A)^{-1} \, d\lambda ,$$

where D is an open bounded set such that $\sigma(A) \subset D$, $\bar{D} \subset \Delta$, D has a finite number of components, ∂D is composed of a finite number of simple closed rectifiable curves, no two of which intersect and $+\partial D$ signifies the positively oriented boundary of D (the above integral is well defined, and it does not depend on the particular choice of D, see [TL], page 310). We recall that $\sigma(f(A)) = f(\sigma(A))$ and $\sigma_e(f(A)) = f(\sigma_e(A))$ (see [GL], Theorem 1 and Section 2, and [CPY], (3.2.8)).

We recall that a spectral set of A is a subset α of its spectrum that is both open and closed in the relative topology of $\sigma(A)$. Let $\varrho_{s-F}(A)$ denote the set of all points $\lambda \in \boldsymbol{C}$ such that $\lambda I_X - A$ is a semi-Fredholm operator (see [K], page 230). It is immediate to verify that $\varrho(A) \subset \varrho_{s-F}(A)$ and $\operatorname{ind}(\lambda I_X - A) = 0$ for any $\lambda \in \varrho(A)$ (where, for any semi-Fredholm operator $T \in L_c(X)$, ind T denotes the semi-Fredholm index of T, see [K], IV, (5.1) and 5.13). Then, if we put $\varrho_{s-F}^{\pm}(A) = \{\lambda \in \varrho_{s-F}(A) : \operatorname{ind}(\lambda I_X - A) \neq \neq 0\}$, it follows immediately that $\varrho_{s-F}^{\pm}(A) \subset \sigma(A)$ (and, consequently, also

$\mathrm{cl}\, \varrho_{s-F}^{\pm}(A) \subset \sigma(A)$). For any $n \in \boldsymbol{Z} \cup \{-\infty, +\infty\}$, we set $\varrho_{s-F}^{n}(A) = \{\lambda \in \varrho_{s-F}(A)$: $\mathrm{ind}\,(\lambda I_X - A) = n\}$. If $\varrho_F(A)$ denotes the set of all points $\lambda \in \boldsymbol{C}$ such that $\lambda I_X - A$ is a Fredholm operator (see [K], page 230), it follows that $\varrho_F(A) = \bigcup_{n \in \boldsymbol{Z}} \varrho_{s-F}^{n}(A)$. Moreover, by [CPY], (3.2.8), $\varrho_F(A) = \boldsymbol{C} \setminus \sigma_e(A)$. By [K], IV, 5.17, $\varrho_{s-F}^{n}(A)$ is open for any $n \in \boldsymbol{Z} \cup \{-\infty, +\infty\}$. Consequently, also $\varrho_F(A)$, $\varrho_{s-F}(A)$, $\varrho_{s-F}^{\pm}(A)$ are open subsets of $\boldsymbol{C}$, and the semi-Fredholm index is constant on any component of $\varrho_{s-F}(A)$. If V is a component of $\varrho_{s-F}(A)$, let $i_A(V)$ denote $\mathrm{ind}\,(\lambda I_X - A)$ as $\lambda \in V$. We set $\sigma_{s-F}(A) = \boldsymbol{C} \setminus \varrho_{s-F}(A)$. Obviously, $\sigma_{s-F}(A) \subset \sigma_e(A) = \sigma_{s-F}(A) \cup \varrho_{s-F}^{+\infty}(A) \cup \varrho_{s-F}^{-\infty}(A)$. Since $\varrho_{s-F}(A)$ is open, it follows that $\partial \varrho_{s-F}(A) \subset \sigma_{s-F}(A)$. Moreover, since obviously $\partial \varrho_F(A)$, $\partial \varrho_{s-F}^{\pm}(A)$, $\partial \varrho_{s-F}^{n}(A)$ (where $n \in \boldsymbol{Z} \cup \{-\infty, +\infty\}$) and ∂V (where V is a component of $\varrho_{s-F}(A)$) are contained in $\partial \varrho_{s-F}(A)$, they are also contained in $\sigma_{s-F}(A)$.

Let $\psi(A)$ denote the set of all points $\lambda \in \sigma(A)$ such that $\{\lambda\}$ is a component of $\sigma(A)$; obviously, $\mathrm{cl}\, \psi(A) \subset \sigma(A)$.

If X is a complex nonzero Banach space, we put $\Sigma_0(X) = \{A \in L_c(X)$: $\sigma(A) = \mathrm{cl}\,(\varrho_{s-F}^{\pm}(A) \cup \psi(A))\}$ (see [B2], 3.1). Let $\Sigma(X)$ denote the set of all continuity points of the spectrum function $\sigma: L_c(X) \to \boldsymbol{K_C}$ (where $\boldsymbol{K_C}$ is the set of all compact nonempty subsets of $\boldsymbol{C}$, endowed with the Hausdorff metric). I recall that $\Sigma_0(X) \subset \Sigma(X)$ (see [B2], 1.15). If X is a separable Hilbert space, also the opposite inclusion holds, so that $\Sigma_0(X) = \Sigma(X)$ (see [B2], 1.15).

If we put $\tau(X) = \{A \in L_c(X)$: $\sigma(A) = \mathrm{cl}\, \psi(A)\}$ (see [B1], 2.1 and 2.4), it follows obviously that $\tau(X) \subset \Sigma_0(X)$. If X is finite-dimensional, it is immediate to remark that $L_c(X) = \tau(X) = \Sigma_0(X) = \Sigma(X)$; the problem of invariance of $\Sigma_0(X)$ under holomorphic functions is therefore trivial in this case.

We recall that, for any complex nonzero Banach space X, $\tau(X)$ is closed with respect to holomorphic functions (see [B1], 2.13), so that, in particular, it is closed with respect to translations and powers. The behaviour of $\Sigma_0(X)$ under holomorphic functions is more complex, as this paper shows.

1.

Definition 1.1. If X is a complex nonzero Banach space, $A \in L_c(X)$ and f is a complex-valued function, holomorphic on an open neighbourhood Δ of $\sigma(A)$, let $Z_f(A)$ denote the set of all points $\lambda \in \sigma(A)$ such that $f'(\lambda) = 0$.

We remark that, as $\sigma(A)$ is compact, only a finite number of components of Δ have nonempty intersection with $\sigma(A)$. Therefore, as f is holomorphic and any set which has no accumulation points in Δ intersects $\sigma(A)$ at most in a finite number of points, it follows that $f(Z_f(A))$ is finite and, for any $\lambda \notin f(\Delta_f(A))$, $f^{-1}(\{\lambda\}) \cap \sigma(A)$ is finite, too (see [R], 10.18).

Definition 1.2. Let X and Y be topological spaces; for any $f: X \to Y$ let I_f denote the set of all points $x \in X$ such that there exists a neighbourhood U_x of x such that

$f(U_x)$ is a neighborhood of $f(x)$ and $f_x\colon U_x \to f(U_x)$ (where $f_x(y) = f(y)$ for any $y \in U_x$) is a homeomorphism.

We remark that, if A is a linear and continuous operator on a complex nonzero Banach space and f is a complex-valued function, holomorphic on an open neighborhood of $\sigma(A)$, it results that $\sigma(A) \setminus Z_f(A) \subset I_f$ (see [R], 10.34), so that $\sigma(A) \setminus I_f \subset Z_f(A)$.

Lemma 1.3. *If X is a topological space, F is a compact subset of X, with empty interior, Y is a Hausdorff space and $f\colon X \to Y$ is a continuous function such that $f^\circ(F \setminus I_f) = \emptyset$, then $f^\circ(F) = \emptyset$.*

Proof. Suppose that X is a topological space, F is a compact subset of X, Y is a Hausdorff space and $f\colon X \to Y$ is a continuous function such that $f^\circ(F \setminus I_f) = \emptyset$ and $f^\circ(F) \neq \emptyset$. We prove that, in this case, $F^\circ \neq \emptyset$.

As $f^\circ(F) \neq \emptyset$, there exists an open nonempty subset G of Y such that $G \subset f(F)$; as $f^\circ(F \setminus I_f) = \emptyset$, there exists $y \in G \setminus f(F \setminus I_f)$, so that $f^{-1}(\{y\}) \cap F \subset I_f$. Therefore, for any $x \in f^{-1}(\{y\}) \cap F$, there exists an open neighborhood U_x of x such that $f(U_x)$ is open and $f_x\colon U_x \to f(U_x)$ (where $f_x(z) = f(z)$ for any $z \in U_x$) is a homeomorphism. Since Y is a Hausdorff space and f is continuous, $f^{-1}(\{y\})$ is closed, so that, as F is compact, $f^{-1}(\{y\}) \cap F$ is compact, too. Consequently, there exist a positive integer n and $x_1, \ldots, x_n \in f^{-1}(\{y\}) \cap F$ such that $f^{-1}(\{y\}) \cap F \subset \bigcup_{j=1}^{n} U_{x_j}$. Since for any $x \in f^{-1}(\{y\}) \cap F$ there exists $j \in \{1, \ldots, n\}$ such that $x \in U_{x_j}$ (so that, as $f_{x_j}(x) = f(x) = y = f_{x_j}(x_j)$ and f_{x_j} is a homeomorphism, $x = x_j$) it follows that $f^{-1}(\{y\}) \cap F = \{x_1, \ldots, x_n\}$. For any $j = 1, \ldots, n$ we define $U_j = U_{x_j}$ and $f_j = f_{x_j}$. Since F is compact and U_j is open for any $j = 1, \ldots, n$, $F \setminus (\bigcup_{j=1}^{n} U_j)$ is compact, so that, as f is continuous, $f(F \setminus (\bigcup_{j=1}^{n} U_j))$ is compact, too. As $y \notin f(F \setminus (\bigcup_{j=1}^{n} U_j))$ (because $f^{-1}(\{y\}) \cap F = \{x_1, \ldots, x_n\} \subset \bigcup_{j=1}^{n} U_j$) and Y is a Hausdorff space, it follows that there exists an open neighborhood U of y, contained in $G \cap (\bigcap_{j=1}^{n} f(U_j))$, such that $U \cap f(F \setminus (\bigcup_{j=1}^{n} U_j)) = \emptyset$. Therefore $f^{-1}(U) \cap F \subset \bigcup_{j=1}^{n} U_j$.

