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OSCILLATION OF DIFFERENTIAL EQUATIONS AND v-DERIVATIVES

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The aim of this paper is to present results concerning oscillatory and asymptotic properties of solutions of n-th order $(n > 1)$ differential equations of the form

$$(1) \qquad (r(t) \ldots (r(t)\,(r(t)\,y'(t))')' \ldots)' + p(t)\,f(y(g(t))) = 0\,,$$

where r, p, g are real-valued and continuous functions on an interval $[t_0, \infty)$, $r(t) > 0$, $g(t) \to \infty$ as $t \to \infty$, and f is a real-valued and continuous function on $(-\infty, \infty)$. The technique used in the paper is based on the notion of the v-derivative of a function. The main tool in establishing the results is the following assertion, which is a special case of Theorem 2.1.

Theorem. *Let the above conditions on r, p, g, f be satisfied. A function $y(t)$ is a solution of the equation*

$$y^{(n)}(t) + r(\Phi(t))\,p(\Phi(t))\,f(y(R(g(\Phi(t))))) = 0$$

on $[t_1, \infty)$ if and only if the function $u(t) = y(R(t))$ is a solution of the equation (1) on $[t_2, \infty)$, where

$$R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)} \quad \text{for} \quad t \geqq t_0\,, \quad R(t) \to \infty \quad \text{as} \quad t \to \infty\,,$$

Φ is the inverse function to R, and $t_1(\geqq 0)$ is such that $g(\Phi(t)) \geqq t_0$ if $t \geqq t_1$, $t_2(\geqq t_0)$ is such that $R(t) \geqq t_1$ if $t \geqq t_2$.

As we shall see, this theorem permits to obtain information about oscillatory and asymptotic properties of solutions of differential equations of the form (1) from the results of this kind known for differential equations of the form

$$y^{(n)}(t) + a(t)\,f(y(h(t))) = 0\,.$$

Examples of such applications of our theorem are presented in Section 3 of this paper.

Suppose throughout this section that $f, g, v, v_1, \ldots$ are real-valued functions of one real variable. Let the interval $(-\infty, \infty)$ be denoted by $\mathbb{R}$.

Definition 1.1. Let functions f *and* v be defined on a neighborhood $O(t)$ of a point $t \in \mathbb{R}$ and let the conditions $x \in O(t)$, $x \neq t$ imply $v(x) \neq v(t)$. If the limit

$$\lim_{x \to t} \frac{f(x) - f(t)}{v(x) - v(t)}$$

is finite, then it is called the *v-derivative of the function f* at the point t and is denoted by $f'_v(t)$ or $\mathrm{d}f(t)/\mathrm{d}v$.

Remark 1.1. It follows from Definition 1.1 that $f'_f(t) = 1$ for every t such that f is defined on $O(t)$ and the conditions $x \in O(t)$, $x \neq t$ imply $f(x) \neq f(t)$.

Using the above definition we can build the v-differential calculus similar to the ordinary differential calculus. Here we introduce two results that we shall need later.

Theorem 1.1. *Let the following conditions be satisfied:*

(i) *a function v is continuous at a point t,*

(ii) *a function g has the v-derivative at the point t,*

(iii) *a function f has the ordinary derivative at the point $g(t)$.*

Then the composite function $f(g)$ has the v-derivative at the point t and

$$(f(g))'_v(t) = f'(g(t))\, g'_v(t)\,.$$

Theorem 1.2. *Let there exist $v'(t) \neq 0$ on an interval I. Then for $t \in I$ the v-derivative $f'_v(t)$ exists if and only if the derivative $f'(t)$ exists. Moreover,*

$$f'_v(t) = \frac{f'(t)}{v'(t)}\,.$$

The proofs of the above two theorems are simple and therefore omitted.

Now we introduce v-derivatives of higher orders.

Definition 1.2. Let $n > 1$ be a natural number. Let functions v_n and $f^{(n-1)}_{v_1,v_2,\ldots,v_{n-1}}$ be defined on a neighborhood $O(t)$ of a point $t \in \mathbb{R}$ and let the conditions $x \in O(t)$, $x \neq t$ imply $v_n(x) \neq v_n(t)$. If the limit

$$\lim_{x \to t} \frac{f^{(n-1)}_{v_1,v_2,\ldots,v_{n-1}}(x) - f^{(n-1)}_{v_1,v_2,\ldots,v_{n-1}}(t)}{v_n(x) - v_n(t)}$$

is finite, then it is called the n-th *v-derivative of the function f* at the point t and denoted by

$$f^{(n)}_{v_1,v_2,\ldots,v_n}(t) \quad \text{or} \quad \frac{\mathrm{d}^n f(t)}{\mathrm{d}v_n \ldots \mathrm{d}v_2\, \mathrm{d}v_1}\,.$$

In the case $v_1 = \ldots = v_m = v$ and $v_{m+1} = \ldots = v_n = u$ we shall write

$$f^{(n)}_{v^m, u^{n-m}}(t) \quad \text{or} \quad \frac{d^n f(t)}{du^{n-m}\, dv^m}.$$

For instance, if we put $n = 2$ and $t \in I$ in Definition 1.2, then by Theorem 1.2 we have

$$\frac{d^2 f(t)}{dv_2\, dv_1} = \frac{d}{dv_2} \frac{f'(t)}{v_1'(t)} \quad \text{whenever} \quad v_1'(t) \neq 0 \quad \text{on} \quad I,$$

$$\frac{d^2 f(t)}{dv_2\, dv_1} = \frac{1}{v_2'(t)} \frac{d}{dt} f'_{v_1}(t) \quad \text{whenever} \quad v_2'(t) \neq 0 \quad \text{on} \quad I,$$

$$\frac{d^2 f(t)}{dv_2\, dv_1} = \frac{1}{v_2'(t)} \frac{d}{dt} \frac{f'(t)}{v_1'(t)} \quad \text{whenever} \quad v_i'(t) \neq 0 \quad (i = 1, 2) \quad \text{on} \quad I,$$

$$\frac{d^2 f(t)}{dv_2\, dv_1} = \frac{f''(t)\, v_1'(t) - f'(t)\, v_1''(t)}{(v_1'(t))^2\, v_2'(t)} \quad \text{whenever} \quad v_i'(t) \neq 0 \quad (i = 1, 2)$$

on I and there exist $f''(t)$ and $v_1''(t)$ on I.

2. THE v-TRANSFORMATION OF A DIFFERENTIAL EQUATION

In the sequel we shall use the term "v-differential equation" for a differential equation (may be with deviating arguments) in which all derivatives of the unknown function have been replaced by its v-derivatives. Specifically, a v-differential equation will be refered to as a $1v$-differential equation if all the derivatives of the unknown function in this equation are v-derivatives with respect to the same function v. Thus a $1v$-differential equation with deviating arguments is an equation of the form

$$(1) \qquad G(t, y(t), y_v'(t), \ldots, y^{(n_0)}_{v^{n_0}}(t), y(h_1(t)), y_v'(h_1(t)), \ldots, y^{(n_1)}_{v^{n_1}}(h_1(t)), \ldots$$
$$\ldots, y(h_k(t)), y_v'(h_k(t)), \ldots, y^{(n_k)}_{v^{n_k}}(h_k(t))) = 0,$$

where $G: \mathbb{R}^m \to \mathbb{R}^n$, $y: \mathbb{R} \to \mathbb{R}^p$, $v: \mathbb{R} \to \mathbb{R}$, $h_i: \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, \ldots, k)$; m, n, p, k and n_j $(j = 0, 1, \ldots, k)$ are nonnegative integers.

