Czechoslovak Mathematical Journal

Dimitrios A. Kandilakis; Nikolaos S. Papageorgiou

On the properties of the Aumann integral with applications to differential inclusions and control systems

Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 1, 1–15

Persistent URL: `http://dml.cz/dmlcz/102274`

CZECHOSLOVAK MATHEMATICAL JOURNAL

Mathematical Institute of the Czechoslovak Academy of Sciences

V. *39* (*114*), Praha 30. 3. 1989, No 1

ON THE PROPERTIES OF THE AUMANN INTEGRAL WITH APPLICATIONS TO DIFFERENTIAL INCLUSIONS AND CONTROL SYSTEMS

DIMITRIOS A. KANDILAKIS and NIKOLAOS S. PAPAGEORGIOU*), Davis

(Received April 3, 1986)

1. INTRODUCTION

In the past, several authors have defined integrals for set valued functions (multifunctions). Let us mention Aumann [3], Debreu [17] and Artstein-Burns [2]. Aumann's integral was defined using integrable selectors, while Debreu considered compact and convex valued multifunctions and so using Radström's imbedding theorem he viewed the set valued integral as a Bochner integral. Finally Artstein and Burns following an idea due to Kurzweil [26] [27], used a Riemann type definition. Very recently Jarník and Kurzweil [23], improved the definition of Artstein and Burns and provided applications to differential inclusions.

The purpose of this paper is to study some properties of the Aumann integral for Banach space valued multifunctions and to provide applications in control theory and differential inclusions. In particular we obtain results closely related to the well known "bang-bang principle" of control theory, for infinite dimensional systems.

2. DEFINITIONS AND PRELIMINARIES

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will use the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{nonempty, closed, (convex)}\}$$

$$P_{(w)k(c)}(X) = \{A \subseteq X\colon \text{nonempty, } (w\text{-})\text{compact, (convex)}\}\,.$$

For $A \in 2^X \setminus \{\emptyset\}$, the norm $|A|$ and the support function $\sigma_A(\cdot)$ of A are defined by:

$$|A| = \sup_{x \in A} \|x\| \quad \text{and} \quad \sigma_A(x^*) = \sup_{x \in A} (x^*, x)\,, \quad x^* \in X^*\,.$$

Also $h(\cdot, \cdot)$ will denote the Hausdorff distance of sets.

*) Research supported by N.S.F. Grant D.M.S. 8403135.

A multifunction $F: \Omega \to P_f(X)$ is said to be *measurable* if it satisfies one of the following two equivalent conditions:

(i) $\omega \to d_{F(\omega)}(x) = \inf_{z \in F(\omega)} \|x - z\|$ is measurable for all $x \in X$.

(ii) there exist $\{f_n(\cdot)\}_{n \geq 1}$ measurable selectors of $F(\cdot)$ s.t. for all $\omega \in \Omega$

$$F(\omega) = \operatorname{cl} \{f_n(\omega)\}_{n \geq 1} \quad \text{(Castaing representation)}\,.$$

If there exists a complete σ-finite measure $\mu(\cdot)$ on Σ, then (i) and (ii) above are equivalent to:

(iii) $\operatorname{Gr}F = \{(\omega, x) \in \Omega \times X: x \in F(\omega)\} \in \Sigma \times B(X)$ (graph measurable). (Here $B(X)$ denotes the Borel σ-field of X).

Further results on the theory of measurable multifunctions can be found in: Castaing-Valadier [7], Himmelberg [21], Rockafellar [32] and Wagner [35].

For any multifunction $F: \Omega \to 2^X \setminus \{\emptyset\}$, let

$$S_F^1 = \{f(\cdot) \in L_X^1(\Omega): f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}\,.$$

If $F(\cdot)$ is graph measurable then S_F^1 is nonempty if and only if $\inf_{x \in F(\omega)} \|x\| \in L_+^1(\Omega)$. Also if $F(\cdot)$ is closed valued, then it is easy to see that S_F^1 is a strongly closed subset of $L_X^1(\Omega)$. Using this set we can define an integral for the multifunction $F(\cdot)$. So let

$$\int_\Omega F(\omega)\, d\mu(\omega) = \{\int_\Omega f(\omega)\, d\mu(\omega): f(\cdot) \in S_F^1\}$$

where $\int_\Omega f(\omega)\, d\mu(\omega)$ is the usual Bochner integral. This multivalued integral is known as Aumann's integral.

We will say that $F: \Omega \to P_f(X)$ is *integrably bounded* if it is measurable and $|F(\cdot)| \in L_+^1$.

Finally suppose that Y, Z are Hausdorff topological spaces and $F: Y \to 2^Z \setminus \{\emptyset\}$. We say that $F(\cdot)$ is u.s.c. (resp. l.s.c.) if for all $V =$ open subset of Z, the set $\{y \in Y: F(y) \subseteq V\}$ (resp. $\{y \in Y: F(y) \cap V \neq \emptyset\}$) is open. If $F(\cdot)$ is both u.s.c. and l.s.c. then is said to *be continuous*.

We will close this section with two general results about measurable multifunctions that we will need in the sequel.

The first is a result about weak compactness in the Lebesgue-Bochner space $L_X^1(\Omega)$ and was first obtained by the second author in [28].

Theorem 2.1 [28]. *If $F: \Omega \to P_{wkc}(X)$ is integrably bounded then S_F^1 is a nonempty, convex, w-compact subset of $L_X^1(\Omega)$.*

Remarks. (1) If (Ω, Σ, μ) is nonatomic and X^* is separable too, then there is a converse to that theorem (see theorem 4.2 in [29]).

(3) An immediate interesting consequence of the theorem is that $\int_\Omega F(\omega)\, d\mu(\omega) \in P_{wkc}(X)$.

The second result is a rather folklore theorem for people working on multifunctions. Here we state it and prove it in the most general possible form for Banach space valued multifunctions. For a similar result for multifunctions taking values in a Souslin localy convex space look at [30] (theorem 4.1). So let (Ω, Σ, μ) be a σ-finite measure space with Σ being a Souslin family (see Wagner [35]) and X a separable Banach space.