We construct now, by induction on k, a finite family $\{W_k\}_{k=1,\ldots,n}$ of open nonempty subsets of U in the following way: we set $W_1 = U$. For any $k = 1, \ldots, n-1$, let W_k be an open nonempty subset of U. Then, if $f_{k+1}^{-1}(W_k) \cap (\bigcup_{j=1}^{k} f_j^{-1}(W_k)) = \emptyset$, we define $W_{k+1} = W_k$; if, instead, there exists $j(k) \in \{1, \ldots, k\}$ such that $f_{k+1}^{-1}(W_k) \cap f_{j(k)}^{-1}(W_k) \neq \emptyset$, we define $W_{k+1} = f(f_{k+1}^{-1}(W_k) \cap f_{j(k)}^{-1}(W_k))$. It is easy to verify, by induction on k, that, for any $k = 1, \ldots, n$ and for any $j, h \in \{1, \ldots, k\}$, if $f_j^{-1}(W_k) \cap f_h^{-1}(W_k) \neq \emptyset$ it follows that $f_j^{-1}(W_k) = f_h^{-1}(W_k)$; then, if we define $W = W_n$, it follows that, for any $j, k \in \{1, \ldots, n\}$, $f_j^{-1}(W) \cap f_k^{-1}(W) = \emptyset$ or $f_j^{-1}(W) = f_k^{-1}(W)$.

Since W is an open nonempty subset of U, $U \subset f(F)$, $f(F)$ is compact and Y is a Hausdorff space, there exists a compact set V, with nonempty interior, such that $V \subset W$; obviously, as Y is Hausdorff, V is closed, so that $f^{-1}(V)$ is closed, too.

For any $j = 1, \ldots, n$ we define $V_j = f_j^{-1}(V)$ and $I_j = \{k \in \{1, \ldots, n\}: f_k^{-1}(W) \cap \cap f_j^{-1}(W) = \emptyset\}$; for any $k \in \{1, \ldots, n\} \setminus I_j$ it results obviously that $f_k^{-1}(W) = f_j^{-1}(W)$. As $F \cap f^{-1}(U) \subset \bigcup_{j=1}^{n} U_j$ and $W \subset U$, it follows that

$$\begin{aligned} F \cap (f^{-1}(V) \setminus (\bigcup_{k \in I_j} f_k^{-1}(W))) &= (F \cap f^{-1}(W)) \cap (f^{-1}(V) \setminus (\bigcup_{k \in I_j} f_k^{-1}(W))) = \\ &= F \cap f^{-1}(V) \cap ((\bigcup_{k \in \{1,\ldots,n\} \setminus I_j} f_k^{-1}(W)) \setminus (\bigcup_{k \in I_j} f_k^{-1}(W))) = \\ &= F \cap f^{-1}(V) \cap (f_j^{-1}(W) \setminus (\bigcup_{k \in I_j} f_k^{-1}(W))) = F \cap f^{-1}(V) \cap f_j^{-1}(W) = \\ &= F \cap f_j^{-1}(V) = F \cap V_j . \end{aligned}$$

Consequently, as F is compact, $f^{-1}(V)$ is closed and $f_k^{-1}(W)$ is open for any $k \in I_j$, $F \cap V_j$ is compact, so that, as f is continuous and Y is a Hausdorff space, $f(F \cap V_j)$ is closed.

Since $V \subset U \subset f(F)$ and $f^{-1}(U) \cap F \subset \bigcup_{j=1}^{n} U_j$, it follows that

$$V = V \cap f(F) = f(F \cap f^{-1}(V)) = f(\bigcup_{j=1}^{n} (F \cap f_j^{-1}(V))) = \bigcup_{j=1}^{n} f(F \cap V_j) .$$

Consequently, as V has nonempty interior, $f(F \cap V_j)$ is closed for any $j = 1, \ldots, n$ and a finite union of closed sets with empty interior has empty interior, there exists $p \in \{1, \ldots, n\}$ such that $f^\circ(F \cap V_p) \neq \emptyset$. As $f(F \cap V_p) = f_p(F \cap V_p)$, f_p is a homeomorphism and U_p is open, it follows that $F \cap^\circ V_p \neq \emptyset$; therefore $F^\circ \neq \emptyset$.

Lemma 1.4. *Let X be a connected topological space and let K be a proper, infinite and compact subset of X; then, if $D(K)$ denotes the set of all accumulation points of K, it follows that $\partial K \cap D(K) \neq \emptyset$.*

Proof. We prove that if K is an infinite compact subset of a connected space X, such that $\partial K \cap D(K) = \emptyset$, it follows that $K = X$.

Since $D(K) \cap \partial K = \emptyset$, any accumulation point of K has a neighborhood which has empty intersection with $X \setminus K$, so that $D(K) \subset K^\circ$; therefore $\bar{K} = K \cup D(K) = K$, so that $\partial K \subset K$ and, consequently, as ∂K is closed, $D(\partial K) \subset D(K) \cap \partial K = \emptyset$.

Since $\partial K \subset K$ and K is compact, ∂K is compact, too, so that, as any infinite compact set has at least one accumulation point and $D(\partial K) = \emptyset$, ∂K is finite. Consequently, as $K = K^\circ \cup \partial K$ and K is infinite, $K^\circ \neq \emptyset$.

Since $\partial K \cap D(K) = \emptyset$, for any $x \in \partial K$ there exists an open neighborhood U_x of x such that $U_x \cap K = \{x\}$. Therefore $K \setminus (\bigcup_{x \in \partial K} U_x) = K \setminus (\bigcup_{x \in \partial K} (U_x \cap K)) = K \setminus (\bigcup_{x \in \partial K} \{x\}) = K \setminus \partial K = K^\circ$, so that, as K is closed and U_x is open for any $x \in \partial K$, K° is closed. Consequently, as $K^\circ \neq \emptyset$ and X is connected, $K^\circ = X$; it follows immediately that $K = X$.

2.

Definition 2.1. Let Δ be an open nonempty subset of C and let $f\colon \Delta \to C$ be a holomorphic function. We shall denote the union of all components of Δ on which f is not constant with $\Omega_f(\Delta)$.

We point out that f is an open map on $\Omega_f(\Delta)$ (see [R], 10.32). We remark that, for any $\lambda \in C$, $f^{-1}(\{\lambda\}) \cap \Omega_f(\Delta)$ is discrete (see [R], 10.18). Hence $f^{-1}(\{\lambda\}) \cap \cap K \cap \Omega_f(\Delta)$ is finite for any $\lambda \in C$ and for any compact set $K \subset \Delta$. Let X be a complex nonzero Banach space and let $A \in L_c(X)$ be such that $\sigma(A) \subset \Delta$. It follows that $f^{-1}(\{\lambda\}) \cap \sigma(A) \cap \Omega_f(\Delta)$ is finite for any $\lambda \in C$. We also remark that, since any component of Δ is open and closed in Δ, the intersection of any component of Δ with $\sigma(A)$ is a spectral set of A. Consequently, since $\sigma(A)$ intersects only a finite number of components of Δ, $\sigma(A) \cap \Omega_f(\Delta)$ is a spectral set of A.

The following theorem is an easy consequence of the results of [GL]. Part i) has already been impliedly proved by the authors of [BHOP] in the particular case of a separable Hilbert space (see [BHOP], 3.4 and 1.(1)). However, the proof of [BHOP], 3.4 cannot be used in the general case, as it is based upon the properties of the left and right essential spectra, whereas the characterization of $\sigma_{s-F}(A)$ as the intersection of the left and right essential spectra may not hold in the algebra of linear and continuous operators on a Banach space which is not a Hilbert one (see, for example, [B2], 1.1). Henceforth, we shall agree that the sum of an empty set of numbers is equal to zero.

Theorem 2.2. *Let X be a complex nonzero Banach space, let $A \in L_c(X)$ and let f be a complex-valued function, holomorphic on an open neighborhood Δ of $\sigma(A)$. It follows that:*

i) $\sigma_{s-F}(f(A)) = f(\sigma_{s-F}(A)) \cup (f(\varrho_{s-F}^{+\infty}(A)) \cap f(\varrho_{s-F}^{-\infty}(A)))$;

ii) *for any $\lambda \in C$ such that $f^{-1}(\{\lambda\}) \subset \varrho_{s-F}(A)$ (and hence, in particular, for any $\lambda \in \varrho_{s-F}(f(A))$), $f^{-1}(\{\lambda\}) \cap \sigma(A)$ is finite and $f^{-1}(\{\lambda\}) \cap D(\sigma(A)) \subset \Omega_f(\Delta)$;*

iii) $\operatorname{ind}(\lambda I_X - f(A)) = \sum_{\mu \in f^{-1}(\{\lambda\}) \cap \sigma^\circ(A)} m_f(\mu) \operatorname{ind}(\mu I_X - A)$ *for any* $\lambda \in \varrho_{s-F}(f(A))$ *(where $m_f(\mu)$ denotes the order of μ as a zero of $f - f(\mu)$ for any $\mu \in \Omega_f(\Delta)$).*

Proof. First of all, we prove i). We recall that Theorem 1 of [GL] can be applied also to the semigroups $\Phi^- = \{T \in L_c(X)\colon T$ is semi-Fredholm and $\operatorname{ind} T < +\infty\}$ and $\Phi^+ = \{T \in L_c(X)\colon T$ is semi-Fredholm and $\operatorname{ind} T > -\infty\}$ (see [GL], Theorem 1 and Section 2). Therefore $\sigma_{s-F}(f(A)) \cup \varrho_{s-F}^{+\infty}(f(A)) = f(\sigma_{s-F}(A) \cup \varrho_{s-F}^{+\infty}(A))$ and $\sigma_{s-F}(f(A)) \cup \varrho_{s-F}^{-\infty}(f(A)) = f(\sigma_{s-F}(A) \cup \varrho_{s-F}^{-\infty}(A))$. It follows that $\sigma_{s-F}(f(A)) = (\sigma_{s-F}(f(A)) \cup \varrho_{s-F}^{+\infty}(f(A))) \cap (\sigma_{s-F}(f(A)) \cup \varrho_{s-F}^{-\infty}(f(A))) = (f(\sigma_{s-F}(A)) \cup f(\varrho_{s-F}^{+\infty}(A))) \cap (f(\sigma_{s-F}(A)) \cup f(\varrho_{s-F}^{-\infty}(A))) = f(\sigma_{s-F}(A)) \cup (f(\varrho_{s-F}^{+\infty}(A)) \cap f(\varrho_{s-F}^{-\infty}(A)))$. We have thus proved i).