Now suppose that the following conditions are satisfied:

a) $k \geqq 0,\ n \geqq 1,\ p \geqq 1$ are integers,

b) $n_i \geqq 0$ $(i = 0, 1, \ldots, k)$ are integers, $\sum_{i=0}^{k} n_i = N \geqq 1$,

c) I and I_1 are intervals in $\mathbb{R}$,

d) $g_i \in C(I)$ $(i = 0, 1, \ldots, k)$, $g_0(t) \equiv t$ and $g_i: I \to I_2 \subset I$,

e) $v \in C(I_1)$, v is a strictly monotonous function, $v: I_1 \to I$,

f) φ is the inverse function to v,

g) $y = (y_1, y_2, \ldots, y_p)$,
h) $F : I \times \mathbb{R}^{(N+k+1)p} \to \mathbb{R}^n$.

Consider the differential equation

$$\tag{2} F(t, y(t), y'(t), \ldots, y^{(n_0)}(t), y(g_1(t)), y'(g_1(t)), \ldots \\ \ldots, y^{(n_1)}(g_1(t)), \ldots, y(g_k(t)), y'(g_k(t)), \ldots, y^{(n_k)}(g_k(t))) = 0\,, \quad t \in I\,.$$

If, in the equation (2), the independent variable t is replaced by the function $v(t)$ and

$$y(g_j(t)),\ y'(g_j(t)), \ldots, y^{(n_j)}(g_j(t))$$

are replaced by

$$y(\varphi(g_j(v(t)))),\ y'_v(\varphi(g_j(v(t)))), \ldots, y^{(n_j)}_{v^{n_j}}(\varphi(g_j(v(t))))$$

for $j = 0, 1, \ldots, k$ in the sense that $v(t)$ replaces even the independent variable as an argument of the function with respect to which the derivatives of the unknown function are calculated, then (2) is transformed into the equation

$$\tag{3} F(v(t), y(t), y'_v(t), \ldots, y^{(n_0)}_{v^{n_0}}(t), y(\varphi(g_1(v(t)))), y'_v(\varphi(g_1(v(t)))), \ldots \\ \ldots, y^{(n_1)}_{v^{n_1}}(\varphi(g_1(v(t)))), \ldots, y(\varphi(g_k(v(t)))), y'_v(\varphi(g_k(v(t)))), \ldots, y^{(n_k)}_{v^{n_k}}(\varphi(g_k(v(t))))) = 0\,, \\ t \in I_1\,.$$

In the sequel we shall call the above mentioned process of obtaining (3) from (2) a *v-transformation of a differential equation.* It is readily evident that a v-transformation preserves the (non-)linearity and the order of equations. Note that (3) is a $1v$-differential equation since it is always possible to write (3) in the form (1).

It is also useful to note that a φ-transformation of (3) leads again to (2).

It is easy to see that, for example, if v is an increasing function and for some $j \in \{1, 2, \ldots, k\}$ we have $g_j(t) \leqq t$ ($g_j(t) \geqq t$), $t \in I$ then $\varphi(g_j(v(t))) \leqq t$ ($\varphi(g_j(v(t))) \geqq t$), $t \in I_1$. Thus it is clear that if v is increasing, then the v-transformation of a retarded (advanced) differential equation gives a retarded (advanced) $1v$-differential equation.

Finally, we note that the equation (2) (the equation (3)) includes an ordinary differential (a $1v$-differential) equation without deviating argument as well as a differential (a $1v$-differential) equation with one or several deviating arguments and also a system of differential ($1v$-differential) equations without or with deviating arguments.

Now we can establish the following result.

Theorem 2.1. *Let the conditions* a)–h) *be satisfied. The function $u(t)$ is a solution of the equation* (2) *on I if and only if the function $u(v(t))$ is a solution of the equation* (3) *on I_1.*

Proof. We put $v(t) = s$, $v(\sigma) = x$ and $u(v(t)) = z(t)$ for $t, \sigma \in I_1$. Then, according to Definition 1.1 or according to Theorem 1.1 and Remark 1.1, for the components of the functions u and z we have

$$
(4) \qquad \begin{aligned}
&z_i(\varphi(g_j(v(t)))) = u_i(g_j(s)), \\
&(z_i)'_v(\varphi(g_j(v(t)))) = (u_i)'(g_j(s)), \\
&(z_i)''_{v^2}(\varphi(g_j(v(t)))) = (u_i)''(g_j(s)), \\
&\vdots \\
&(z_i)^{(n_j)}_{v^{n_j}}(\varphi(g_j(v(t)))) = (u_i)^{(n_j)}(g_j(s))
\end{aligned}
$$

for $t \in I_1$, $s = v(t) \in I$; $i = 1, 2, \ldots, p$; $j = 0, 1, \ldots, k$.

From (4) we see that

$$
\begin{aligned}
&z(\varphi(g_j(v(t)))) = u(g_j(s)), \\
&z'_v(\varphi(g_j(v(t)))) = u'(g_j(s)), \\
&z''_{v^2}(\varphi(g_j(v(t)))) = u''(g_j(s)), \\
&\vdots \\
&z^{(n_j)}_{v^{n_j}}(\varphi(g_j(v(t)))) = u^{(n_j)}(g_j(s))
\end{aligned}
$$

for $t \in I_1$, $s \in I$, $j = 0, 1, \ldots, k$. Therefore

$$
\begin{aligned}
&F(v(t), z(t), z'_v(t), \ldots, z^{(n_0)}_{v^{n_0}}(t), z(\varphi(g_1(v(t)))), z'_v(\varphi(g_1(v(t)))), \ldots \\
&\ldots, z^{(n_1)}_{v^{n_1}}(\varphi(g_1(v(t)))), \ldots, z(\varphi(g_k(v(t)))), z'_v(\varphi(g_k(v(t)))), \ldots \\
&\ldots, z^{(n_k)}_{v^{n_k}}(\varphi(g_k(v(t))))) = F(s, u(s), u'(s), \ldots, u^{(n_0)}(s), u(g_1(s)), u'(g_1(s)), \ldots \\
&\ldots, u^{(n_1)}(g_1(s)), \ldots, u(g_k(s)), u'(g_k(s)), \ldots, u^{(n_k)}(g_k(s)))
\end{aligned}
$$

for $t \in I_1$ and $s = v(t) \in I$.

Thus we see that u is a solution of (2) on I if and only if $z = u(v)$ is a solution of (3) on I_1 and the proof is completed.

Remark 2.1. It is easy to see from the proof of Theorem 2.1 that if (2) and (3) are scalar equations, then we can write

$$F(t, y(t), \ldots, y^{(n_k)}(g_k(t))) \leqq 0 \quad (\geqq 0)$$

instead of (2) and

$$F(v(t), y(t), \ldots, y^{(n_k)}_{v^{n_k}}(\varphi(g_k(v(t))))) \leqq 0 \quad (\geqq 0)$$

instead of (3), and Theorem 2.1 remains valid.