Theorem 2.2. *If* $F: \Omega \to 2^X \setminus \{\emptyset\}$ *is graph measurable and* $S_F^1 \neq \emptyset$ *then for all* $x^* \in X^*$, $\sigma_{\int_\Omega F}(x^*) = \int_\Omega \sigma_{F(\omega)}(x^*) \, \mathrm{d}\mu(\omega)$.

Proof. Our proof follows Rockafellar [32] (theorem 3A).

From theorem 5.10 of Wagner [35] we know that $F(\cdot)$ admits a Castaing representation $\{f_n(\cdot)\}_{n \geqq 1}$. Hence:

$$\sigma_{F(\omega)}(x^*) = \sup_{n \geqq 1} (x^*, f_n(\omega)) \Rightarrow \omega \to \sigma_{F(\omega)}(x^*) \quad \text{is measurable}.$$

Also since for $f(\cdot) \in S_F^1 (x^*, f(\omega)) \leqq \sigma_{F(\omega)}(x^*)$ μ-a.e. and $(x^*, f(\cdot)) \in L^1$ we deduce that for all $x^* \in X^*$, $\omega \to \sigma_{F(\omega)}(x^*)$ is quasintegrable.

Directly from the definitions it is easy to see that we always have:

$$\sigma_{\int_\Omega F}(x^*) \leqq \int_\Omega \sigma_{F(\omega)}(x^*) \, \mathrm{d}\mu(\omega).$$

Fix $x^* \in X^*$ and let $\beta < \int_\Omega \sigma_{F(\omega)}(x^*) \, \mathrm{d}\mu(\omega)$. Our goal is to show that there exists $\hat{f}(\cdot) \in S_F^1$ s.t.

$$\beta < \int_\Omega (x^*, \hat{f}(\omega)) \, \mathrm{d}\mu(\omega).$$

Take $\{\Omega_n\}_{n \geqq 1} \subseteq \Sigma$ s.t. $\mu(\Omega_n) < \infty$ and $\Omega = \bigcup_{n \geqq 1} \Omega_n$ and let $p(\cdot) \in L_+^1$, $p(\omega) > 0$ for all $\omega \in \Omega$. For $n \geqq 1$ define

$$A_n = \{\omega \in \Omega: \sigma_{F(\omega)}(x^*) \leqq n\} \cap \Omega_n.$$

Then set:

$$g_n(\omega) = \sigma_{F(\omega)}(x^*) - \frac{1}{n} p(\omega) \quad \text{for} \quad \omega \in A_n$$

$$= (x^*, f(\omega)) - \frac{1}{n} p(\omega) \quad \text{for} \quad \omega \in \Omega \setminus A_n.$$

Clearly $g_n(\cdot) \in L^1(\Omega)$ and $g_n(\omega) \uparrow \sigma_{F(\omega)}(x^*)$ μ-a.e. So an application of the monotone convergence theorem tells us that for all $n \geqq n_0$, $\int_\Omega g_n(\omega) \, \mathrm{d}\mu(\omega) > \beta$. Set $g(\cdot) = g_{n_0}(\cdot)$. Let $R(\omega) = \{x \in F(\omega): g(\omega) \leqq (x^*, x)\}$. Because $g(\omega) < \sigma_{F(\omega)}(x^*)$ μ-a.e., we see that for all $\omega \in \Omega$, $R(\omega) \neq \emptyset$. Also $\mathrm{Gr}R = \{(\omega, x) \in \Omega \times L: g(\omega) - (x^*, x) \leqq 0\} \cap \mathrm{Gr}F \in \Sigma \times B(X)$. Hence we can apply Aumann's selection theorem to find $h: \Omega \to X$ measurable s.t. $h(\omega) \in R(\omega)$ μ-a.e. Clearly $h(\cdot)$ is μ-a.e. a measurable selector of $F(\cdot)$, but it is not necessarily in the space $L_X^1(\Omega)$. Furthermore note that:

$$\beta < \int_\Omega (x^*, h(\omega)) \, \mathrm{d}\mu(\omega).$$

Next let $B_n = \{\omega \in \Omega: \|h(\omega)\| \leqq n\} \cap A_n$ and define

$$\hat{f}_n(\cdot) = \chi_{B_n} h(\cdot) + \chi_{\Omega \setminus B_n} f(\cdot).$$

Clearly $\{\hat{f}_n(\cdot)\}_{n \geqq 1} \subseteq S_F^1$ and

$$\int_\Omega (x^*, \hat{f}_n(\omega)) \, \mathrm{d}\mu(\omega) = \int_{B_n} (x^*, h(\omega)) \, \mathrm{d}\mu(\omega) + \int_{\Omega \setminus B_n} (x^*, f(\omega)) \, \mathrm{d}\mu(\omega) \geqq$$

$$\geqq \int_{B_n} g(\omega)\,\mathrm{d}\mu(\omega) + \int_{\Omega\setminus B_n} (x^*, f(\omega))\,\mathrm{d}\mu(\omega) =$$
$$= \int_\Omega g(\omega)\,\mathrm{d}\mu(\omega) + \int_{\Omega\setminus B_n} [x^*, f(\omega)) - g(\omega)]\,\mathrm{d}\mu(\omega)$$

Recalling that $\int_\Omega g(\omega)\,\mathrm{d}\mu(\omega) > \beta$ and that $\mu(\Omega \setminus B_n) \downarrow 0$ we finally have that for large enough n

$$\int_\Omega (x^*, \hat{f}_n(\omega))\,\mathrm{d}\mu(\omega) > \beta\,.$$

Since $\hat{f}_n(\cdot) \in S_F^1$, the proof is completed.

Q.E.D.

Remark. If $\mu(\cdot)$ is complete, then Σ is a Souslin family.

3. THE AUMANN INTEGRAL

One of the major results in the seminal paper of Aumann [3] was that if $F\colon \Omega \to P_f(\mathbb{R}^n)$ is integrably bounded, then $\int_\Omega F(\omega)\,\mathrm{d}\mu(\omega) = \int_\Omega \operatorname{conv} F(\omega)\,\mathrm{d}\mu(\omega)$. This result is a consequence of the well known Lyapunov's convexity theorem and has important applications in control theory (see Clarke [10], theorem 5.1.6, p. 115). In infinite dimensional Banach spaces, we know that Lyapunov's theorem holds only in an approximate form. So the above result of Aumann must be modified accordingly. Previous infinite dimensional versions of this result were obtained by Datko [13] and Hiai-Umegaki [20]. Our theorem extends and sharpens those results.