Now we prove ii). Let $\lambda \in C$. Suppose that $f^{-1}(\{\lambda\}) \cap \sigma(A)$ is infinite: we prove that consequently $f^{-1}(\{\lambda\}) \not\subset \varrho_{s-F}(A)$. As $\sigma(A)$ is compact, it has nonempty intersection with only a finite number of components of Δ. Hence Δ has a component Δ_0 such that $\Delta_0 \cap f^{-1}(\{\lambda\}) \cap \sigma(A)$ is infinite. We put $\sigma_0 = \sigma(A) \cap \Delta_0$. As we remarked above, σ_0 is a spectral set of A, and so it is compact. Consequently, as $\Delta_0 \cap \cap f^{-1}(\{\lambda\}) \cap \sigma(A)$ is infinite and contained in σ_0, $f^{-1}(\{\lambda\}) \cap \Delta_0$ is not discrete, so that (see [R], 10.18), as Δ_0 is connected, f is constant on Δ_0. Therefore $\Delta_0 \subset \subset f^{-1}(\{\lambda\})$. Since σ_0 is infinite, from Lemma 1.4 it follows that $\partial\sigma_0 \cap D(\sigma_0) \neq \emptyset$; since $\partial\sigma_0 = \partial\sigma(A) \cap \Delta_0$ and $D(\sigma_0) = D(\sigma(A)) \cap \Delta_0$ there exists $\mu \in D(\sigma(A)) \cap \cap \partial\sigma(A) \cap \Delta_0$. From [K], IV, 5.6 and 5.31 it follows that $\partial\sigma(A) \cap \varrho_{s-F}(A)$ consists of isolated points, so that $D(\sigma(A)) \cap \partial\sigma(A) \cap \varrho_{s-F}(A) = \emptyset$. Therefore, obviously, $\mu \notin \varrho_{s-F}(A)$. Since $\Delta_0 \subset f^{-1}(\{\lambda\})$, we have thus proved that $f^{-1}(\{\lambda\}) \not\subset \varrho_{s-F}(A)$. Hence, for any $\lambda \in \mathbf{C}$ such that $f^{-1}(\{\lambda\}) \subset \varrho_{s-F}(A)$, $f^{-1}(\{\lambda\}) \cap \sigma(A)$ is finite. Moreover, for any $\mu \in f^{-1}(\{\lambda\}) \cap D(\sigma(A))$, the intersection of any neighbourhood of μ with $\sigma(A)$ is infinite. It follows easily that $f^{-1}(\{\lambda\}) \cap D(\sigma(A)) \subset \Omega_f(\Delta)$.

Finally, we prove iii). Let $\lambda \in \varrho_{s-F}(f(A))$.

Since $\varrho^n_{s-F}(A)$ is an open subset of $\sigma(A)$ for any $n \in (\mathbf{Z} \setminus \{0\}) \cup \{-\infty, +\infty\}$, $f^{-1}(\{\lambda\}) \cap \sigma(A) \subset \varrho_{s-F}(A)$ by i) and $f^{-1}(\{\lambda\}) \cap \sigma^\circ(A) \subset \Omega_f(\Delta)$ by ii), it follows that

$$f^{-1}(\{\lambda\}) \cap \Big(\bigcup_{n\in(\mathbf{Z}\setminus\{0\})\cup\{-\infty,+\infty\}} \varrho^n_{s-F}(A)\Big) \cap \Omega_f(\Delta) \subset f^{-1}(\{\lambda\}) \cap \sigma^\circ(A) \subset$$

$$\subset \Big(f^{-1}(\{\lambda\}) \cap \Big(\bigcup_{n\in(\mathbf{Z}\setminus\{0\})\cup\{-\infty,+\infty\}} \varrho^n_{s-F}(A)\Big) \cap \Omega_f(\Delta)\Big) \cup$$

$$\cup \left(f^{-1}(\{\lambda\}) \cap \sigma(A) \cap \varrho^0_{s-F}(A) \cap \Omega_f(\Delta)\right).$$

Therefore

$$\sum_{\mu\in f^{-1}(\{\lambda\})\cap\sigma^\circ(A)} m_f(\mu) \operatorname{ind}(\mu I_X - A) =$$

$$= \sum_{\mu\in f^{-1}(\{\lambda\})\cap(\bigcup_{n\in(\mathbf{Z}\setminus\{0\})\cup\{-\infty,+\infty\}} \varrho^n_{s-F}(A))\cap\Omega_f(\Delta)} m_f(\mu) \operatorname{ind}(\mu I_X - A).$$

By [GL], Theorem 1 (see [TL], IV, 3.1),

$$\operatorname{ind}(\lambda I_X - f(A)) = \sum_{n\in\mathbf{Z}\cup\{-\infty,+\infty\}} n \sum_{\mu\in f^{-1}(\{\lambda\})\cap\sigma(A)\cap\varrho^n_{s-F}(A)\cap\Omega_f(\Delta)} m_f(\mu) =$$

$$= \sum_{n\in(\mathbf{Z}\setminus\{0\})\cup\{-\infty,+\infty\}} \sum_{\mu\in f^{-1}(\{\lambda\})\cap\varrho^n_{s-F}(A))\cap\Omega_f(\Delta)} m_f(\mu) \operatorname{ind}(\mu I_X - A) =$$

$$= \sum_{\mu\in f^{-1}(\{\lambda\})\cap(\bigcup_{n\in(\mathbf{Z}\setminus\{0\})\cup\{-\infty,+\infty\}} \varrho^n_{s-F}(A))\cap\Omega_f(\Delta)} m_f(\mu) \operatorname{ind}(\mu I_X - A) =$$

$$= \sum_{\mu\in f^{-1}(\{\lambda\})\cap\sigma^\circ(A)} m_f(\mu) \operatorname{ind}(\mu I_X - A).$$

Lemma 2.3. *Let Δ be an open nonempty subset of $\mathbf{C}$, let $f\colon \Delta \to \mathbf{C}$ be a holomorphic function, let K be a compact nonempty subset of Δ and let V be an open subset of Δ, such that $\partial V \subset K$ and $f(V) \not\subset f(K)$. Then:*

i) *$f(V) \setminus f(K)$ is open and $f^{-1}(\{\lambda\}) \cap V \subset \Omega_f(\Delta)$ for any $\lambda \in \mathbf{C} \setminus f(K)$;*

ii) *for any $\lambda \in f(V \cap \Omega_f(\Delta)) \setminus f(K \cap \Omega_f(\Delta))$, there exists a component of $f(V) \setminus \setminus f(K)$ which contains a punctured neighborhood of λ.*

If, in addition, V is bounded, it follows that:

iii) *any component of $f(V) \setminus f(K)$ is also a component of $\boldsymbol{C} \setminus f(K)$ and $U \subset f(W)$ for any component U of $\boldsymbol{C} \setminus f(K)$ and for any component W of V such that $f(W) \cap U \neq \emptyset$;*

iv) *$f^{-1}(\{\lambda\}) \cap V \cap \Omega_f(\Delta)$ is finite for any $\lambda \in \boldsymbol{C}$, $f^{-1}(\{\lambda\}) \cap V$ is finite for any $\lambda \in \boldsymbol{C} \setminus f(K)$ and the function $N_f((\cdot), V)$: $\lambda \in \boldsymbol{C} \setminus f(K) \to N_f(\lambda, V) = \sum_{\mu \in f^{-1}(\{\lambda\}) \cap V} m_f(\mu) \in \boldsymbol{N}$ is constant on any component of $\boldsymbol{C} \setminus f(K)$;*

v) *if $z \in f(V \cap \Omega_f(\Delta)) \setminus f(K \cap \Omega_f(\Delta))$ and U_z is the component of $\boldsymbol{C} \setminus f(K)$ which contains a punctured neighborhood of z, we have that $N_f(\lambda, V) = \sum_{\mu \in f^{-1}(\{z\}) \cap V \cap \Omega_f(\Delta)} m_f(\mu)$ for any $\lambda \in U_z$.*

Proof. First of all, we prove i). Since any component of Δ is the intersection of Δ with a closed subset of $\boldsymbol{C}$ and $V \cup K$ is closed (as $\partial V \subset K$) and contained in Δ, the intersection of $V \cup K$ with any component of Δ is closed in $\boldsymbol{C}$. Since the intersection of any component of Δ, which does not intersect K, with $V \cup K$ is an open proper subset of $\boldsymbol{C}$, it follows that any component of Δ which intersects V intersects also K. Consequently, since $\Delta \setminus \Omega_f(\Delta)$ is the union of all components of Δ on which f is constant, $f(V \cap (\Delta \setminus \Omega_f(\Delta)) \subset f(K)$. Hence $f(V) \setminus f(K) = f(V \cap \Omega_f(\Delta)) \setminus f(K)$, so that, since f is an open map on $\Omega_f(\Delta)$ and K is compact, $f(V) \setminus f(K)$ is open. Moreover for any $\lambda \in \boldsymbol{C} \setminus f(K)$, $f^{-1}(\{\lambda\}) \cap V \subset \Omega_f(\Delta)$.