Note that the v-transformation of an ordinary differential equation may be used to extend the set of differential equations with known solutions. Indeed, if we put $k = 0$, $n = 1$, $p = 1$ in condition a) and require that $v \in C^{n_0}(I_1)$ with $v'(t) \neq 0$ then the v-derivatives $y'_v(t), y''_{v^2}(t), \ldots, y^{(n_0)}_{v^{n_0}}(t)$ may be expressed in terms of ordinary derivatives, e.g.

$$y'_v(t) = \frac{y'(t)}{v'(t)}, \quad y''_{v^2}(t) = \frac{y''(t)\,v'(t) - y'(t)\,v''(t)}{v'^3(t)}, \ldots .$$

Thus, in the case $k = 0$, $n = 1$, $p = 1$ and $v \in C^{n_0}(I_1)$, $v'(t) \neq 0$, the equation (3) becomes an ordinary differential equation, e.g.

$$F_1(t, y(t), y'(t), \ldots, y^{(n_0)}(t)) = 0\,, \quad t \in I_1\,. \tag{5}$$

Since (3) and (5) are merely two ways of writing the same equation, they have the same solutions. Thus (2) has been transformed to (5). Moreover, we know that according to Theorem 2.1, u is a solution of (2) if and only if the composite function $u(v)$ is a solution of (5). Therefore, for example, if $y_1(t)$ and $y_2(t)$ are linearly independent solutions of

$$y''(t) + p(t)\, y'(t) + q(t)\, y(t) = 0\,, \quad t \in I \tag{6}$$

and $v \in C^2(I_1)$, $v' \neq 0$ and $v: I_1 \to I$, then

$$y_1(v(t)) \quad \text{and} \quad y_2(v(t))$$

are linearly independent solutions of

$$y''(t) + \left[v'(t)\, p(v(t)) - \frac{v''(t)}{v'(t)}\right] y'(t) + v'^2(t)\, q(v(t))\, y(t) = 0\,, \quad t \in I_1\,. \tag{7}$$

Conversely, if v is such that $u_1(t)$ and $u_2(t)$ are linearly independent solutions of (7), then

$$u_1(\varphi(t)) \quad \text{and} \quad u_2(\varphi(t))$$

are linearly independent solutions of (6).

Recently, there has been increasing interest in studying the oscillatory character and the asymptotic behaviour of solutions of n-th order differential equations with or without deviating arguments involving the so-called quasi-derivatives of the unknown function. In the following sections of this paper we shall be concerned with oscillatory and asymptotic properties of solutions of some special cases of such equations.

In the sequel we shall restrict our attention to those solutions of the equations considered which exist on some ray $[T, \infty)$ and are non-trivial in any neighborhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros, and nonoscillatory otherwise. An equation is said to be oscillatory if all of its solutions are oscillatory; otherwise it is said to be nonoscillatory.

3. OSCILLATION OF THE n-TH ORDER $1v$-DIFFERENTIAL EQUATIONS

Consider the n-th order differential equation with deviating argument of the form

$$(r(t) \ldots (r(t)\,(r(t)\, y'(t))')' \ldots)' + p(t) f(y(g(t))) = 0 \tag{1}$$

for $t \geqq t_0$, where $r \in C([t_0, \infty))$, $r(t) > 0$. We shall state the conditions on the functions

p, f, g later. It is evident that the above equation is a special case of the equation

$$(2) \qquad (r_{n-1}(t) \dots (r_2(t) (r_1(t) y'(t))')' \dots)' + p(t) f(y(g(t))) = 0 ,$$

where $t \geqq t_0$, $r_i \in C([t_0, \infty))$, $r_i(t) > 0$ $(i = 1, 2, \dots, n - 1)$.

Now, if we define

$$R_i(t) = \int_{t_0}^{t} \frac{ds}{r_i(s)} , \quad t \geqq t_0 , \quad i = 1, 2, \dots, n - 1 ,$$

then we can introduce the well-known properties (A) and (B) of the equation (2) in the following form:

Definition 3.1. The equation (2) is said *to have the property* (A) if, for n even, the equation (2) is oscillatory, and for n odd, every solution $y(t)$ of (2) is either oscillatory or

$$(3) \qquad \frac{d^i y(t)}{dR_i \dots dR_2 \, dR_1} \to 0 \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \dots, n - 1) .$$

Definition 3.2. The equation (2) is said *to have the property* (B) if, for n even, every solution $y(t)$ of (2) is oscillatory or satisfies the condition (3) or satisfies the condition

$$(4) \qquad \left| \frac{d^i y(t)}{dR_i \dots dR_2 \, dR_1} \right| \to \infty \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \dots, n - 1) ,$$

and for n odd, every solution $y(t)$ of (2) is either oscillatory or satisfies the condition (4).

Without mentioning it again, the following notation will be used throughout this paper:

$$R(t) = \int_{t_0}^{t} \frac{ds}{r(s)} , \quad t \geqq t_0 ,$$

Φ is the inverse function to R .

Remark 3.1. It is easy to see that in the case

$$r_i(t) = r(t) , \quad i = 1, 2, \dots, n - 1$$

the conditions (3) and (4) assume the form

$$(3_1) \qquad \frac{d^i y(t)}{dR^i} \to 0 \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \dots, n - 1)$$

and

$$(4_1) \qquad \left| \frac{d^i y(t)}{dR^i} \right| \to \infty \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \dots, n - 1) ,$$

and in the case $r(t) \equiv 1$ the conditions (3_1) and (4_1) assume the form

$$(3_2) \qquad y^{(i)}(t) \to 0 \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \dots, n - 1)$$

and

$$(4_2) \qquad |y^{(i)}(t)| \to \infty \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \dots, n-1).$$

In the sequel we shall use the terms "the property (A)", "the property (B)" for the equation of the form (1) or of the form $y^{(n)}(t) + p(t) f(y(g(t))) = 0$ in the sense of Remark 3.1.

Lemma 3.1. *Let $R(t) \to \infty$ as $t \to \infty$.*

a) *If a function $y(t)$ is oscillatory on an interval $[t_1, \infty)$, then the function $u(t) = y(R(t))$ is oscillatory on the interval $[t_2, \infty)$, where t_2 is such that $R(t_2) = t_1$.*

b) *If $y(t)$ satisfies the condition (3_2) $((4_2))$, then $u(t) = y(R(t))$ satisfies the condition (3_1) $((4_1))$.*

Proof. Since $R(t) \to \infty$ as $t \to \infty$, the assertion a) is evident. Now, because (by Theorem 1.1 and Remark 1.1)

$$u(t) = y(R(t)), \quad \frac{\mathrm{d}u(t)}{\mathrm{d}R} = y'(R(t)),$$

$$\frac{\mathrm{d}^2 u(t)}{\mathrm{d}R^2} = y''(R(t)), \dots, \frac{\mathrm{d}^{n-1} u(t)}{\mathrm{d}R^{n-1}} = y^{(n-1)}(R(t)),$$

we see that the assertion b) holds and the proof is complete.

First we consider the n-th order linear ordinary differential equation

$$(5) \qquad (r(t) \dots (r(t)(r(t)\, y'(t))')' \dots)' + p(t)\, y(t) = 0, \quad t \geqq t_0,$$

where $r, p \in C([t_0, \infty))$, $r(t) > 0$.