We will start by establishing the convexity of the Aumann integral. So let (Ω, Σ, μ) be a nonatomic, σ-finite measure space, with Σ being a Souslin family and X is a separable Banach space.

Proposition 3.1. *If $F\colon \Omega \to 2^X \setminus \{\emptyset\}$ is a graph measurable and $S_F^1 \neq \emptyset$ then* $\operatorname{cl} \int_\Omega F(\omega)\,\mathrm{d}\mu(\omega)$ *is convex.*

Proof. Let $x_1, x_2 \in \operatorname{cl} \int_\Omega F_1(\omega)\,\mathrm{d}\mu(\omega)$. Then given $\varepsilon > 0$ there exist $f_1(\cdot), f_2(\cdot) \in S_F^1$ s.t. $\|x_1 - \int_\Omega f_1(\omega)\,\mathrm{d}\mu(\omega)\| < \varepsilon/2$ and $\|x_2 - \int_\Omega f_2(\omega)\,\mathrm{d}\mu(\omega)\| < \varepsilon/2$.

Consider the vector valued measure $r\colon \Sigma \to X \times X$ defined by

$$r(A) = \left(\int_A f_1(\omega)\,\mathrm{d}\mu(\omega),\ \int_A f_2(\omega)\,\mathrm{d}\mu(\omega)\right).$$

Because $\mu(\cdot)$ is nonatomic, corollary 1 (p. 98) of Kluvanek-Knowles [25], tells us that the norm closure of the range of $r(\cdot)$ is convex. Note that $r(\emptyset) = (0, 0)$ and $r(\Omega) = (\int_\Omega f_1(\omega)\,\mathrm{d}\mu(\omega), \int_\Omega f_2(\omega)\,\mathrm{d}\mu(\omega))$. Hence for $\lambda \in (0, 1)$ there exists $A \in \Sigma$ s.t.

$$\|r(A) - \lambda\, r(\Omega)\| < \varepsilon/4 \quad \text{and} \quad \|r(\Omega \setminus A) - (1 - \lambda)\, r(\Omega)\| < \varepsilon/4 \Rightarrow$$
$$\Rightarrow \left\|\int_A f_i(\omega)\,\mathrm{d}\mu(\omega) - \lambda \int_\Omega f_i(\omega)\,\mathrm{d}\mu(\omega)\right\| < \varepsilon/4$$

and

$$\left\|\int_{\Omega\setminus A} f_i(\omega)\,\mathrm{d}\mu(\omega) - (1 - \lambda) \int_\Omega f_i(\omega)\,\mathrm{d}\mu(\omega)\right\| < \varepsilon/4$$

for $i = 1, 2$. Set $f = \chi_A f_1 + \chi_{\Omega\setminus A} f_2$. Clearly $f(\cdot) \in S_F^1$. Then we have:

$$\left\|\lambda x_1 + (1 - \lambda)\, x_2 - \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega)\right\| \leqq$$

$$\leqq \|\lambda x_1 - \lambda \int_\Omega f_1(\omega)\, d\mu(\omega)\| + \|\lambda \int_\Omega f_1(\omega)\, d\mu(\omega) - \int_A f_1(\omega)\, d\mu(\omega)\| +$$
$$+ \|(1-\lambda)\, x_2 - (1-\lambda) \int_\Omega f_2(\omega)\, d\mu(\omega)\| +$$
$$+ \|(1-\lambda) \int_\Omega f_2(\omega)\, d\mu(\omega) - \int_{\Omega \setminus A} f_2(\omega)\, d\mu(\omega)\| <$$
$$< \lambda\varepsilon/2 + \varepsilon/4 + (1-\lambda)\, \varepsilon/2 + \varepsilon/4 = \varepsilon\,.$$

This proves that indeed $\mathrm{cl} \int_\Omega F(\omega)\, d\mu(\omega)$ is convex.

Q.E.D.

Now we are ready for the infinite dimensional generalization of Aumann's theorem. Let (Ω, Σ, μ) be a nonatomic complete, σ-finite measure space while X is still a separable Banach space.

Theorem 3.1. *If $F\colon \Omega \to 2^X \setminus \{\emptyset\}$ is graph measurable and $S_F^1 \neq \emptyset$ then* $\mathrm{cl} \int_\Omega F(\omega)\, d\mu(\omega) = \mathrm{cl} \int_\Omega \mathrm{cl\,conv}\, F(\omega)\, d\mu(\omega)$.

Proof. Using theorem 2.2 we have for all $x^* \in X^*$

$$\sigma_{\mathrm{cl} \int_\Omega F}(x^*) = \sigma_{\int_\Omega F}(x^*) = \int_\Omega \sigma_{F(\omega)}(x^*)\, d\mu(\omega) = \int_\Omega \sigma_{\mathrm{cl\,conv}\, F(\omega)}(x^*)\, d\mu(\omega)\,.$$

Using theorem 4.4 of Debreu [17] (see also theorem 3.4 of Himmelberg [21]) we have that $\omega \to \mathrm{cl\,conv}\, F(\omega)$ is measurable. So a new application of theorem 2.2 gives us:

$$\int_\Omega \sigma_{\mathrm{cl\,conv}\, F(\omega)}(x^*)\, d\mu(\omega) = \sigma_{\int_\Omega \mathrm{cl\,conv}\, F}(x^*) \Rightarrow \sigma_{\mathrm{cl} \int_\Omega F}(x^*) = \sigma_{\mathrm{cl} \int_\Omega \mathrm{cl\,conv}\, F}(x^*)$$

for all $x^* \in X^*$. But from proposition 3.1 we know that $\mathrm{cl} \int_\Omega F$ is convex. So finally we have:

$$\mathrm{cl} \int_\Omega F(\omega)\, d\mu(\omega) = \mathrm{cl} \int_\Omega \mathrm{cl\,conv}\, F(\omega)\, d\mu(\omega)\,.$$

Q.E.D.