Now we prove ii). Let $\lambda \in f(V \cap \Omega_f(\Delta)) \setminus f(K \cap \Omega_f(\Delta))$. Since $\Omega_f(\Delta)$ is closed in the relative topology of Δ, $K \cap \Omega_f(\Delta)$ is compact and $\partial(V \cap \Omega_f(\Delta)) \subset K \cap \Omega_f(\Delta)$. Moreover, since obviously $V \cap \Omega_f(\Delta) \neq \emptyset$ and, as we proved above, any component of Δ which intersects V intersects also K, we have $K \cap \Omega_f(\Delta) \neq \emptyset$. Hence $f(V \cap \Omega_f(\Delta)) \setminus f(K \cap \Omega_f(\Delta))$ is open, by i). Therefore, if U denotes the component of $f(V \cap \Omega_f(\Delta)) \setminus f(K \cap \Omega_f(\Delta))$ which contains λ, it follows that U is open. Since K is compact, it intersects only a finite number of components of Δ. Consequently, since $\Delta \setminus \Omega_f(\Delta)$ is the union of all components of Δ on which f is constant, $f(K \cap (\Delta \setminus \Omega_f(\Delta)))$ is finite. If we set $U_0 = U \setminus (f(K \cap (\Delta \setminus \Omega_f(\Delta))) \cup \{\lambda\})$, it follows that U_0 is a connected punctured neighborhood of λ. Moreover, $U_0 \subset (f(V) \setminus f(K \cap \Omega_f(\Delta))) \cap (\boldsymbol{C} \setminus f(K \cap (\Delta \setminus \Omega_f(\Delta))) = f(V) \setminus (f(K \cap \Omega_f(\Delta)) \cup f(K \cap (\Delta \setminus \Omega_f(\Delta)))) = f(V) \setminus f(K)$. Hence there exists a component of $f(V) \setminus f(K)$ which contains U_0.

We have thus proved ii). Now we suppose V to be bounded.

If we remark that $f(V) \setminus f(K) = f(V \setminus K) \setminus f(K)$ and any component of $V \setminus K$ is a bounded component of $\boldsymbol{C} \setminus K$, the first assertion of iii) is an easy consequence of [BHOP], 3.1. However, we can give a very simple direct proof. Since V is bounded and $\partial V \subset K$, it follows that $V \cup K$ is compact. Consequently, $f(V) \setminus f(K) = (\boldsymbol{C} \setminus f(K)) \cap f(V \cup K)$ is closed in the relative topology of $\boldsymbol{C} \setminus f(K)$. Since $f(V) \setminus f(K)$ is open in $\boldsymbol{C}$, it follows immediately that any component of $f(V) \setminus f(K)$ is also a component of $\boldsymbol{C} \setminus f(K)$.

Since, by what we have just proved, any component of $f(W)\setminus f(K)$ is also a component of $\mathbf{C}\setminus f(K)$ for any component W of V such that $f(W)\not\subset f(K)$ (we point out that $\partial W\subset\partial V\subset K$), the second assertion of iii) can be easily proved in the same way of the first assertion of [BHOP], 3.5.

Since V is bounded and $\partial V\subset K$, so that $V\cup K$ is a compact subset of Δ, $f^{-1}(\{\lambda\})\cap\cap V\cap\Omega_f(\Delta)$ is finite for any $\lambda\in\mathbf{C}$. If, moreover, $\lambda\in\mathbf{C}\setminus f(K)$, it follows that $f^{-1}(\{\lambda\})\cap V$ is finite, as $f^{-1}(\{\lambda\})\cap V\subset\Omega_f(\Delta)$ by i). Hence the first two assertions of iv) are proved.

The third assertion of iv) is an easy consequence of the theory of topological degree and of the Cauchy-Riemann conditions. However, it has already been proved, in a more direct way, by the authors of [BHOP], in the particular case of V being a hole of $\sigma_e(T)$, contained in $\sigma(T)$ (where T is a linear continuous operator on a separable Hilbert space), $K=\sigma_e(T)$ and f being a nonconstant complex-valued function, holomorphic on a connected open neighborhood of $\sigma(T)$ (see [BHOP], 3.5). The proof of [BHOP], 3.5 can be repeated in this case.

Finally, we prove v). Let $z\in f(V\cap\Omega_f(\Delta))\setminus f(K\cap\Omega_f(\Delta))$. Then, by ii), there exists a component U_z of $\mathbf{C}\setminus f(K)$ which contains a punctured neighborhood of z. Moreover, $U_z\cup\{z\}$ is contained in a component of $f(V\cap\Omega_f(\Delta))\setminus f(K\cap\Omega_f(\Delta))$. Hence, by iv), $N_f(\lambda, V\cap\Omega_f(\Delta))=N_f(z, V\cap\Omega_f(\Delta))=\sum_{\mu\in f^{-1}(\{z\})\cap V\cap\Omega_f(\Delta)} m_f(\mu)$ for any $\lambda\in U_z$. Since $U_z\subset\mathbf{C}\setminus f(K)$, and consequently $f^{-1}(U_z)\cap V\subset\Omega_f(\Delta)$ by i), it follows that $N_f(\lambda, V)=N_f(\lambda, V\cap\Omega_f(\Delta))=\sum_{\mu\in f^{-1}(\{z\})\cap V\cap\Omega_f(\Delta)} m_f(\mu)$ for any $\lambda\in U_z$.

Let K be a nonempty compact subset of $\mathbf{C}$, let V be an open subset of $\mathbf{C}$, with $\partial V\subset K$, and let f be a complex-valued function, holomorphic on an open neighborhood Δ of $V\cup K$. Since $f(V)\setminus f(K)=f(V\cap\Omega_f(\Delta))\setminus f(K)$ and $f(V)\setminus f(K)$ is open, it is easy to verify that $f(V)\setminus f(K)\neq\emptyset$ if and only if $f(V\cap\Omega_f(\Delta))\setminus\setminus f(K\cap\Omega_f(\Delta))\neq\emptyset$.

Suppose V to be bounded. Then, by Lemma 2.3, iv), $N_f((\cdot), V)$ is constant on U for any nonempty set U which is contained in a component of $\mathbf{C}\setminus f(K)$. We shall denote the value of $N_f((\cdot), V)$ on U by $N_f(U, V)$. Obviously, if U_1 and U_2 are nonempty subsets of the same component of $\mathbf{C}\setminus f(K)$, it follows that $N_f(U_1, V)==N_f(U_2, V)$.

Let X be a complex nonzero Banach space, let $A\in L_c(X)$, let V be a component of $\varrho_F(A)$, contained in $\sigma(A)$ and let f be a complex-valued function, holomorphic on an open neighborhood of $\sigma(A)$ and such that $f(V)\not\subset f(\sigma_e(A))$. Then, since $\partial V\subset\sigma_e(A)$, if we put $K=\sigma_e(A)$ all the hypotheses of Lemma 2.3 (including the boundedness of V) are satisfied. Hence, since any component of $\varrho_F(A)$ which intersects $\sigma^\circ(A)$ is contained in $\sigma(A)$ (see [K], IV, 5.6 and 5.31), the following result is an immediate consequence of Theorem 2.2 and Lemma 2.3. It has already been proved by the authors of [BHOP] in the particular case of X being a separable

Hilbert space and Δ being connected (see [BHOP], 3.7). The proof of [BHOP], 3.7 can be repeated without changes in the general case.

Theorem 2.4. *Let X be a complex nonzero Banach space, let $A \in L_c(X)$, let f be a complex-valued function, holomorphic on an open neighborhood of $\sigma(A)$ and let U be a component of $\varrho_F(f(A)) = \mathbf{C} \setminus f(\sigma_e(A))$. Then, if $C(U)$ denotes the family of all components V of $\varrho_F(A)$ such that $V \subset \sigma(A)$ and $f(V) \cap U \neq \emptyset$, it follows that $C(U)$ is finite and $i_{f(A)}(U) = \sum_{V \in C(U)} N_f(U, V)\, i_A(V)$.*

Theorem 2.5. *Let X be a complex nonzero Banach space, let $A \in \Sigma_0(X)$ and let f be a complex-valued function, holomorphic on an open neighborhood Δ of $\sigma(A)$. Then the following conditions are equivalent:*

i) $f(A) \in \Sigma_0(X)$;

ii) $f(\varrho_{s-F}^{+\infty}(A)) \cap f(\varrho_{s-F}^{-\infty}(A))$ *has empty interior and, for any component U of $\mathbf{C} \setminus f(\sigma_e(A))$ such that $U \subset f(\sigma(A))$,* $\sum_{V \in C(U)} N_f(U, V)\, i_A(V) \neq 0$;

iii) *for any* $\lambda \in f(\varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta))$, $\{-\infty, +\infty\} \not\subset \{\operatorname{ind}(\mu I_X - A) : \mu \in f^{-1}(\{\lambda\}) \cap \varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta)\}$ *and, if* $\lambda \notin f(\partial \varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta))$,
$$\sum_{\mu \in f^{-1}(\{\lambda\}) \cap \varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta)} m_f(\mu) \operatorname{ind}(\mu I_X - A) \neq 0;$$

iv) $\{\lambda \in f(\varrho_{s-F}^{\pm}(A)) \setminus f(Z_f(A)) : f^{-1}(\{\lambda\}) \cap \sigma(A) \subset \varrho_{s-F}^{\pm}(A)$ *and, if* $\lambda \notin f(\varrho_{s-F}^{+\infty}(A)) \cap f(\varrho_{s-F}^{-\infty}(A))$, $\sum_{\mu \in f^{-1}(\{\lambda\}) \cap \sigma(A)} \operatorname{ind}(\mu I_X - A) = 0\}$ *has empty interior;*

v) $(f(\varrho_{s-F}^{+\infty}(A)) \cap f(\varrho_{s-F}^{-\infty}(A))) \setminus f(Z_f(A) \cup \partial \varrho_{s-F}^{\pm}(A) \cup \operatorname{cl} \psi(A))$ *has empty interior and, for any component U of $\mathbf{C} \setminus f(\sigma_e(A))$, contained in $f(\sigma(A))$, there exists $\lambda \in U$ such that* $\sum_{\mu \in f^{-1}(\{\lambda\}) \cap \sigma^\circ(A)} m_f(\mu) \operatorname{ind}(\mu I_X - A) \neq 0$.