According to Theorem 1.2 we know that the equation (5) may be written in the form

$$(5_1) \qquad \frac{\mathrm{d}^n y(t)}{\mathrm{d}R^n} + r(t)\, p(t)\, y(t) = 0, \quad t \geqq t_0.$$

The v-transformation of (5_1) with $v = \Phi$ leads to the equation

$$(6) \qquad y^{(n)}(t) + r(\Phi(t))\, p(\Phi(t))\, y(t) = 0, \quad t \in [0, R(\infty)).$$

Before proving our first result we introduce the following comparison results due to V. A. Kondratev [4] and T. A. Čanturija [2].

Theorem A. *Let $n \geqq 3$. Let functions a and b be integrable on every finite and closed subinterval of the interval $[0, \infty)$.*

A_1) *If $a(t) \geqq b(t) \geqq 0$ for $t \in [0, \infty)$ and the equation*

$$(7) \qquad u^{(n)}(t) + b(t)\, u(t) = 0$$

has the property (A), *then the equation*

$$(8) \qquad u^{(n)}(t) + a(t)\, u(t) = 0$$

has the property (A).

A$_2$) *If* $a(t) \leqq b(t) \leqq 0$ *for* $t \in [0, \infty)$ *and the equation* (7) *has the property* (B), *then the equation* (8) *has the property* (B).

Also, it will be useful to observe the following well-known fact.

A system of fundamental solutions of the equation

$$(9) \qquad y^{(n)}(t) + \alpha\, y(t) = 0\,, \quad \alpha \in R\,, \quad \alpha \neq 0$$

is formed by real and imaginary parts of complex functions of one real variable of the form

$$(10) \qquad y(t) = e^{kt}\,,$$

where

$$(11) \qquad k = (\sqrt[n]{|-\alpha|})\left(\cos\frac{\arg(-\alpha) + 2s\pi}{n} + i\sin\frac{\arg(-a) + 2s\pi}{n}\right),$$

$s = 0, 1, \ldots, n-1$ and $\arg(-\alpha)$ means the principal value of the argument of the number $-\alpha$.

Theorem 3.1. *Let* $n \geqq 3$. *Let* $r, p \in C([t_0, \infty))$, $r(t) > 0$ *and* $R(t) \to \infty$ *as* $t \to \infty$. *Then*

a) *the equation* (5) *has the property* (A) *if* $\liminf_{t\to\infty} r(t)\, p(t) > 0$,

b) *the equation* (5) *has the property* (B) *if* $\limsup_{t\to\infty} r(t)\, p(t) < 0$.

Proof. It follows from (10) and (11) that the equation (9) has the property (A) if $\alpha > 0$.

Now suppose that $\liminf_{t\to\infty} r(t)\, p(t) = a > 0$. Since $R(t) \to \infty$ if and only if $t \to \infty$, so

$$\liminf_{t\to\infty} r(\Phi(t))\, p(\Phi(t)) = \liminf_{t\to\infty} r(t)\, p(t) = a$$

and then for any $\alpha \in (0, a)$ there exists $T_\alpha \geqq 0$ such that

$$r(\Phi(t))\, p(\Phi(t)) \geqq \alpha \quad \text{for} \quad t \geqq T_\alpha\,.$$

Applying Theorem A to the equations (6) and (9) (with $\alpha > 0$) we see that the equation (6) has the property (A). But the v-transformation of (6) with $v = R$ leads to the equation (5), and thus according to Theorem 2.1 and Lemma 3.1 we know that the equation (5) has the property (A).

In the case $\alpha < 0$ the equation (9) has the property (B) and the second part of the theorem can be proved analogously. This completes the proof.

T. A. Čanturija [1] has presented the following results (Theorem 2.1 to 2.4 in [1]).

Theorem B. *Let* $n \geqq 3$ *and* $a \in C([0, \infty))$.

B$_1$) *Let* $a(t) \geqq 0$ *and*

$$\liminf_{t\to\infty} t^{n-1} \int_t^\infty a(s)\, ds > \frac{M_n^*}{n-1}\,,$$

where M_n^* *is the maximum of all local maxima of the polynomial*

$$P_n^*(x) = -x(x-1)\dots(x-n+1).$$

Then the equation (8) *has the property* (A).

B$_2$) *Let* $a(t) \leqq 0$ *and*

$$\liminf_{t\to\infty} t^{n-1} \int_t^\infty |a(s)| \, ds > \frac{M_{*n}}{n-1},$$

where M_{*n} *is the maximum of all local maxima of the polynomial*

$$P_{*n}(x) = x(x-1)\dots(x-n+1).$$

Then the equation (8) *has the property* (B).

B$_3$) *Let* $a(t) \geqq 0$ *and*

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-2} a(s) \, ds > (n-1)!.$$

Then the equation (8) *has the property* (A).

B$_4$) *Let* $a(t) \leqq 0$. *Let* n *be odd* (n even) *and*

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-2} |a(s)| \, ds > (n-1)! \quad (>2(n-2)!).$$

Then the equation (8) *has the property* (B).

As we shall see, Theorem 2.1 enables us to extend the above results to the equation (5).

Theorem 3.2. *Let* $n \geqq 3$. *Let* $r, p \in C([t_0, \infty))$, $r(t) > 0$ *and* $R(t) \to \infty$ *as* $t \to \infty$.

a) *Suppose that* $p(t) \geqq 0$ *and*

$$\liminf_{t\to\infty} [R(t)]^{n-1} \int_t^\infty p(s) \, ds > \frac{M}{n-1},$$

where M *is the maximum of all local maxima of the polynomial*

$$P(t) = -t(t-1)\dots(t-n+1).$$

Then the equation (5) *has the property* (A).

b) *Suppose that* $p(t) \leqq 0$ *and*

$$\liminf_{t\to\infty} [R(t)]^{n-1} \int_t^\infty |p(s)| \, ds > \frac{K}{n-1},$$

where K *is the maximum of all local maxima of the polynomial*

$$Q(t) = t(t-1)\dots(t-n+1).$$

Then the equation (5) *has the property* (B).

c) *Suppose that* $p(t) \geqq 0$ *and*

$$\limsup_{t\to\infty} R(t) \int_t^\infty [R(s)]^{n-2} p(s) \, ds > (n-1)!.$$

Then the equation (5) *has the property* (A).

d) *Suppose that* $p(t) \leqq 0$. *Let* n *be odd* (n *even*) *and*

$$\limsup_{t\to\infty} R(t) \int_t^\infty [R(s)]^{n-2} |p(s)| \, ds > (n-1)! \quad (>2(n-2)!) .$$

Then the equation (5) *has the property* (B).

Proof. Since

$$\liminf_{t\to\infty} [R(t)]^{n-1} \int_t^\infty p(s) \, ds = \liminf_{t\to\infty} t^{n-1} \int_t^\infty r(\Phi(s)) \, p(\Phi(s)) \, ds ,$$

so according to Theorem B (part B_1), the assumptions of our theorem (part a) ensure that the equation (6) has the property (A). By Theorem 2.1 and Lemma 3.1 we see that the equation (5) has the property (A), and part a) of Theorem 3.2 is proved.