This leads us to the first "bang-bang" type result concerning the Aumann integral. By $\mathrm{ext}\, A$ we will denote the extreme points of the set $A \subseteq X$. As before (Ω, Σ, μ) is a nonatomic complete, σ-finite measure space and X a separable Banach space.

Theorem 3.2. *If $F\colon \Omega \to P_{wk}(X)$ is measurable and S_F^1, $S_{\mathrm{ext}\, F}^1$ are nonempty*

$$\textit{then}\ \mathrm{cl} \int_\Omega F(\omega)\, d\mu(\omega) = \mathrm{cl} \int_\Omega \mathrm{ext}\, F(\omega)\, d\mu(\omega)\,.$$

Proof. From the Krein-Milman theorem we know that for all $\omega \in \Omega$ we have $\mathrm{cl\,conv}\, F(\omega) = \mathrm{cl\,conv\,ext}\, F(\omega)$. From Benamara [4] we know that $\omega \to \mathrm{ext}\, F(\omega)$ is a graph measurable multifunction. So applying theorem 3.1 twice we get

$$\mathrm{cl} \int_\Omega F(\omega)\, d\mu(\omega) = \mathrm{cl} \int_\Omega \mathrm{ext}\, F(\omega)\, d\mu(\omega)\,.$$

Q.E.D.

Remarks. (1) If $F(\cdot)$ is in addition convex valued and integrably bounded then the above result together with theorem 2.1 tell us that $\int_\Omega F(\omega)\, d\mu(\omega) = \mathrm{cl} \int_\Omega \mathrm{ext}\, F(\omega)\, d\mu(\omega)$. Also in this case the hypothesis S_F^1, $S_{\mathrm{ext}\, F}^1 \neq \emptyset$ is automatically satisfied.

(2) If X has the Radon-Nikodym property (R.N.P.), we can assume that $F(\cdot)$ has

only nonempty, closed, bounded and convex values (see Diestel-Uhl [18], p. 218).

Motivated from control theory we make the following definition.

We will say that $g\colon \Omega \to X$ is a *"bang-bang" function for* $F(\cdot)$ if and only if $g(\cdot) \in S_{\mathrm{ext}\,F} = \{$set of all measurable selectors of $\omega \to \mathrm{ext}\, F(\omega)\}$.

Using theorem 3.2 we can prove the following interesting density property that "bang-bang" functions have. The spaces (Ω, Σ, μ) and X remain as before.

Proposition 3.2. *If* $F\colon \Omega \to P_{wkc}(X)$ *is measurable,* $S_F^1, S_{\mathrm{ext}\,F}^1$ *are nonempty,* $f(\cdot) \in S_F^1$ *and* $\varepsilon > 0$

then there exists a bang-bang function $g(\cdot)$ *for* $F(\cdot)$ *s.t.*

$$\left\|\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) - \int_\Omega g(\omega)\,\mathrm{d}\mu(\omega)\right\| < \varepsilon\,.$$

Proof. From theorem 3.2 we know that $\mathrm{cl} \int_\Omega F(\omega)\,\mathrm{d}\mu(\omega) = \mathrm{cl} \int_\Omega \mathrm{ext}\, F(\omega)\,\mathrm{d}\mu(\omega) \Rightarrow$ $\Rightarrow \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) \in \mathrm{cl} \int_\Omega \mathrm{ext}\, F(\omega)\,\mathrm{d}\mu(\omega)$. So we can find $g(\cdot) \in S_{\mathrm{ext}\,F}^1$ s.t.

$$\left\|\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) - \int_\Omega g(\omega)\,\mathrm{d}\mu(\omega)\right\| < \varepsilon\,.$$

Q.E.D.

An interesting particular case of this proposition, which often appears in applications, is the following. Suppose $\{f_k(\cdot)\}_{k=1}^n \subseteq L_X^1(\Omega)$.

Proposition 3.3. *If* $f\colon \Omega \to X$ *is a measurable function s.t.* $f(\omega) \in \mathrm{conv}\,\{f_k(\omega)\}_{k=1}^n$ *and* $\varepsilon > 0$

then there exists $\{A_k\}_{k=1}^n$ *a measurable partition of* Ω *s.t.*

$$\left\|\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) - \sum_{k=1}^n \int_{A_k} f_k(\omega)\,\mathrm{d}\mu(\omega)\right\| < \varepsilon\,.$$

Proof. Consider the multifunction $F(\omega) = \mathrm{conv}\,\{f_k(\omega)\}_{k=1}^n$. It is easy to see that this is a $P_{kc}(X)$-valued multifunction which is integrably bounded. Furthermore for all $\omega \in \Omega$, $\mathrm{ext}\, F(\omega) \subseteq \{f_k(\omega)\}_{k=1}^n$ and also $f(\cdot) \in S_F^1$. So applying proposition 3.2 we get the desired result.

Q.E.D.

Another interesting consequence of proposition 3.2, is the following result, which has important applications in control theory and optimization. Our proposition generalizes theorem 3 of Cesari [9].

Proposition 3.4. *If* $\{f_k(\cdot)\}_{k=1}^n \subseteq L_X^1(\Omega)$

$$\textit{then } \mathrm{cl}\left\{\int_\Omega g(\omega)\,\mathrm{d}\mu(\omega)\colon g(\cdot) = \sum_{k=1}^n \chi_{A_k} f_k(\cdot),\ \{A_k\}_{k=1}^n \textit{ is a } \Sigma\textit{-partition of } \Omega\right\}$$

is a convex and weakly compact subset of X.

Remark. If X is finite dimensional, then the closure is redundant.

Actually we can say more about the bang-bang approximations. Assume that (Ω, Σ, μ) is a nonatomic, complete finite measure space and X a separable Banach space.