Proof. First of all, we prove that $f^\circ(Z_f(A) \cup \partial \varrho_{s-F}^{\pm}(A) \cup \operatorname{cl} \psi(A)) = \emptyset$. Since the boundary of any open set has empty interior and $\varrho_{s-F}^{\pm}(A)$ is an open subset of $\mathbf{C}$, $\partial^\circ \varrho_{s-F}^{\pm}(A) = \emptyset$. Besides, since $\mathbf{C}$ is a locally connected space, it follows that $\sigma^\circ(A) \cap \operatorname{cl} \psi(A) = \emptyset$, and so $(\operatorname{cl} \psi(A))^\circ = \emptyset$. Consequently, since both $\partial \varrho_{s-F}^{\pm}(A)$ and $\operatorname{cl} \psi(A)$ are closed and the union of a finite number of closed sets with empty interior has empty interior, $\partial \varrho_{s-F}^{\pm}(A) \cup \operatorname{cl} \psi(A)$ has empty interior, too. Since $(\partial \varrho_{s-F}^{\pm}(A) \cup \operatorname{cl} \psi(A)) \setminus I_f \subset \sigma(A) \setminus I_f \subset Z_f(A)$ and $f(Z_f(A))$ at most consists of a finite number of points, it follows that $f^\circ((\partial \varrho_{s-F}^{\pm}(A) \cup \operatorname{cl} \psi(A)) \setminus I_f) = \emptyset$. Hence, since obviously $\mathbf{C}$ is a Hausdorff space and $\partial \varrho_{s-F}^{\pm}(A) \cup \operatorname{cl} \psi(A)$ is compact, from Lemma 1.3 it follows that $f^\circ(\partial \varrho_{s-F}^{\pm}(A) \cup \operatorname{cl} \psi(A)) = \emptyset$. Consequently, since $f(Z_f(A))$ is finite, $f^\circ(Z_f(A) \cup \partial \varrho_{s-F}^{\pm}(A) \cup \operatorname{cl} \psi(A)) = \emptyset$.

Now we prove that i) implies ii). Suppose that there exists an open nonempty subset G of the complex plane such that $G \subset f(\varrho_{s-F}^{+\infty}(A)) \cap f(\varrho_{s-F}^{-\infty}(A))$. Since $f(\varrho_{s-F}^{+\infty}(A)) \cap f(\varrho_{s-F}^{-\infty}(A)) \subset \sigma_{s-F}(f(A))$ by Theorem 2.2, it follows that $G \cap \varrho_{s-F}^{\pm}(f(A)) = \emptyset$. Morevoer, $G \subset \sigma^\circ(f(A))$, so that, since $\mathbf{C}$ is locally connected, $G \cap \psi(f(A)) = \emptyset$. Therefore $\varrho_{s-F}^{\pm}(f(A)) \cup \psi(f(A))$ is not dense in $\sigma(f(A))$, and consequently $f(A) \notin \Sigma_0(X)$.

Suppose now that there exists a component U of $C \setminus f(\sigma_e(A)) = \varrho_F(f(A))$, contained in $f(\sigma(A)) = \sigma(f(A))$, such that $\sum_{V \in C(U)} N_f(U, V)\, i_A(V) = 0$. From Theorem 2.4 it follows that $i_{f(A)}(U) = 0$. Therefore $U \subset \sigma^\circ(f(A)) \cap \overset{0}{\varrho}_{s-F}(f(A))$, and consequently $U \cap (\psi(f(A)) \cup \varrho_{s-F}^{\pm}(f(A))) = \emptyset$. Hence $\psi(f(A)) \cup \varrho_{s-F}^{\pm}(f(A))$ is not dense in $\sigma(f(A))$, so that $f(A) \notin \Sigma_0(X)$.

We have thus proved that i) implies ii). We prove that ii) implies iii).

Suppose that there exists $\lambda_0 \in f(\varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta))$ such that $\{-\infty, +\infty\} \subset \subset \{\operatorname{ind}(\mu I_X - A) \colon \mu \in f^{-1}(\{\lambda_0\}) \cap \varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta)\}$. Let $\mu_1 \in f^{-1}(\{\lambda_0\}) \cap \varrho_{s-F}^{+\infty}(A) \cap \cap\, \Omega_f(\Delta)$ and $\mu_2 \in f^{-1}(\{\lambda_0\}) \cap \varrho_{s-F}^{-\infty}(A) \cap \Omega_f(\Delta)$. Since $\varrho_{s-F}^{+\infty}(A)$ and $\varrho_{s-F}^{-\infty}(A)$ are open, there exist a neighborhood U_1 of μ_1 and a neighborhood U_2 of μ_2 such that $U_1 \subset \subset \varrho_{s-F}^{+\infty}(A)$ and $U_2 \subset \varrho_{s-F}^{-\infty}(A)$. Moreover, since μ_1 and μ_2 belong to $\Omega_f(\Delta)$ and f is an open map on $\Omega_f(\Delta)$, it follows that there exists an open neighborhood U of λ_0 such that $U \subset f(U_1) \cap f(U_2) \subset f(\varrho_{s-F}^{+\infty}(A)) \cap f(\varrho_{s-F}^{-\infty}(A))$. Hence $f(\varrho_{s-F}^{+\infty}(A)) \cap \cap\, f(\varrho_{s-F}^{-\infty}(A))$ has nonempty interior.

Suppose now that there exists $\lambda_1 \in f(\varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta)) \setminus f(\partial\varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta))$ such that $\{-\infty, +\infty\} \not\subset \{\operatorname{ind}(\mu I_X - A) \colon \mu \in f^{-1}(\{\lambda_1\}) \cap \Omega_f(\Delta) \cap \varrho_{s-F}^{\pm}(A)\}$ and $\sum_{\mu \in f^{-1}(\{\lambda_1\}) \cap \varrho_{s-F}(A) \cap \Omega_f(\Delta)} m_f(\mu) \operatorname{ind}(\mu I_X - A) = 0$. Then $\{\operatorname{ind}(\mu I_X - A) \colon \mu \in \in f^{-1}(\{\lambda_1\}) \cap \Omega_f(\Delta) \cap \varrho_{s-F}^{\pm}(A)\} \subset Z$, $f(\varrho_{s-F}^{\pm}(A)) \setminus f(\partial\varrho_{s-F}^{\pm}(A)) \neq \emptyset$ and, by Lemma 2.3, there exists a component U_{λ_1} of $f(\varrho_{s-F}^{\pm}(A)) \setminus f(\partial\varrho_{s-F}^{\pm}(A))$ (which is also a component of $C \setminus f(\partial\varrho_{s-F}^{\pm}(A))$ as $\varrho_{s-F}^{\pm}(A)$ is bounded) that contains a punctured neighborhood of λ_1. We point out that $U_{\lambda_1} \subset f(\sigma(A)) = \sigma(f(A))$. Let χ denote the family of all components V of $\varrho_{s-F}^{\pm}(A)$ such that $f(V) \cap U_{\lambda_1} \neq \emptyset$. From Lemma 2.3 it follows that, for any $V \in \chi$, $U_{\lambda_1} \subset f(V)$ (so that $V \subset \Omega_f(\Delta)$) and, since $\partial V \subset \partial\varrho_{s-F}^{\pm}(A)$, $N_f(U_{\lambda_1}, V) = \sum_{\mu \in f^{-1}(\{\lambda_1\}) \cap V} m_f(\mu)$. Morevoer, $\{\lambda_1\} \cup U_{\lambda_1}$ is contained in a component of $C \setminus f(\partial\varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta))$, so that χ coincides with the family of all components V of $\varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta)$ such that $\lambda_1 \in f(V)$. Consequently, $f^{-1}(\{\lambda_1\}) \cap \varrho_{s-F}^{\pm}(A) \cap \cap\, \Omega_f(\Delta) = f^{-1}(\{\lambda_1\}) \cap (\bigcup_{V \in \chi} V)$ and $\{i_A(V) \colon V \in \chi\} = \{\operatorname{ind}(\mu I_X - A) \colon \mu \in f^{-1}(\{\lambda_1\}) \cap \cap\, \varrho_{s-F}^{\pm}(A) \cap \Omega_f(\Delta)\} \subset Z$ (so that $f^{-1}(U_{\lambda_1}) \cap \varrho_{s-F}^{\pm}(A) \subset \bigcup_{V \in \chi} V \subset \varrho_F(A)$). Since $A \in \in \Sigma_0(X)$, so that $\sigma(A) = \operatorname{cl}(\varrho_{s-F}^{\pm}(A)) \cup \operatorname{cl}\psi(A)$, and $U_{\lambda_1} \subset C \setminus f(\partial\varrho_{s-F}^{\pm}(A))$, it follows that $f^{-1}(U_{\lambda_1}) \cap \sigma_e(A) \subset \operatorname{cl}\psi(A)$. Hence $U_{\lambda_1} \setminus f(\sigma_e(A)) \supset U_{\lambda_1} \setminus f(\operatorname{cl}\psi(A))$, so that, since $f^\circ(\operatorname{cl}\psi(A)) = \emptyset$ by what we have proved above and $\partial\varrho_{s-F}^{\pm}(A) \subset \sigma_e(A)$ (which implies $C \setminus f(\sigma_e(A)) \subset C \setminus f(\partial\varrho_{s-F}^{\pm}(A))$), there exists a component U of $C \setminus f(\sigma_e(A))$ such that $U \subset U_{\lambda_1} \subset f(\sigma(A))$.

Since $U_{\lambda_1} \subset f(V)$ and $V \subset \varrho_F(A)$ for any $V \in \chi$, it follows that $\chi \subset C(U)$. Since $\sigma^\circ(A) \cap \operatorname{cl}\psi(A) = \emptyset$ and $A \in \Sigma_0(X)$, it follows that $\sigma^\circ(A) \cap \varrho_F(A) \subset \varrho_{s-F}^{\pm}(A)$. Consequently, $V \subset \varrho_{s-F}^{\pm}(A)$ for any $V \in C(U)$, so that $C(U) \subset \chi$.

We have thus proved that $C(U) = \chi$.

In addition

$$\sum_{V \in C(U)} N_f(U, V)\, i_A(V) = \sum_{V \in \chi} N_f(U_{\lambda_1}, V)\, i_A(V) =$$

$$= \sum_{V \in \chi} \Big(\sum_{\mu \in f^{-1}(\{\lambda_1\}) \cap V} m_f(\mu) \Big) \, i_A(V) = \sum_{V \in \chi} \sum_{\mu \in f^{-1}(\{\lambda_1\}) \cap V} m_f(\mu) \operatorname{ind}(\mu I_X - A) =$$

$$= \sum_{\mu \in f^{-1}(\{\lambda_1\}) \cap (\bigcup_{V \in \chi} V)} m_f(\mu) \operatorname{ind}(\mu I_X - A) =$$

$$= \sum_{\mu \in f^{-1}(\{\lambda_1\}) \cap \varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta)} m_f(\mu) \operatorname{ind}(\mu I_X - A) = 0 \, .$$

Hence condition ii) is not satisfied.