It is evident that the other parts of Theorem 3.2 may be proved similarly and thus the proof is complete.

Now we consider the n-th order linear differential equation with retarded argument

$$(12) \qquad (r(t) \ldots (r(t)(r(t) \, y'(t))')' \ldots)' + p(t) \, y(g(t)) = 0 , \quad t \geqq t_0$$

where the following conditions will be assumed to be fulfilled.

(i_1) n is even,
(i_2) $r \in C([t_0, \infty))$, $r(t) > 0$, $R(t) \to \infty$ as $t \to \infty$,
(i_3) $p \in C([t_0, \infty))$, $p(t) > 0$,
(i_4) $g \in C^1([t_0, \infty))$, $g(t) \leqq t$, $g'(t) > 0$, $\lim_{t\to\infty} g(t) = \infty$.

Similarly as before we can write (12) in the form

$$(13) \qquad \frac{d^n y(t)}{dR^n} + r(t) \, p(t) \, y(g(t)) = 0 , \quad t \geqq t_0 .$$

By the v-transformation of (13) with $v = \Phi$ we obtain the retarded differential equation

$$(14) \qquad y^{(n)}(t) + r(\Phi(t)) \, p(\Phi(t)) \, y(R(g(\Phi(t)))) = 0$$

which we shall study for $t \in [t_1, \infty)$, where t_1 ($\geqq 0$) is such that

$$g(\Phi(t)) \geqq t_0 \quad \text{if} \quad t \geqq t_1 .$$

Recently M. Naito [7] has established the following results (Theorems 2,4 and 5 in [7]).

Theorem C. *Let* (i_1), (i_3), (i_4) *be satisfied.*

C_1) (i) *The equation*

$$(15) \qquad x^{(n)}(t) + p(t) \, x(g(t)) = 0$$

is oscillatory if

$$(16) \qquad \int^\infty [g(s)]^{n-2} \, p(s) \, ds = \infty .$$

(ii) *Suppose that* (16) *fails to hold. Then the equation* (15) *is oscillatory if*

$$\limsup_{t\to\infty} g(t) \int_t^\infty [g(s)]^{n-2} p(s)\,ds > (n-1)!\,,$$

or if

$$\liminf_{t\to\infty} g(t) \int_t^\infty [g(s)]^{n-2} p(s)\,ds > \frac{(n-1)!}{4}\,.$$

C_2) *Suppose that*

$$\int^\infty s^{n-2} p(s)\,ds < \infty\,.$$

Then the equation (15) *is nonoscillatory if*

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-2} p(s)\,ds < \frac{(n-2)!}{4}\,.$$

C_3) *Assume that*

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0\,.$$

(i) *The equation* (15) *is strongly oscillatory if and only if either*

$$\int^\infty s^{n-2} p(s)\,ds = \infty$$

or

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-2} p(s)\,ds = \infty\,.$$

(ii) *The equation* (15) *is strongly nonoscillatory if and only if*

$$\int^\infty s^{n-2} p(s)\,ds < \infty$$

and

$$\lim_{t\to\infty} t \int_t^\infty s^{n-2} p(s)\,ds = 0\,.$$

The above mentioned notions of strong oscillation and strong nonoscillation are defined as follows: An equation of the form (12) is said to be *strongly oscillatory* if the related equation

$$(17)\qquad (r(t)\ldots(r(t)\,(r(t)\,y'(t))')'\ldots)' + \lambda\, p(t)\, y(g(t)) = 0\,,\quad t \geqq t_0$$

is oscillatory for all positive values of λ. An equation of the form (12) is said to be *strongly nonoscillatory* if (17) is nonoscillatory for all positive λ's.

The purpose of the following theorem is to extend Theorem C to the equation (12).

Theorem 3.3. *Let* $(i_1)-(i_4)$ *be satisfied.*

a) (i) *The euqation* (12) *is oscillatory if*

$$(18)\qquad \int^\infty [R(g(t))]^{n-2} p(t)\,dt = \infty\,.$$

(ii) *Suppose that* (18) *fails to hold. Then the equation* (12) *is oscillatory if*

$$\limsup_{t\to\infty} R(g(t)) \int_t^\infty [R(g(s))]^{n-2} p(s)\,ds > (n-1)!\,,$$

or if

$$\liminf_{t\to\infty} R(g(t)) \int_t^\infty [R(g(s))]^{n-2} p(s)\,\mathrm{d}s > \frac{(n-1)!}{4}.$$

b) *Suppose that*

$$\int^\infty [R(t)]^{n-2} p(t)\,\mathrm{d}t < \infty .$$

Then the equation (12) *is nonoscillatory if*

$$\limsup_{t\to\infty} R(t) \int_t^\infty [R(s)]^{n-2} p(s)\,\mathrm{d}s < \frac{(n-2)!}{4}.$$

c) *Assume that*

$$\liminf_{t\to\infty} \frac{R(g(t))}{R(t)} > 0 .$$

(i) *The equation* (12) *is strongly oscillatory if and only if either*

$$\int^\infty [R(t)]^{n-2} p(t)\,\mathrm{d}t = \infty$$

or

$$\limsup_{t\to\infty} R(t) \int_t^\infty [R(s)]^{n-2} p(s)\,\mathrm{d}s = \infty .$$

(ii) *The equation* (12) *is strongly nonoscillatory if and only if*

$$\int^\infty [R(t)]^{n-2} p(t)\,\mathrm{d}t < \infty$$

and

$$\lim_{t\to\infty} R(t) \int_t^\infty [R(s)]^{n-2} p(s)\,\mathrm{d}s = 0 .$$

Proof. As we can see,

$$(19)\qquad [R(g(\Phi(t)))]' = \frac{r(\Phi(t))}{r(g(\Phi(t)))}\, g'(\Phi(t)) > 0 \quad \text{for} \quad t \geqq t_1$$

if $g'(t) > 0$ for $t \geqq t_0$. Moreover,

$$(20)\qquad \int^\infty [R(g(t))]^{n-2} p(t)\,\mathrm{d}t = \int^\infty [R(g(\Phi(t)))]^{n-2}\, r(\Phi(t))\, p(\Phi(t))\,\mathrm{d}t ,$$

$$\limsup_{t\to\infty} R(g(t)) \int_t^\infty [R(g(s))]^{n-2} p(s)\,\mathrm{d}s =$$
$$= \limsup_{t\to\infty} R(g(\Phi(t))) \int_t^\infty [R(g(\Phi(s)))]^{n-2}\, r(\Phi(s))\, p(\Phi(s))\,\mathrm{d}s ,$$

and the same is true when we write lim inf instead of lim sup. Therefore, according to Theorem C, part C_1), the assumptions of our theorem, part a), ensure that the equation (14) is oscillatory. By Theorem 2.1 and Lemma 3.1 we see that the equation (12) is oscillatory and the assertion a) of Theorem 3.3 is proved. It is clear that the assertions b) and c) of Theorem 3.3 can be proved similarly. To prove the assertion c), we note that

$$\liminf_{t\to\infty} \frac{R(g(t))}{R(t)} = \liminf_{t\to\infty} \frac{R(g(\Phi(t)))}{t} .$$

The proof is complete.

Note that part a) of Theorem 3.3 covers Theorem 1 in [9] and, of course, also all results which are covered by this theorem.