Theorem 3.3. *If $F: \Omega \to P_{wkc}(X)$ is integrably bounded and $g(\cdot) \in S_F^1$ then we can find $\{g_n(\cdot)\}_{n \geq 1}$ bang-bang functions for $F(\cdot)$ s.t.*

$$g_n(\cdot) \to^{w-L_X^1} g(\cdot) \quad as \quad n \to \infty .$$

Proof. We need to show that $\mathrm{cl}_w S^1_{\mathrm{ext}\,F} = S_F^1$. From theorem 2.1 we know that S_F^1 is w-compact in $L_X^1(\Omega)$. So $\mathrm{cl}_w S^1_{\mathrm{ext}\,F} \subseteq S_F^1$. Hence we need to show that given $f(\cdot) \in S_F^1$, $\{u_i(\cdot)\}_{i=1}^m \subseteq [L_X^1(\Omega)]^* = L^\infty_{X^*_{w^*}}(\Omega)$ and $\varepsilon > 0$ we can find $\hat{f}(\cdot) \in S^1_{\mathrm{ext}\,F}$ s.t. $|\langle u_i, f - \hat{f}\rangle| < \varepsilon$ for all $i = 1, \ldots, m$, where $\langle \cdot, \cdot \rangle$ indicates the duality brackets between L_X^1 and $L^\infty_{X^*_{w^*}} = (L_X^1)^*$. To this end let $R: \Omega \to P_{kc}(\mathbb{R}^m)$ be the multifunction defined by

$$R(\omega) = \{r = (u_i(\omega), z)_{i=1}^m \in \mathbb{R}^m : z \in F(\omega)\} .$$

A straightforward application of Castaing's representation theorem tells us that $R(\cdot)$ is measurable and clearly it is integrably bounded. Then using theorem 3.2 of this paper and theorem 4 of [3] we get that

$$\int_\Omega R(\omega)\, \mathrm{d}\mu(\omega) = \int_\Omega \mathrm{ext}\, R(\omega)\, \mathrm{d}\mu(\omega) .$$

So there exists $r(\cdot) \in S^1_{\mathrm{ext}\,R}$ s.t.

$$\int_\Omega (u_i(\omega), f(\omega))_{i=1}^m\, \mathrm{d}\mu(\omega) = \int_\Omega r(\omega)\, \mathrm{d}\mu(\omega) .$$

Consider the multifunction $M: \Omega \to 2^{\mathbb{R}^m} \setminus \{\emptyset\}$ defined by:

$$M(\omega) = \{x \in F(\omega) : (u_i(\omega), x)_{i=1}^m = r(\omega)\} .$$

Note that $(\omega, x) \to (u_i(\omega), x)_{i=1}^m - r(\omega)$ is a Caratheodory function from $\Omega \times X$ into $\mathbb{R}^m$ and so it is jointly measurable. Thus

$$\mathrm{Gr} M = \{(\omega, x) \in \Omega \times X : (u_i(\omega), x)_{i=1}^m - r(\omega) = 0\} \cap \mathrm{Gr} F \in \Sigma \times B(X) .$$

Apply Aumann's selection theorem to find $\hat{f}: \Omega \to X$ measurable s.t. $\hat{f}(\omega) \in M(\omega)$ for all $\omega \in \Omega$. Then clearly $\hat{f}(\cdot) \in S_F^1$ and

$$(u_i(\omega), \hat{f}(\omega))_{i=1}^m = r(\omega)$$

for all $\omega \in \Omega$. Our claim is that $\hat{f}(\cdot) \in \mathrm{ext}\, S_F^1$. From Benamara [5] we know that $\mathrm{ext}\, S_F^1 = S^1_{\mathrm{ext}\,F}$. So if $\hat{f}(\cdot) \notin \mathrm{ext}\, S_F^1$, then there exists $A \in \Sigma$ with $\mu(A) > 0$ s.t. $\hat{f}(\omega) \notin$ $\notin \mathrm{ext}\, F(\omega)$ for all $\omega \in A$. Now consider $G: A \to 2^{X \times X} \setminus \{\emptyset\}$ defined by

$$G(\omega) = \{(x, y) \in F(\omega) \times F(\omega) : \hat{f}(\omega) = \tfrac{1}{2}(x + y)\} .$$

The X-valued map $(\omega, x, y) \to \hat{f}(\omega) - \frac{1}{2}(x + y)$ is a Caratheodory map, so it is jointly measurable. Hence

$$\mathrm{Gr} G = \{(\omega, x, y) \in A \times X \times X : \hat{f}(\omega) - \tfrac{1}{2}(x + y) = 0\} \cap$$
$$\cap\, \mathrm{Gr}(F \times F) \in \Sigma_A \times B(X \times X) .$$

But $B(X \times X) = B(X) \times B(X)$. Therefore $\mathrm{Gr} G \in \Sigma_A \times B(X) \times B(X)$ (recall $\Sigma_A =$ $= \Sigma \cap A$). Apply Aumann's selection theorem to find $x, y: A \to X$ measurable s.t. for all $\omega \in A (x(\omega), y(\omega)) \in G(\omega) \Rightarrow \hat{f}(\omega) = \frac{1}{2}[x(\omega) + y(\omega)]$. But then for $\omega \in A$

we will have:

$$r(\omega) = \left(u_i(\omega), \tfrac{1}{2}(x(\omega) + y(\omega))\right)_{i=1}^m = \tfrac{1}{2}\left(u_i(\omega), x(\omega)\right)_{i=1}^m + \tfrac{1}{2}\left(u_i(\omega), y(\omega)\right)_{i=1}^m$$

and since $\left(u_i(\omega), x(\omega)\right)_{i=1}^m$ and $\left(u_i(\omega), y(\omega)\right)_{i=1}^m \in R(\omega)$ we have a contradiction to the fact that $r(\cdot) \in \operatorname{ext} S_R^1 = S_{\operatorname{ext} R}^1$. Therefore we deduce that $\hat{f}(\cdot) \in S_{\operatorname{ext} F}^1 = \operatorname{ext} S_F^1$ and $\hat{f}(\cdot) \in W_{u_1,\ldots,u_m}(f) = \{v(\cdot) \in L_X^1 : |\langle u_i, f - v\rangle| < \varepsilon,\ i = 1, \ldots, m\}$. Since $f(\cdot) \in S_F^1$, $\{u_i(\cdot)\}_{i=1}^m \subseteq L_{X^*, w^*}^\infty(\Omega)$ and $\varepsilon > 0$ were arbitrary, we conclude that $S_F^1 = \operatorname{cl}_w S_{\operatorname{ext} F}^1$. Since S_F^1 is w-compact, by the Eberlein-Smulian theorem is sequentially w-compact. So we can find $\{g_n(\cdot)\}_{n \geq 1} \subseteq S_{\operatorname{ext} F}^1$ s.t. $g_n(\cdot) \to^{w-Lx^1} g(\cdot)$ as $n \to \infty$.