We have thus proved that ii) implies iii).

We prove that iii) implies iv). Suppose that condition iii) is satisfied. Then, since, for any $\lambda \in \boldsymbol{C} \setminus f(Z_f(A)), f^{-1}(\{\lambda\}) \cap \sigma(A) \subset \Omega_f(\Delta)$ and $m_f(\mu) = 1$ for any $\mu \in f^{-1}(\{\lambda\}) \cap \sigma(A)$, $\{\lambda \in f(\varrho^{\pm}_{s-F}(A)) \setminus f(Z_f(A)) : f^{-1}(\{\lambda\}) \cap \sigma(A) \subset \varrho^{\pm}_{s-F}(A)$ and, if $\lambda \notin f(\varrho^{+\infty}_{s-F}(A)) \cap f(\varrho^{-\infty}_{s-F}(A))$, $\sum_{\mu \in f^{-1}(\{\lambda\}) \cap \sigma(A)} \operatorname{ind}(\mu I_X - A) = 0\} \subset \{\lambda \in f(\varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta)) \setminus f(\partial \varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta)) : \sum_{\mu \in f^{-1}(\{\lambda\}) \cap \varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta)} m_f(\mu) \operatorname{ind}(\mu I_X - A) = 0$ if $\{-\infty, +\infty\} \not\subset \{\operatorname{ind}(\mu I_X - A) : \mu \in f^{-1}(\{\lambda\}) \cap \varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta)\}\} = \emptyset$, so that also condition iv) is satisfied.

We prove that iv) implies v). Since $A \in \Sigma_0(X)$, so that $\sigma(A) = \operatorname{cl} \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A)$, it follows that $(f(\varrho^{+\infty}_{s-F}(A)) \cap f(\varrho^{-\infty}_{s-F}(A))) \setminus f(Z_f(A) \cup \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A)) \subset \{\lambda \in f(\varrho^{\pm}_{s-F}(A)) : f^{-1}(\{\lambda\}) \cap \sigma(A) \subset \varrho^{\pm}_{s-F}(A)\}$. Therefore, if condition iv) is satisfied it follows that $(f(\varrho^{+\infty}_{s-F}(A)) \cap f(\varrho^{-\infty}_{s-F}(A))) \setminus f(Z_f(A) \cup \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A))$ has empty interior.

Suppose now that there exists a component U of $\boldsymbol{C} \setminus f(\sigma_e(A))$, contained in $f(\sigma(A))$, such that $\sum_{\mu \in f^{-1}(\{\lambda\}) \cap \sigma^{\circ}(A)} m_f(\mu) \operatorname{ind}(\mu I_X - A) = 0$ for any $\lambda \in U$. We recall that $f^{\circ}(Z_f(A) \cup \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A)) = \emptyset$. Consequently, $U \setminus f(Z_f(A) \cup \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A))$ is open and nonempty. Moreover, $f^{-1}(U \setminus f(Z_f(A) \cup \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A))) \cap \sigma(A) \subset \varrho^{\pm}_{s-F}(A)$, so that $f^{-1}(U \setminus f(Z_f(A) \cup \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A))) \cap \sigma(A) = f^{-1}(U \setminus f(Z_f(A) \cup \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A))) \cap \sigma^{\circ}(A)$ and $U \setminus f(Z_f(A) \cup \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A)) \subset f(\varrho^{\pm}_{s-F}(A))$. Since $m_f(\mu) = 1$ for any $\mu \in \sigma(A) \setminus f(Z_f(A))$, it follows that $U \setminus f(Z_f(A) \cup \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A)) \subset \{\lambda \in f(\varrho^{\pm}_{s-F}(A)) \setminus f(Z_f(A)) : f^{-1}(\{\lambda\}) \cap \sigma(A) \subset \varrho^{\pm}_{s-F}(A)$ and, if $\lambda \notin f(\varrho^{+\infty}_{s-F}(A)) \cap f(\varrho^{-\infty}_{s-F}(A))$, $\sum_{\mu \in f^{-1}(\{\lambda\}) \cap \sigma(A)} \operatorname{ind}(\mu I_X - A) = 0\}$, which consequently has nonempty interior. Hence condition iv) is not satisfied.

We have thus proved that iv) implies v).

Finally, we prove that v) implies i). Suppose that condition v) is satisfied. Since $A \in \Sigma_0(X)$, it follows that $\sigma_{s-F}(A) \subset \partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A)$. Consequently, since $f^{\circ}(\partial \varrho^{\pm}_{s-F}(A) \cup \operatorname{cl} \psi(A)) = \emptyset$, $f^{\circ}(\sigma_{s-F}(A)) = \emptyset$. Therefore, since

$$f(\varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta)) \text{ is open } \operatorname{cl}(f(\varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta)) \setminus f(\sigma_{s-F}(A))) =$$
$$= \operatorname{cl}(f(\varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta))) = \operatorname{cl}(f(\varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta))) = f(\operatorname{cl}(\varrho^{\pm}_{s-F}(A)) \cap \Omega_f(\Delta)) \, .$$

Moreover, since $\partial \varrho^{\pm}_{s-F}(A) \subset \sigma_{s-F}(A)$, $f(\varrho^{\pm}_{s-F}(A)) \setminus f(\sigma_{s-F}(A)) = f(\varrho^{\pm}_{s-F}(A) \cap \Omega_f(\Delta)) \setminus f(\sigma_{s-F}(A))$ (see Lemma 2.3).

It follows that $\sigma(f(A)) = f(\sigma(A)) = (f(\text{cl}\,(\psi(A))) \setminus f(\text{cl}^{\pm}_{s-F}(A) \cap \Omega_f(\Delta))) \cup \cup\, \text{cl}((f(\varrho^{\pm}_{s-F}(A)) \setminus f(\sigma_{s-F}(A)))$.

It can be proved that $f(\text{cl}\,\psi(A)) \setminus f(\text{cl}\,(\varrho^{\pm}_{s-F}(A))) \cap \Omega_f(\Delta)) \subset \text{cl}\,\psi(f(A))$ (since $f(\text{cl}\,\varrho^{\pm}_{s-F}(A)) \setminus \Omega_f(\Delta)) \subset f(Z_f(A))$, the proof is analogous to the proof of [B1], 2.13).

We prove that $f(\varrho^{\pm}_{s-F}(A)) \setminus f(\sigma_{s-F}(A)) \subset \varrho^{\pm}_{s-F}(f(A))$. Let U be a component of $f(\varrho^{\pm}_{s-F}(A)) \setminus f(\sigma_{s-F}(A))$. Then $U \subset f(\sigma(A))$ and, by Lemma 2.3, U is a component of $\boldsymbol{C} \setminus f(\sigma_{s-F}(A))$ and $f^{-1}(U) \cap \varrho^{\pm}_{s-F}(A) \subset \Omega_f(\Delta)$. It follows that $f(\varrho^{+\infty}_{s-F}(A)) \cap \cap f(\varrho^{-\infty}_{s-F}(A)) \cap U$ is open, and therefore, since $(f(\varrho^{+\infty}_{s-F}(A)) \cap f(\varrho^{-\infty}_{s-F}(A))) \setminus f(Z_f(A) \cup \cup\, \partial\varrho^{\pm}_{s-F}(A) \cup \text{cl}\,\psi(A))$ and $f(Z_f(A) \cup \partial\varrho^{\pm}_{s-F}(A) \cup \text{cl}\,\psi(A))$ have empty interior and $f(Z_f(A) \cup \partial\varrho^{\pm}_{s-F}(A) \cup \text{cl}\,\psi(A))$ is closed, $f(\varrho^{+\infty}_{s-F}(A)) \cap f(\varrho^{-\infty}_{s-F}(A)) \cap U = \emptyset$.

Hence U is a component of $\boldsymbol{C} \setminus (f(\sigma_{s-F}(A)) \cup (f(\varrho^{+\infty}_{s-F}(A)) \cap f(\varrho^{-\infty}_{s-F}(A)))) = = \varrho_{s-F}(f(A))$ (see Theorem 2.2).

If $i_{f(A)}(U) \in \{-\infty, +\infty\}$, obviously $U \subset \varrho^{\pm}_{s-F}(f(A))$. If, instead, $i_{f(A)}(U) \in \boldsymbol{Z}$, it follows that U is a component of $\varrho_F(f(A)) = \boldsymbol{C} \setminus f(\sigma_e(A))$. Since $U \subset f(\sigma(A))$ and condition v) is satisfied, there exists $\lambda \in U$ such that $\sum_{\mu \in f^{-1}(\{\lambda\}) \cap \sigma^\circ(A)} m_f(\mu)\, \text{ind}\,(\mu I_X - - A) \neq 0$. Hence, by Theorem 2.2, $i_{f(A)}(U) = \text{ind}\,(\lambda I_X - f(A)) = = \sum_{\mu \in f^{-1}(\{\lambda\}) \cap \sigma^\circ(A)} m_f(\mu)\, \text{ind}\,(\mu I_X - A) \neq 0$, so that $U \subset \varrho^{\pm}_{s-F}(f(A))$.

We have thus proved that $f(\varrho^{\pm}_{s-F}(A|) \setminus f(\sigma_{s-F}(A)) \subset \varrho^{\pm}_{s-F}(f(A))$. It follows immediately that $\sigma(f(A)) = \text{cl}\,\psi(f(A)) \cup \text{cl}\,\varrho^{\pm}_{s-F}(f(A))$, and therefore $f(A) \in \Sigma_0(X)$.

The proof is now complete.