We finish our investigation of the retarded equation (12) with a result which extends the following one due to R. Oláh [8].

Theorem D (Theorem 2 in [8]). *Let* $n \geqq 3$. *Suppose that* $p \in C([t_0, \infty))$, $p(t) \geqq 0$, $g \in C^1([t_0, \infty))$, $g(t) \leqq t$, $g(t) \to \infty$ *as* $t \to \infty$ *and* $0 \leqq g'(t) \leqq 1$. *If*

$$\limsup_{t \to \infty} g(t) \int_t^\infty [g(s)]^{n-2} p(s) \, ds > (n-1)!,$$

then the equation (15) *has the property* (A).

Theorem 3.4. *Let* $n \geqq 3$. *Suppose that* $r \in C([t_0, \infty))$, $r(t) > 0$, $r(t)$ *is nonincreasing*, $p \in C([t_0, \infty))$, $p(t) \geqq 0$, $g \in C^1([t_0, \infty))$, $g(t) \leqq t$, $g(t) \to \infty$ *as* $t \to \infty$ *and* $0 \leqq g'(t) \leqq 1$. *If*

$$\limsup_{t \to \infty} R(g(t)) \int_t^\infty [R(g(s))]^{n-2} p(s) \, ds > (n-1)!,$$

then the equation (12) *has the property* (A).

Proof. If we take into account (19) and (20) and notice that

$$[R(g(\Phi(t)))]' = \frac{r(\Phi(t))}{r(g(\Phi(t)))} g'(\Phi(t)) \leqq 1 \quad \text{for} \quad t \geqq t_1$$

provided $g'(t) \leqq 1$ for $t \geqq t_0$ and $r(t)$ is nonincreasing, we conclude from Theorem D that the equation (14) has the property (A). By Theorem 2.1 and Lemma 3.1 we know that the equation (12) has the property (A) and the proof is complete.

Now we shall proceed in the investigation of the equation (12) with an advanced argument, i.e., the following conditions will be assumed to be fulfilled.

(ii_1) n is even,
(ii_2) $r \in C([t_0, \infty))$, $r(t) > 0$, $R(t) \to \infty$ as $t \to \infty$,
(ii_3) $p \in C([t_0, \infty))$, $p(t) > 0$,
(ii_4) $g \in C^1([t_0, \infty))$, $g'(t) > 0$, $g(t) \geqq t$.

Note that the condition $g(t) \geqq t$ allows us to deal with (14) on the interval $[0, \infty)$ if necessary.

Recently, T. Kusano [5] has proved the following theorems (Theorem 2, 3 and 4 in [5]).

Theorem E. *Let* (ii_1), (ii_3), (ii_4) *be satisfied.*

E_1) *Suppose that*

$$\limsup_{t \to \infty} \frac{g(t)}{t} < \infty .$$

The equation (15) *is strongly oscillatory if and only if*

$$\limsup_{t \to \infty} t \int_t^\infty s^{n-2} p(s) \, ds = \infty ,$$

and the equation (15) *is strongly nonoscillatory if and only if*

$$\lim t \int_t^\infty s^{n-2}\, p(s)\, ds = 0\,.$$

E$_2$) *The equation* (15) *is oscillatory if*

$$\int^\infty t^{n-2}\, p(t)\, dt = \infty \tag{22}$$

or if (22) *fails to hold but one of the following inequalities holds*:

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-2}\, p(s)\, ds > (n-1)!\,,$$

$$\liminf_{t\to\infty} t \int_t^\infty s^{n-2}\, p(s)\, ds > \frac{(n-1)!}{4}\,.$$

E$_3$) *The equation* (15) *is nonoscillatory if*

$$\limsup_{t\to\infty} g(t) \int_t^\infty s^{n-2}\, p(s)\, ds < \frac{(n-2)!}{4}\,.$$

In a similar way as before, using Theorem E, we obtain the following three results.

Theorem 3.5. *Let* (ii$_1$)$-$(ii$_4$) *be satisfied.*

a) *Suppose that*

$$\limsup_{t\to\infty} \frac{R(g(t))}{R(t)} < \infty\,.$$

The equation (12) *is strongly oscillatory if and only if*

$$\limsup_{t\to\infty} R(t) \int_t^\infty [R(s)]^{n-2}\, p(s)\, ds = \infty\,,$$

and the equation (12) *is strongly nonoscillatory if and only if*

$$\lim_{t\to\infty} R(t) \int_t^\infty [R(s)]^{n-2}\, p(s)\, ds = 0\,.$$

b) *The equation* (12) *is oscillatory if*

$$\int^\infty [R(t)]^{n-2}\, p(t)\, dt = \infty \tag{23}$$

or if (23) *fails to hold but one of the following inequalities does*:

$$\limsup_{t\to\infty} R(t) \int_t^\infty [R(s)]^{n-2}\, p(s)\, ds > (n-1)!\,,$$

$$\liminf_{t\to\infty} R(t) \int_t^\infty [R(s)]^{n-2}\, p(s)\, ds > \frac{(n-1)!}{4}\,.$$

c) *The equation* (12) *is nonoscillatory if*

$$\limsup_{t\to\infty} R(g(t)) \int_t^\infty [R(s)]^{n-2}\, p(s)\, ds < \frac{(n-2)!}{4}\,.$$

For the class of nonlinear differential equations we have several remarkable papers concerning oscillation and asymptotic behavior of solutions of differential equations of the form

$$y^{(n)}(t) + p(t) f(y(g(t))) = 0\,. \tag{24}$$

Here we extend some results from the equation (24) to the n-th order equation of the form

$$(r(t) \dots (r(t)(r(t)\, y'(t))')' \dots)' + p(t) f(y(g(t))) = 0\,, \quad t \geqq t_0\,. \tag{25}$$

For the sake of brevity we shall say that the equation (25) ((24)) has the property (C) if every solution $y(t)$ of (25) ((24)) is either oscillatory or such that

$$R^i(t) \frac{d^i y(t)}{dR^i} \to 0 \quad \text{as} \quad t \to \infty \quad (i = 0, 1, \dots, n-1)$$

$$(t^i\, y^{(i)}(t) \to 0 \text{ as } t \to \infty \ (i = 0, 1, \dots, n-1))\,.$$

Consider the equation (25), where

(iii$_1$) $n \geqq 2$,
(iii$_2$) $r \in C([t_0, \infty))$, $r(t) > 0$, $R(t) \to \infty$ as $t \to \infty$,
(iii$_3$) $p \in C([t_0, \infty))$, $p(t) \geqq 0$,
(iii$_4$) $g \in C([t_0, \infty))$, $g(t) \leqq t$, $g(t) \to \infty$ as $t \to \infty$,
(iii$_5$) $f \in C(\mathbb{R})$, $x f(x) > 0$ if $x \neq 0$.

As we already know the equation (25) is just another form of the equation

$$\frac{d^n y(t)}{dR^n} + r(t)\, p(t) f(y(g(t))) = 0\,, \quad t \geqq t_0 \tag{26}$$

and the v-transformation of (26) with $v = \Phi$ leads to the equation

$$y^{(n)}(t) + r(\Phi(t))\, p(\Phi(t)) f(y(R(g(\Phi(t))))) = 0\,, \tag{27}$$

which we shall consider for $t \in [t_1, \infty)$, where t_1 ($\geqq 0$) si such that $g(\Phi(t)) \geqq t_0$ if $t \geqq t_1$.