Q.E.D.

A special case of the above result generalizes significantly theorem 2 of Datko [12].

Corollary I. *If* $\{f_k(\cdot)\}_{k=1}^n \subseteq L_X^1(\Omega)$ *and* $f(\omega) \in \operatorname{conv}\{f_k(\omega)\}_{k=1}^n$ μ-a.e. *then there exist functions* $g_m(\cdot) = \sum_{k=1}^n \chi_{A_{mk}} f_k(\cdot)$, *where* $\{A_{mk}\}_{k=1}^n$ $m \geqq 1$ *is a measurable partition of* Ω *s.t.* $g_m(\cdot) \to^{w-Lx^1} g(\cdot)$ *as* $m \to \infty$.

Remark. Even when we specialize to the case $X = \mathbb{R}^n$, this corollary is more general than theorem 2 of Datko, who required that the f_k's take values in a compact convex set K (i.e. for all $\omega \in \Omega$, $f_k(\omega) \in K$ for all $k = 1, \ldots, n$).

Another important byproduct of theorem 3.3 is the following:

Corollary II. *If* $F: \Omega \to P_{wkc}(X)$ *is integrably bounded*

$$\textit{then} \quad S_F^1 = \operatorname{cl}_w \operatorname{ext} S_F^1 = \operatorname{cl}_w S_{\operatorname{ext} F}^1 .$$

We have two more useful results concerning the extremal properties of the Aumann integral. So let (Ω, Σ, μ) be a complete, σ-finite measure space and X a separable Banach space.

Theorem 3.4. *If* $F: \Omega \to 2^X \setminus \{\emptyset\}$ *is graph measurable and* S_F^1, $S_{\operatorname{ext} F}^1$ *are nonempty*

$$\textit{then} \quad \operatorname{ext} \int_\Omega F(\omega)\, d\mu(\omega) \subseteq \int_\Omega \operatorname{ext} F(\omega)\, d\mu(\omega) .$$

Proof. Let $x \in \operatorname{ext} \int_\Omega F(\omega)\, d\mu(\omega)$. Then there exists $f(\cdot) \in S_F^1$ s.t. $x = \int_\Omega f(\omega)\, d\mu(\omega)$. Suppose $f(\omega) \notin \operatorname{ext} F(\omega)$ for all $\omega \in A \in \Sigma$ with $\mu(A) > 0$. Then this means that $f(\cdot) \notin S_{\operatorname{ext} F}^1 = \operatorname{ext} S_F^1$. Hence we can find $f_1(\cdot), f_2(\cdot) \in S_F^1$ s.t. $f = \frac{1}{2} f_1 + \frac{1}{2} f_2 \Rightarrow x = \frac{1}{2} \int_\Omega f_1(\omega)\, d\mu(\omega) + \frac{1}{2} \int_\Omega f_2(\omega)\, d\mu(\omega) \Rightarrow x = \frac{1}{2} x_1 + \frac{1}{2} x_2$ with $x_1, x_2 \in \int_\Omega F(\omega)\, d\mu(\omega)$, a contradiction to the choice of x.

Q.E.D.

Remark. The above inclusion may be strict as the following simple counter-example shows. Let $\Omega = [0, 1]$ with the Lebesgue measure $\lambda(\cdot)$. Let $F: \Omega \to P_{kc}(\mathbb{R})$ be defined by $F(\omega) = [0, 1]$ for all $\omega \in \Omega$. Then $\operatorname{ext} F(\omega) = \{0, 1\}$. Also

$$\int_\Omega F(\omega)\, d\lambda(\omega) = [0, 1] \Rightarrow \operatorname{ext} \int_\Omega F(\omega)\, d\lambda(\omega) = \{0, 1\}$$

while $\int_\Omega \operatorname{ext} F(\omega)\, d\lambda(\omega) = [0, 1]$.

Our final result on the bang-bang type properties of the Aumann integral is the following. Now (Ω, Σ, μ) is a complete, finite measure space and X a separable Banach space.

Theorem 3.5. *If $F_1, F_2 \colon \Omega \to P_{wkc}(X)$ are integrably bounded and*

$$x \in \int_\Omega \operatorname{ext} \{F_1(\omega) + F_2(\omega)] \, d\mu(\omega)$$

then there exist unique $f_1(\cdot), f_2(\cdot) \in L^1_X(\Omega)$ s.t. $f_1(\omega) \in \operatorname{ext} F_1(\omega)$ μ-a.e.

$$f_2(\omega) \in \operatorname{ext} F_2(\omega) \ \mu\text{-a.e. and } x = \int_\Omega (f_1(\omega) + f_2(\omega)) \, d\mu(\omega) .$$

Proof. By definition $x = \int_\Omega f(\omega) \, d\mu(\omega)$, where $f(\cdot) \in S^1_{\operatorname{ext}(F_1+F_2)}$. But recall that $S^1_{\operatorname{ext}(F_1+F_2)} = \operatorname{ext} S^1_{F_1+F_2}$. Also from theorem 1.4 of Hiai-Umegaki [20] we know that $S^1_{F_1+F_2} = \operatorname{cl}[S^1_{F_1} + S^1_{F_2}]$. From theorem 2.1 we have that $S^1_{F_1}$ and $S^1_{F_2}$ are w-compact, convex. So $\operatorname{cl}[S^1_{F_1} + S^1_{F_2}] = S^1_{F_1} + S^1_{F_2} \Rightarrow \operatorname{ext} S^1_{F_1+F_2} = \operatorname{ext}(S^1_{F_1} + S^1_{F_2})$. Applying theorem 1 (p. 5) of Kluvanek-Knowles [25], we get that there exist unique $f_1(\cdot) \in \operatorname{ext} S^1_{F_1} = S^1_{\operatorname{ext} F_1}$ and $f_2(\cdot) \in \operatorname{ext} S^1_{F_2} = S^1_{\operatorname{ext} F_2}$ s.t. $f = f_1 + f_2$. Then $x = \int_\Omega (f_1(\omega) + f_2(\omega)) \, d\mu(\omega)$, with $f_1(\omega) \in \operatorname{ext} F_1(\omega)$ μ-a.e. and $f_2(\omega) \in \operatorname{ext} F_2(\omega)$ μ-a.e.
Q.E.D.