Corollary 2.6. *Let X be a complex nonzero Banach space, let $A \in \Sigma_0(X)$ and let f be a complex-valued function, holomorphic in a neighborhood of $\sigma(A)$ and one-to-one on $\sigma(A) \setminus Z_f(A)$. Then $f(A) \in \Sigma_0(X)$.*

Proof. Since f is one-to-one on $\sigma(A) \setminus Z_f(A)$, $f^{-1}(\{\lambda\}) \cap \sigma(A)$ consists of a single point $\mu_\lambda \in \varrho^{\pm}_{s-F}(A)$ for any $\lambda \in f(\varrho^{\pm}_{s-F}(A)) \setminus f(Z_f(A))$, so that, for any $\lambda \in f(\varrho^{\pm}_{s-F}(A)) \setminus \setminus f(Z_f(A))$, $\{-\infty, +\infty\} \not\subset \{\text{ind}\,(\mu I_X - A) \colon \mu \in f^{-1}(\{\lambda\})\} \subset \{0, \text{ind}\,(\mu_\lambda I_X - A)\}$ and

$$\sum_{\mu \in f^{-1}(\{\lambda\}) \cap \sigma(A)} \text{ind}\,(\mu I_X - A) = \text{ind}\,(\mu_\lambda I_X - A) \neq 0\,.$$

Therefore, in particular, condition iv) of Theorem 2.5 is satisfied, and so $f(A) \in \Sigma_0(X)$.

The following result is an immediate consequence of Corollary 2.6.

Corollary 2.7. *Let X be a complex nonzero Banach space and let $A \in \Sigma_0(X)$. Then $\lambda I_X + \mu A \in \Sigma_0(X)$ for any $\lambda, \mu \in \boldsymbol{C}$.*

We have proved indeed that $\Sigma_0(X)$ is closed with respect to polynomials of first degree. Nevertheless, generally speaking, $\Sigma_0(X)$ is not closed with respect to polynomials, as we shall show afterwards.

Corollary 2.8. *Let X be a complex nonzero Banach space, let $p \geqq 2$ be an integer and let $A \in \Sigma_0(X)$ be such that $\sigma(A) \cap (\bigcup_{k=1}^{p-1} e^{2k\pi i/p}\sigma(A)) \subset \{0\}$. Then $A^p \in \Sigma_0(X)$.*

Proof. Let f_p denote the function defined by $f_p(\lambda) = \lambda^p$ for any $\lambda \in \boldsymbol{C}$. Obviously f_p is holomorphic and, as p is positive, $Z_{f_p}(A) = \sigma(A) \cap \{0\}$. For any $\varrho > 0$ and for any $\phi \in [0, 2\pi)$, $f_p^{-1}(\{\varrho e^{i\phi}\}) = \{\varrho^{1/p} e^{i(\phi/p + 2k\pi/p)}: k = 0, \ldots, p-1\}$. For any $j \in \{0, \ldots, p-1\}$ such that $\varrho^{1/p} e^{i(\phi/p + 2j\pi/p)} \in \sigma(A)$, it follows that $(f_p^{-1}(\{\varrho e^{i\phi}\}) \setminus \{\varrho^{1/p} e^{i(\phi/p + 2\pi j/p)}\}) \cap \sigma(A) = \sigma(A) \cap (\bigcup_{k=1}^{p-1} e^{2k\pi i/p} \{\varrho^{1/p} e^{i(\phi/p + 2j\pi/p)}\}) \subset$ $\subset (\sigma(A) \setminus \{0\}) \cap (\bigcup_{k=1}^{p-1} e^{2k\pi i/p} \sigma(A)) = \emptyset$. Therefore, for any $\lambda \in \boldsymbol{C} \setminus \{0\}$, $f_p^{-1}(\{\lambda\}) \cap$ $\cap\, \sigma(A)$ consists at most of a single point. Since $f_p(\sigma(A) \setminus Z_f(A)) = f_p(\sigma(A) \setminus$ $\setminus (\{0\} \cap \sigma(A))) = f_p \sigma(A) \setminus \{0\}) \subset \boldsymbol{C} \setminus \{0\}$, we have thus proved that f_p is one-to-one on $\sigma(A) \setminus Z_{f_p}(A)$. From Corollary 2.6 it follows that $A^p = f_p(A) \in \Sigma_0(X)$.

Theorem 2.9. *Let X be a complex infinite-dimensional Hilbert space, let Δ be an open nonempty subset of $\boldsymbol{C}$ nad let $f: \Delta \to \boldsymbol{C}$ be an holomorphic function. Then the following conditions are equivalent:*

i) *$f(A) \in \Sigma_0(X)$ for any $A \in \Sigma_0(X)$ such that $\sigma(A) \subset \Delta$;*

ii) *f is one-to-one on $\Omega_f(\Delta)$;*

iii) *f is one-to-one on $(f')^{-1}(\boldsymbol{C} \setminus \{0\})$.*

Proof. First of all, we prove that i) implies ii). We set $\Omega = \Omega_f(\Delta)$.

Suppose that f is not one-to-one on Ω. Consequently, there exist $\lambda_1, \lambda_2 \in \Omega$ such that $\lambda_1 \neq \lambda_2$ and $f(\lambda_1) = f(\lambda_2)$. Since $\lambda_1, \lambda_2 \in \Omega$, from [R], 10.18 it follows that there exists $r > 0$ such that $B_{\boldsymbol{C}}(\lambda_1, r) \cap B_{\boldsymbol{C}}(\lambda_2, r) = \emptyset$ and, for any $j = 1, 2$, $\mathrm{cl}\, B_{\boldsymbol{C}}(\lambda_j, r) \subset \Omega$ and $f(\lambda) \neq f(\lambda_j)$ for any $\lambda \in \mathrm{cl}\, B_{\boldsymbol{C}}(\lambda_j, r) \setminus \{\lambda_j\}$.

We define $m_j = m_f(\lambda_j)$ for any $j = 1, 2$. Since X is infinite-dimensional, it contains two closed, separable and infinite-dimensional subspaces X_1 and X_2 such that $X_1 \perp X_2$. Hence, if we define $X_0 = (X_1 \oplus X_2)^{\perp}$ and, for any $j = 0, 1, 2$, P_j denotes the orthogonal projection of X onto X_j, it obviously follows that $P_j P_k = 0$ if $k \neq j$ and $\sum_{j=0}^{2} P_j = I_X$. Since $\bigcup_{j=1}^{2} \mathrm{cl}\, B_{\boldsymbol{C}}(\lambda_j, r) \subset \Omega$ and Ω is open, $\Omega \setminus (\bigcup_{j=1}^{2} \mathrm{cl}\, B_{\boldsymbol{C}}(\lambda_j, r)) \neq \emptyset$. Since f is not constant on any component of Ω, there exists $\lambda_0 \in \Omega \setminus (\bigcup_{j=1}^{2} \mathrm{cl}\, B_{\boldsymbol{C}}(\lambda_j, r))$ such that $f(\lambda_0) \neq f(\lambda_1) = f(\lambda_2)$.

For any $j = 1, 2$ let $S_j \in L_c(X_j)$ be an unilateral shift. Then, for any $j = 1, 2$, it results that $\sigma(S_j) = \sigma(S_j^{(*)}) = \mathrm{cl}\, B_{\boldsymbol{C}}(0, 1)$ see [H], Sol. 67), $\varrho_{s-F}^{\pm}(S_j^{(*)}) = B_{\boldsymbol{C}}(0, 1)$ and $\mathrm{ind}\,(\lambda I_{X_j} - S_j^{(*)}) = 1$ for $\lambda \in B_{\boldsymbol{C}}(0, 1)$ (see [K], IV, 5.24). Besides, for any positive integer p, considering that, for any $\varrho \geqq 0$ and for any $\theta \in [0, 2\pi)$, $\varrho e^{i\theta} I_{X_j} -$ $- (S_j^{(*)})^p = (-1)^{p-1} \prod_{k=0}^{p-1} (\varrho^{1/p} e^{i(\theta/p + 2k\pi/p)} I_{X_j} - S_j^{(*)})$, from [CPY], (3.2.7) it follows that $B_{\boldsymbol{C}}(0, 1) \subset \varrho_{s-F}^{\pm}((S_j^{(*)})^p)$ and $\mathrm{ind}\,(\lambda I_{X_j} - (S_j^{(*)})^p)) = p$ for any $\lambda \in B_{\boldsymbol{C}}(0, 1)$. Consequently, $B_{\boldsymbol{C}}(0, 1) \subset \varrho_{s-F}^{\pm}(S_j^p)$ and $\mathrm{ind}\,(\lambda I_{X_j} - S_j^p) = -p$ for any $\lambda \in B_{\boldsymbol{C}}(0, 1)$. (see [K], IV, 5.14). Let us consider the operator $A \in L_c(X)$ defined in the following way: $A = \lambda_0 P_0 + (\lambda_1 I_{X_1} + r S_1^{m_2}) P_1 + (\lambda_2 I_{X_2} + r (S_2^{(*)})^{m_1}) P_2$. It is not difficult to verify that $A^{(*)} = \bar{\lambda}_0 P_0 + (\bar{\lambda}_1 I_{X_1} + r (S_1^{(*)})^{m_2}) P_1 + (\bar{\lambda}_2 I_{X_2} + r S_2^{m_1}) P_2$. Since

$\sigma(\lambda_1 I_{X_1} + rS_1^{m_2}) = \operatorname{cl} B_{\mathbf{C}}(\lambda_1, r)$ and $\sigma(\lambda_2 I_{X_2} + r(S_2^{(*)})^{m_1}) = \operatorname{cl} B_{\mathbf{C}}(\lambda_2, r)$, from [TL], V, 5.4 it follows that $\sigma(A) = \sigma(\lambda_0 I_{X_0}) \cup (\bigcup_{j=1}^{2} \operatorname{cl} B_{\mathbf{C}}(\lambda_j, r))$. Then $\bigcup_{j=1}^{2} \operatorname{cl} B_{\mathbf{C}}(\lambda_j, r) \subset$ $\subset \sigma(A) \subset \{\lambda_0\} \cup (\bigcup_{j=1}^{2} \operatorname{cl} B_{\mathbf{C}}(\lambda_j, r))$. Since $\lambda_0 \notin \bigcup_{j=1}^{2} \operatorname{cl} B_{\mathbf{C}}(\lambda_j, r)$, it follows that $\{\lambda_0\} \cap$ $\cap\, \sigma(A) \subset \psi(A)$.