In the sequel we shall use the following notation:

$\mathbb{R}_\alpha = (-\infty, -\alpha] \cup [\alpha, \infty)$, $\alpha \geqq 0$;
$C^*(\mathbb{R}_\alpha) = \{f$ with the property (iii$_5$) $|f$ is of bounded variation on every $[a, b] \subset \subset \mathbb{R}_\alpha\}$.

Lemma 3.2. (Lemma 4 in [6].) *Suppose that f has the property* (iii$_5$). *Then $f \in C^*(\mathbb{R}_\alpha)$ if and only if $f(x) = q(x)\, h(x)$ for all $x \in \mathbb{R}_\alpha$, where q: $\mathbb{R}_\alpha \to (0, \infty)$ is nondecreasing on $(-\infty, -\alpha]$ and nonincreasing on $[\alpha, \infty)$, and h: $\mathbb{R}_\alpha \to \mathbb{R}$ is nondecreasing in $\mathbb{R}_\alpha$.*

Definition 3.3. The function h in Lemma 3.2 will be called the *nondecreasing component of f* while q will be called the *positive component of f*.

Following W. E. Mahfoud [6] we define

$C_I(\mathbb{R}_\alpha) = \{f \in C^*(\mathbb{R}_\alpha) | f$ has a positive component bounded away from zero$\}$

and

$C_D(\mathbb{R}_\alpha) = \{f \in C^*(\mathbb{R}_\alpha) | f$ has a bounded nondecreasing component$\}$.

We are now in a position to introduce several results proved in [6] and then state their extension.

Theorem F. *Let* (iii_1), $(\mathrm{iii}_3)-(\mathrm{iii}_5)$ *be satisfied.*

$\mathrm{F}_1)$ *Let*

$$\liminf_{y \to \pm\infty} |f(y)| > 0 .$$

If $\int^\infty p(t)\,\mathrm{d}t = \infty$, *then for n even* (24) *is oscillatory, while for n odd it has the property* (C).

$\mathrm{F}_2)$ *If* $\int^\infty t^{n-1}\,p(t)\,\mathrm{d}t = \infty$, *then for n even every bounded solution of* (24) *is oscillatory, while for n odd either every bounded solution* $y(t)$ *of* (24) *is oscillatory or* $t^k\,y^{(k)}(t) \to 0$ *as* $t \to \infty$, $k = 0, 1, \ldots, n-1$.

$\mathrm{F}_3)$ *Let* $f \in C_I(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$.
If

$$\int^\infty t^i\,p(t)\,f[\pm c g^{n-i-2}(t)]\,\mathrm{d}t = \pm\infty$$

for every $c > 0$ *and every* $i \in \{0, 1, \ldots, n-2\}$, *then for n even* (24) *is oscillatory, while for n odd it has the property* (C).

$\mathrm{F}_4)$ *Let* $f \in C_I(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$. *For every solution* $y(t)$ *of* (24), *either* $y(t)$ *is oscillatory or* $y^{(n-1)}(t) \to 0$ *as* $t \to \infty$ *if and only if*

$$\int^\infty p(t)\,f[\pm c g^{n-1}(t)]\,\mathrm{d}t = \pm\infty \quad \textit{for every} \quad c > 0 . \tag{28}$$

$\mathrm{F}_5)$ *Let* $f \in C_I(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$ *and let f be bounded above or below.*
For n even, (24) *is oscillatory if and only if* (28) *holds.*
For n odd, (24) *has the property* (C) *if and only if* (28) *holds.*

$\mathrm{F}_6)$ *Let* $f \in C_I(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$ *and*

$$\liminf_{t \to \infty} \frac{g(t)}{t} > 0 .$$

For every solution $y(t)$ *of* (24), *either* $y(t)$ *is oscillatory or* $y^{(n-1)}(t) \to 0$ *as* $t \to \infty$ *if and only if*

$$\int^\infty p(t)\,f[\pm c t^{n-1}]\,\mathrm{d}t = \pm\infty \quad \textit{for every} \quad c > 0 .$$

$\mathrm{F}_7)$ *Let* $f \in C^*(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$. *If*

$$\int^\infty p(t)\,q[\pm c g^{n-1}(t)]\,\mathrm{d}t = \infty$$

for every $c > 0$ *and for some positive component q of f, then for n even* (24) *is oscillatory, while for n odd* (24) *has the property* (C).

$\mathrm{F}_8)$ *Let* $f \in C_D(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$.
For n even, (24) *is oscillatory if and only if* (28) *holds.*
For n odd, (24) *has the property* (C) *if and only if* (28) *holds.*

Theorem 3.6. *Let* $(\mathrm{iii}_1)-(\mathrm{iii}_5)$ *be satisfied.*

a) *Let*

$$\liminf_{t \to \pm\infty} |f(y)| > 0 .$$

If $\int^\infty p(t)\,\mathrm{d}t = \infty$ *then for n even* (25) *is oscillatory, while for n odd it has the property* (C).

b) *If*

$$\int^\infty R^{n-1}(t)\,p(t)\,\mathrm{d}t = \infty\,,$$

then for n even every bounded solution of (25) *is oscillatory, while for n odd every bounded solution* $u(t)$ *of* (25) *is either oscillatory or*

$$R^k(t)\,\frac{\mathrm{d}^k u(t)}{\mathrm{d}R^k} \to 0 \quad as \quad t \to \infty\,, \quad k = 0, 1, \ldots, n-1\,.$$

c) *Let* $f \in C_I(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$. *If*

$$\int^\infty R^i(t)\,p(t)\,f[\pm c(R(g(t)))^{n-i-2}]\,\mathrm{d}t = \pm\infty$$

for every $c > 0$ *and every* $i \in \{0, 1, \ldots, n-2\}$ *then for n even* (25) *is oscillatory while for n odd it has the property* (C).

d) *Let* $f \in C_I(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$. *For every solution* $u(t)$ *of* (25), *either* $u(t)$ *is oscillatory or*

$$\frac{\mathrm{d}^{n-1}u(t)}{\mathrm{d}R^{n-1}} \to 0 \quad as \quad t \to \infty$$

if and only if

(29) $$\int^\infty p(t)\,f[\pm c(R(g(t)))^{n-1}]\,\mathrm{d}t = \pm\infty \quad for\ every \quad c > 0\,.$$

e) *Let* $f \in C_I(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$ *and let f be bounded above or below.*
For n even, (25) *is oscillatory if and only if* (29) *holds.*
For n odd, (25) *has the property* (C) *if and only if* (29) *holds.*

f) *Let* $f \in C_I(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$ *and*

$$\liminf_{t\to\infty} \frac{R(g(t))}{R(t)} > 0\,.$$

For every solution $u(t)$ *of* (25), *either* $u(t)$ *is oscillatory or*

$$\frac{\mathrm{d}^{n-1}u(t)}{\mathrm{d}R^{n-1}} \to 0 \quad as \quad t \to \infty$$

if and only if

$$\int^\infty p(t)\,f(\pm cR^{n-1}(t))\,\mathrm{d}t = \pm\infty \quad for\ every \quad c > 0\,.$$

g) *Let* $f \in C^*(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$. *If*

$$\int^\infty p(t)\,q[\pm c(R(g(t)))^{n-1}]\,\mathrm{d}t = \infty$$

for every $c > 0$ *and for some positive component q of f, then for n even* (25) *is oscillatory, while for n odd it has the property* (C).