4. DIFFERENTIAL INCLUSIONS AND CONTROL SYSTEMS

In this section we will prove some "bang-bang" type theorems for differential inclusions and control systems.

We will start with differential inclusions.

Let $T = [0, b]$, a finite closed interval in $\mathbb{R}_+$ and let $X = \mathbb{R}^n$. Let $F \colon T \times X \to 2^X \setminus \{\emptyset\}$ be a multifunction (also known as orientor field). Consider the following multivalued Cauchy problem:

$$(*) \qquad \dot{x}(t) \in F(t, x(t)) \,, \text{ a.e.,} \qquad x(0) = x_0 \,.$$

By a solution of $(*)$ we understand an absolutely continuous function $x \colon T \to X$, which satisfies $(*)$ almost everywhere. Let $S(x_0)$ be the solution set of $(*)$ and let $R(t) = \{z \in X \colon z = x(t) \text{ for some } x(\cdot) \in S(x_0)\}$ i.e. $R(t)$ is the attainable (reachable) set at time t, of the differential inclusion.

We will call a solution of $(*)$, a *"bang-bang" solution* if its derivative almost all time instants lies on the extreme points of $F(\cdot, x(\cdot))$.

We can prove the following "bang-bang" theorem for the inclusion $(*)$.

Theorem 4.1. *If $F \colon T \times \mathbb{R}^n \to P_{kc}(\mathbb{R}^n)$ is a multifunction s.t.*

1) $(t, x) \to F(t, x)$ is measurable

2) for all $t \in T$, $x \to F(t, x)$ is u.s.c.

then for every $z \in R(t)$, there exists a bang-bang solution

$$y(\cdot) \text{ of } (*) \text{ s.t. } z = y(t) \,.$$

Proof. From theorem 4.2 of Davy [14] we know that for all $t \in T$, $R(t) \neq \emptyset$. Let $x(\cdot) \in S(x_0)$ s.t. $z = x(t)$. Consider the map

$$\phi: S^1_{F(\cdot, x(\cdot))} \to \mathbb{R}^n$$

defined by:

$$\phi(f) = x_0 + \int_0^t f(s)\,\mathrm{d}s\,.$$

Because of hypothesis 1) and theorem 2.1, $S^1_{F(\cdot,x(\cdot))}$ is w-compact and convex in $L^1_n(T)$. Also $\phi(\cdot)$ is affine, continuous when $S^1_{F(\cdot,x(\cdot))}$ has the relative weak L^1_n-topology. Then applying theorem 3.2 of Artstein [1] (although for the result priority should be given to Benamara [5], corollary 1.7, p. 39), we can find $g(\cdot) \in \operatorname{ext} S^1_{F(\cdot,x(\cdot))} = S^1_{\operatorname{ext} F(\cdot,x(\cdot))}$ s.t. $\phi(g) = z = \phi(\dot{x})$. Let $y(\tau) = x_0 + \int_0^\tau g(s)\,\mathrm{d}s$. Then $y(\cdot)$ is a bang-bang solution and $y(t) = z$.

Q.E.D.

Another important subset of the solution set $S(x_0)$ of (*) are the extremal solutions. Recall that $x(\cdot) \in S(x_0)$ is said *extremal at time* $t \in T$, if $x(t) \in \partial R(t)$, where $R(\cdot)$ is the attainability multifunction. As before $T = [0, b]$ and $X = \mathbb{R}^n$.

Theorem 4.2. *If* $F: T \times \mathbb{R}^n \to P_{kc}(\mathbb{R}^n)$ *is a multifunction s.t.*

1) *for all* $x \in \mathbb{R}^n$, $t \to F(t, x)$ *is measurable and*

$$\sigma_{F(t,x)} \leqq K(\|x\|^2 + 1) \quad a.e.$$

2) *for all* $t \in T$, $x \to F(t, x)$ *is u.s.c.*

3) *the attainability multifunction* $R(\cdot)$ *has convex values and if* $\{x_n(\cdot)\}_{n \geqq 1} \subseteq S(x_0)$ *are extremal at* $\{t_n\}_{n\geqq 1}$ *and* $x_n(\tau) \to x(\tau)$ *for all* $\tau \in T$.

then there exists $t \in T$ *s.t.* $x(\cdot)$ *is extremal at* t.

Proof. Since $\{t_n\}_{n\geqq 1} \subseteq T$ and T is compact, by passing to a subsequence if necessary we may assume that $t_n \to t \in T$. Also from theorem 2 (p. 184) of Kikuchi [24] we know that $S(x_0)$ is a compact subset of $C_n(T)$ with the uniform convergence topology. So once more without any loss of generality we may assume that $x_n(\cdot) \to^{C_n} x(\cdot) \in S(x_0) \Rightarrow x_n(t_n) \to x(t)$ as $n \to \infty$. Furthermore from theorem 4 (p. 184) of [24] we know that the attainability multifunction $R(\cdot)$ is continuous. Note that $R(t) = e_t(S(x_0))$ where $e_t: C_n(T) \to X$ is the evaluation map at time t. But $e_t(\cdot)$ is continuous (see Dugundji [19], theorem 1.4, p. 260) and $S(x_0)$ is compact in $C_n(T)$. Thus $R(\cdot)$ is compact valued. Hence $R(\cdot)$ is Hausdorff continuous. Then proposition 2.1 of DeBlasi-Pianigiani [16] tells us that $\partial R(\cdot)$ is Hasudorff continuous. So $\partial R(t_n) \to^h \partial R(t)$ as $n \to \infty$. Since $x_n(t_n) \in \partial R(t_n)$ $n \geqq 1$, we conclude that $x(t) \in \partial R(t)$ i.e. $x(\cdot)$ is extremal at $t \in T$.