Since $B_{\mathbf{C}}(\lambda_1, r) \cap \sigma(\lambda_2 I_{X_2} + r(S_2^{(*)})^{m_1}) = B_{\mathbf{C}}(\lambda_1, r) \cap \operatorname{cl} B_{\mathbf{C}}(\lambda_2, r) = \emptyset =$ $= \operatorname{cl} B_{\mathbf{C}}(\lambda_1, r) \cap B_{\mathbf{C}}(\lambda_2, r) = \sigma(\lambda_1 I_{X_1} + rS_1^{m_2}) \cap B_{\mathbf{C}}(\lambda_2, r)$ and $\lambda_0 \notin \bigcup_{j=1}^{2} B_{\mathbf{C}}(\lambda_j, r)$, from [TL], V, 5.2 it follows that $\operatorname{Ker}(\lambda I_X - A) = \operatorname{Ker}(((\lambda - \lambda_1)/r) I_{X_1} - S_1^{m_2})$, $\operatorname{Ker}(\bar{\lambda} I_X - A^{(*)}) = \operatorname{Ker}((\overline{(\lambda - \lambda_1)}/r) I_{X_1} - (S_1^{(*)})^{m_2})$ and $\operatorname{Im}(\lambda I_X - A) = X_0 \oplus$ $\oplus \operatorname{Im}(((\lambda - \lambda_1)/r) I_{X_1} - S_1^{m_2}) \oplus X_2$ for any $\lambda \in B_{\mathbf{C}}(\lambda_1, r)$ (so that, as $(\lambda - \lambda_1)/r \in$ $\in B_{\mathbf{C}}(0, 1) \subset \varrho_{s-F}(S_1^{m_2})$, $\lambda \in \varrho_{s-F}(A)$ and $\operatorname{ind}(\lambda I_X - A) = \operatorname{ind}(((\lambda - \lambda_1)/r) I_{X_1} -$ $- S_1^{m_2}) = -m_2$) and $\operatorname{Ker}(\lambda I_X - A) = \operatorname{Ker}(((\lambda - \lambda_2)/r) I_{X_2} - (S_2^{(*)})^{m_1})$, $\operatorname{Ker}(\bar{\lambda} I_X - A^{(*)}) = \operatorname{Ker}((\overline{(\lambda - \lambda_2)}/r) I_{X_2} - S_2^{m_1})$ and $\operatorname{Im}(\lambda I_X - A) = X_0 \oplus X_1 \oplus$ $\oplus \operatorname{Im}(((\lambda - \lambda_2)/r) I_{X_2} - (S_2^{(*)})^{m_1})$ for any $\lambda \in B_{\mathbf{C}}(\lambda_2, r)$ (so that, as $(\lambda - \lambda_2)/r \in$ $\in B_{\mathbf{C}}(0, 1) \subset \varrho_{s-F}((S_2^{(*)})^{m_1})$, $\lambda \in \varrho_{s-F}(A)$ and $\operatorname{ind}(\lambda I_X - A) = \operatorname{ind}(((\lambda - \lambda_2)/r) I_{X_2} -$ $-(S_2^{(*)})^{m_1}) = m_1$).

We have thus proved that $\varrho_{s-F}^{\pm}(A) \supset \bigcup_{j=1}^{2} B_{\mathbf{C}}(\lambda_j, r)$. Hence $\sigma(A) = (\sigma(A) \cap \{\lambda_0\}) \cup$ $\cup (\bigcup_{j=1}^{2} \operatorname{cl} B_{\mathbf{C}}(\lambda_j, r)) \subset \psi(A) \cup \operatorname{cl} \varrho_{s-F}^{\pm}(A)$. Since the opposite inclusion is trivial, it follows obviously that $A \in \Sigma_0(X)$.

Since $\sigma(A) \subset \{\lambda_0\} \cup (\bigcup_{j=1}^{2} \operatorname{cl} B_{\mathbf{C}}(\lambda_j, r)) \subset \Omega \subset \Delta$, f is holomorphic in a neighborhood of $\sigma(A)$. We prove that $f(A) \notin \Sigma_0(X)$.

Since $\lambda_1, \lambda_2 \in \varrho_{s-F}^{\pm}(A)$, $f(\lambda_1) = f(\lambda_2)$, $f(\lambda_0) \neq f(\lambda_1)$ and $f(\lambda) \neq f(\lambda_1)$ for any $\lambda \in (\bigcup_{j=1}^{2} \operatorname{cl} B_{\mathbf{C}}(\lambda_j, r)) \setminus \{\lambda_1, \lambda_2\}$ it follows that $f^{-1}(\{f(\lambda_1)\}) \cap \sigma(A) = \{\lambda_1, \lambda_2\} \subset$ $\subset \varrho_{s-F}^{\pm}(A) \subset \Omega$. Therefore

$$f(\lambda_1) \in f(\varrho_{s-F}^{\pm}(A) \cap \Omega) \setminus f(\partial \varrho_{s-F}^{\pm}(A) \cap \Omega)\,,$$

$$\{\operatorname{ind}(\mu I_X - A) \colon \mu \in f^{-1}(\{f(\lambda_1)\}) \cap \varrho_{s-F}^{\pm}(A) \cap \Omega\} = \{m_1, -m_2\} \not\supset \{-\infty, +\infty\}$$

and

$$\sum_{\mu \in f^{-1}(\{f(\lambda_1)\}) \cap \varrho_{s-F}^{\pm}(A) \cap \Omega} m_f(\mu) \operatorname{ind}(\mu I_X - A) = m_1 \operatorname{ind}(\lambda_1 I_X - A) +$$
$$+ m_2 \operatorname{ind}(\lambda_2 I_X - A) = -m_1 m_2 + m_2 m_1 = 0\,.$$

Consequently, by Theorem 2.5, iii), $f(A) \notin \Sigma_0(X)$.

We have thus proved that if f is not one-to-one on Ω there exists $A \in \Sigma_0(X)$ such that $\sigma(A) \subset \Delta$ and $f(A) \notin \Sigma_0(X)$. Hence i) implies ii). Since obviously $(f')^{-1}(\mathbf{C} \setminus \{0\})$ is contained in the union of all components of Δ on which f is not constant, ii) implies iii). Since $(f')^{-1}(\mathbf{C} \setminus \{0\}) \cap \sigma(A) = \sigma(A) \setminus Z_f(A)$ for any $A \in \Sigma_0(X)$ such that $\sigma(A) \subset \Delta$, from Corollary 2.6 it follows immediately that iii) implies i).

The proof is therefore complete.

Corollary 2.10. *Let X be a complex infinite-dimensional Hilbert space and let Δ be an open nonempty subset of* **C**. *Then, for any integer* $p \geqq 2$, $A^p \in \Sigma_0(X)$ *for any* $A \in \Sigma_0(X)$ *such that* $\sigma(A) \subset \Delta$ *iff* $\Delta \cap \left(\bigcup_{k=1}^{p-1} e^{2k\pi i/p}\Delta\right) = \emptyset$.

Proof. If $\Delta \cap \left(\bigcup_{k=1}^{p-1} e^{2k\pi i/p}\Delta\right) = \emptyset$, it follows that $\sigma(A) \cap \left(\bigcup_{k=1}^{p-1} e^{2k\pi i/p}\sigma(A)\right) = \emptyset$ for any $A \in \Sigma_0(X)$ such that $\sigma(A) \subset \Delta$. From Corollary 2.8 it follows that $A^p \in \Sigma_0(X)$ for any $A \in \Sigma_0(X)$ such that $\sigma(A) \subset \Delta$.

Conversely, if $\Delta \cap \left(\bigcup_{k=1}^{p-1} e^{2k\pi i/p}\Delta\right) \neq \emptyset$, as Δ is open there exists $\lambda \in (\Delta \setminus \{0\}) \cap \cap \left(\bigcup_{k=1}^{p-1} e^{2k\pi i/p}\Delta\right)$. Consequently, there exist $\mu \in \Delta \setminus \{0\}$ and $k \in \{1, \ldots, p-1\}$ such that $\lambda = e^{2k\pi i/p}\mu$, so that $\lambda \neq \mu$ and $\lambda^p = \mu^p$, and therefore the function $f_p\colon \Delta \to \mathbf{C}$ (where $f_p(z) = z^p$ for any $z \in \Delta$) is not one-to-one. Since f_p is not constant on any component of Δ, from Theorem 2.9, ii) it follows that there exists $A \in \Sigma_0(X)$ such that $A^p \notin \Sigma_0(X)$.

Corollary 2.10 obviously proves, in particular, that $\Sigma_0(X)$, generally speaking, is not closed with respect to powers.

References

[AFHV] *C. Apostol, L. A. Fialkow, D. A. Herrero, D. Voiculescu:* Approximation of Hilbert space operators, volume II, Research Notes in Mathematics 102 (Pitman, Boston, 1984).

[B1] *L. Burlando:* On two subsets of a Banach algebra that are related to the continuity of spectrum and spectral radius, Linear Algebra Appl., *84*, (1986), 251—269.

[B2] *L. Burlando:* Continuity of spectrum and spectral radius in algebras of operators, to appear.

[BHOP] *C. Bosh, C. Hernandez, E. De Oteyza, C. Pearcy:* Spectral pictures of functions of operators, J. Operator Theory, *8* (1982), 391—400.

[CM] *J. B. Conway, B. B. Morrel:* Operators that are points of spectral continuity, Integral Equations and Operator Theory, *2* (1979), 174—198.

[CPY] *S. R. Caradus, W. E. Pfaffenberger, B. Yood:* Calkin algebras and algebras of operators on Banach spaces, Lecture Notes in Pure and Applied Mathematics 9 (Marcel Dekker, New York, 1974).

[GL] *B. Gramsch, D. Lay:* Spectral mapping theorems for essential spectra, Math. Ann., *192* (1971), 17—32.

[H] *P. R. Halmos:* A Hilbert space problem book (Van Nostrand, Princeton, 1967).

[K] *T. Kato:* Perturbation theory for linear operators (Springer, New York, 1966).

[R] *W. Rudin:* Real and complex analysis (McGraw-Hill, New York, 1966).

[TL] *A. E. Taylor, D. C. Lay:* Introduction to functional analysis, second edition (John-Wiley & Sons, New York, 1980).

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