h) *Let* $f \in C_D(\mathbb{R}_\alpha)$ *for some* $\alpha > 0$.
For n even, (25) *is oscillatory if and only if* (29) *holds.*
For n odd, (25) *has the property* (C) *if and only if* (29) *holds.*

Proof of Theorem 3.6, part c). Since

$$\int^{\infty} R^i(t)\, p(t) f[\pm c(R(g(t)))^{n-i-2}]\, \mathrm{d}t =$$
$$= \int^{\infty} t^i r(\Phi(t))\, p(\Phi(t)) f[\pm c(R(g(\Phi(t))))^{n-i-2}]\, \mathrm{d}t$$

thus, according to Theorem F, part F_3), we see that under the assumptions of Theorem 3.6, part c), for n even (27) is oscillatory while for n odd it has the property (C). We already know that if $y(t)$ is an oscillatory solution of (27) then $y(R(t))$ is the oscillatory solution of (25) and if $y(t)$ is a solution of (27) with the property

$$t^k\, y^{(k)}(t) \to 0 \quad \text{as} \quad t \to \infty\,, \quad k = 0, 1, \ldots, n-1\,,$$

so $u(t) = y(R(t))$ is the solution of (25) with the property

$$R^k(t) \frac{\mathrm{d}^k u(t)}{\mathrm{d}R^k} = R^k(t)\, y^{(k)}(R(t)) \to 0 \quad \text{as} \quad t \to \infty\,, \quad k = 0, 1, \ldots, n-1\,.$$

Thus, using Theorem 2.1 we prove Theorem 3.6, part c). The other parts of Theorem 3.6 can be proved similarly.

It is easy to see that many other results concerning the oscillatory and asymptotic behavior of solutions of differential equations of the type (24) can be generalized to the equation of the type (25).

4. AN EXTENSION OF THE PREVIOUS RESULTS

Here we shall show that the results obtained in Section 3 of this paper for equations of the form

$$(r(t) \ldots (r(t)\,(r(t)\, y'(t))')' \ldots)' + p(t) f(y(g(t))) = 0$$

may be extended to equations of the form

$$(r_{n-1}(t) \ldots (r_2(t)\,(r_1(t)\, y'(t))')' \ldots)' + p(t) f(y(g(t))) = 0$$

by using suitable comparison theorems.

Thus, for instance, in [3] T. A. Čanturija has presented two comparison theorems which we can formulate as follows.

Theorem G. *Let* $n \geqq 3$; $p_i, q_i \in C([0, \infty))$, $p_i(t) > 0$, $q_i(t) > 0$; $f \in C([0, \infty) \times R)$, $h \in C([0, \infty) \times R)$.

G_1) *Let*

(1) $$q_i(t) \geqq p_i(t) \quad \text{for} \quad t \in [0, \infty)\,, \quad (i = 1, 2, \ldots, n-1)\,,$$

(2) $$\int_0^{\infty} \frac{\mathrm{d}t}{q_i(t)} = \infty \quad (i = 1, 2, \ldots, n-1)\,,$$

$$-f(t, u) \operatorname{sgn} u \geqq -h(t, u) \operatorname{sgn} u \geqq 0 \quad \text{for} \quad t \in [0, \infty)\,, \quad u \in R\,.$$

Let the function $-h(t, u)$ *be nondecreasing in* u. *Then the equation*

(3) $$(p_{n-1}(t) \ldots (p_2(t)\,(p_1(t)\, u'(t))')' \ldots)' = f(t, u(t))$$

has the property (A) *if the equation*

$$(4) \qquad (q_{n-1}(t) \dots (q_2(t)(q_1(t)\, v'(t))')' \dots)' = h(t, v(t))$$

has the property (A).

G$_2$) *Let the conditions* (1), (2) *be satisfied and*

$$f(t, u) \operatorname{sgn} u \geqq h(t, u) \operatorname{sgn} u \geqq 0 \quad \textit{for} \quad t \in [0, \infty), \quad u \in \mathbb{R}.$$

Let the function $h(t, u)$ *be nondecreasing in* u. *Then the equation* (3) *has the property* (B) *if the equation* (4) *has the property* (B).

Now we are ready to introduce several examples of extensions of the previous results.

Consider the equation

$$(5) \qquad (r_{n-1}(t) \dots (r_2(t)(r_1(t)\, y'(t))')' \dots)' + p(t)\, y(t) = 0,$$

where

(iv$_1$) $n \geqq 3$,
(iv$_2$) $r_i \in C([t_0, \infty))$, $r_i(t) > 0$ $(i = 1, 2, \dots, n-1)$,
(iv$_3$) $p \in C([t_0, \infty))$.

From Theorem 3.1 and Theorem G we have the following result.

Theorem 4.1. *Let* (iv$_1$), (iv$_2$), (iv$_3$) *be satisfied. Let*

$$(6) \qquad r \in C([t_0, \infty)), \quad r(t) \geqq \max_{i=1,2,\dots,n-1} r_i(t) \quad \textit{for } t \in [t_0\ \infty),$$

and

$$(7) \qquad R(t) = \int_{t_0}^{t} \frac{ds}{r(s)} \to \infty \quad \textit{as} \quad t \to \infty.$$

a) *Suppose* $p(t) \geqq 0$ *and* $\liminf_{t\to\infty} r(t)\, p(t) > 0$. *Then the equation* (5) *has the property* (A).

b) *Suppose* $p(t) \leqq 0$ *and* $\limsup_{t\to\infty} r(t)\, p(t) < 0$. *Then the equation* (5) *has the property* (B).

In a similar way, using successively parts a), b), c), d) of Theorem 3.2 and Theorem G, we obtain the following results.

Theorem 4.2. *Let the conditions* (iv$_1$)–(iv$_3$), (6) *and* (7) *be satisfied.*

a) *Let* $p(t) \geqq 0$ *and*

$$\liminf_{t\to\infty} [R(t)]^{n-1} \int_t^\infty p(s)\, ds > \frac{M}{n-1},$$

where M *has the same meaning as in Theorem* 3.2. *Then the equation* (5) *has the property* (A).

b) *Let* $p(t) \leqq 0$ *and*

$$\liminf_{t\to\infty} [R(t)]^{n-1} \int_t^\infty |p(s)|\, ds > \frac{K}{n-1},$$

where K has the same meaning as in Theorem 3.2. *Then the equation* (5) *has the property* (B).

c) *Let* $p(t) \geqq 0$ *and*

$$\limsup_{t \to \infty} R(t) \int_t^\infty [R(s)]^{n-2} p(s) \, ds > (n-1)! .$$

Then the equation (5) *has the property* (A).

d) *Let* $p(t) \leqq 0$. *Let n be odd* (*n even*) *and*

$$\limsup_{t \to \infty} R(t) \int_t^\infty [R(s)]^{n-2} |p(s)| \, ds > (n-1)! \quad (> 2(n-2)!) .$$

Then the equation (5) *has the property* (B).

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