Q.E.D.

Remarks. (1) Conditions under which hypothesis 3) is satisfied are provided by Blagodatskikh [6]. Specifically if for all $t \in T$, $F(t, \cdot)$ is concave i.e. for all $x, z \in X$ and $\lambda \in [0, 1]$ $\lambda F(t, x) + (1 - \lambda) F(t, z) \subseteq F(t, \lambda x + (1 - \lambda) z)$, then $R(t)$ is convex for all $t \in T$.

(2) Density results of the set of extremal solution in $S(x_0)$ were obtained under stronger hypotheses by Cellina [8] and DeBlasi-Pianigiani [15].

(3) An extremal solution is a bang-bang solution if $F(\cdot, \cdot)$ has strictly convex values.

Now we pass to control systems.

First we will consider the following nonlinear control system defined on $T = [0, b]$ and X = separable reflexive Banach space. By X_w we will denote X with the weak topology.

$$(**) \qquad \begin{aligned} \dot{x}(t) &= f(t, x(t)) + g(t, x(t))\, u(t)\,, \\ u(\cdot) &\in S^1_U\,, \quad x(0) = x_0\,. \end{aligned}$$

The following assumptions will be made concerning $(**)$.

First about the multifunction $U(\cdot)$.

(A_1) $U: T \to P_{wkc}(X)$ is integrably bounded.

Next for the vector fields $f: T \times X \to X$ and $g: T \times X \to \mathbb{R}$.

(A_2) For all $x \in X$, $t \to f(t, x)$ and $t \to g(t, x)$ are measurable and $\|f(t, x)\| \leqq a(t)$ a.e., $|g(t, x)| \leqq M$ where $a(\cdot) \in L^1_+$.

(A_3) For all $t \in T$, $x \to f(t, x)$ is continuous from X_w into X_w and $x \to g(t, x)$ is continuous from X_w into $\mathbb{R}$.

(A_4) For all $u(\cdot) \in S^1_U$, $(**)$ admits a unique solution $x(\cdot, u)$.

By $R(t)$ we will denote the attainable set at time t of $(**)$, while by $R^e(t)$ we will denote the attainable set at time t, when we use only bang-bang controls (i.e. $u(t) \in \operatorname{ext} U(t)$ a.e.).

We have the following approximate bang-bang theorem.

Theorem 4.3. *If* (A_1) *to* (A_4) *hold and* $x \in R(t)$ *then there exist* $\{x_n\}_{n \geqq 1} \subseteq R^e(t)$ *s.t.* $x_n \to^w x$. *In fact* $R(t) = \mathrm{cl}_w\, R^e(t)$.

Proof. Because of hypothesis (A_4) we can define a single valued map $\phi: (S^1_U, w) \to C_{X_w}(T)$ by $\phi(u) = x(\cdot, u)$, where (S^1_U, w) is the set S^1_U endowed with the weak topology. We will show that $\phi(\cdot)$ is in fact continuous. Note that the range of $\phi(\cdot)$ is in the solution set $S(x_0)$ of the differential inclusion

$$\begin{aligned} \dot{x}(t) &\in f(t, x(t)) + g(t, x(t))\, U(t) \\ x(0) &= x_0 \end{aligned}$$

and $S(x_0)$ is compact and metrizable in the relative $C_{X_w}(T)$ topology (see [7] and [31]). Also S^1_U being a w-compact subset in the separable Banach space $L^1_X(T)$, is metrizable. So we can work with sequences to show that $\phi(\cdot)$ is continuous. Let $u_n(\cdot) \to^{w-L_X^1} u(\cdot)$. Then we have:

$$\dot{x}(t, u_n) = \dot{x}_n(t) = f(t, x_n(t)) + g(t, x_n(t))\, u_n(t) \quad \text{a.e.}$$

A straightforward application of the Dunford-Pettis compactness criterion tells us that $\{\dot{x}_n(\cdot)\}_{n \geqq 1}$ is relatively w-compact in $L^1_X(T)$. So by passing a subsequence if necessary we may assume that $\dot{x}_n(\cdot) \to^{w-L_X^1} z(\cdot) \in L^1_X(T)$. Thus for all $t \in T$ we have that

$$\int_0^t \dot{x}_n(s)\, ds \to^w \int_0^t z(s)\, ds \Rightarrow$$

$$\Rightarrow x_n(t) = x_0 + \int_0^t f(s, x_n(s))\, ds + \int_0^t g(s, x_n(s))\, u_n(s)\, ds \to^w x(t)$$

where $x(t) = x_0 + \int_0^t z(s)\, ds$. On the other hand because of (A_3) we have

$$\int_0^t f(s, x_n(s))\, ds + \int_0^t g(s, x_n(s))\, u_n(s)\, ds \to^w \int_0^t f(s, x(s))\, ds + \int_0^t g(s, x(s))\, u(s)\, ds \Rightarrow$$

$$\Rightarrow x(t) = x_0 + \int_0^t f(s, x(s))\, ds + \int_0^t g(s, x(s))\, u(s)\, ds \Rightarrow x(\cdot)$$

solves $(**)$ for the admissible control $u(\cdot)$.

Since $S(x_0)$ is compact, metrizable in $C_{X_w}(T)$, we can find $\{x_{n_k}(\cdot)\}_{k \geqq 1} \subseteq \{x_n(\cdot)\}_{n \geqq 1}$ s.t. $x_{n_k}(\cdot) \to^{C_{X_w}} x(\cdot)$. Because every subsequence has a further subsequence that converges in $C_{X_w}(T)$ to $x(\cdot) = x(\cdot, u)$ we conclude that $x_n(\cdot) \to^{C_{X_w}} x(\cdot) \Rightarrow \phi(\cdot)$ is continuous.

Next let $x(\cdot, u) \in S(x_0)$ s.t. $x = x(t, u)$. Let $\{u_n(\cdot)\}_{n \geqq 1} \subseteq S^1_{\text{ext}\, F} = \text{ext}\, S^1_F$ s.t. $u_n(\cdot) \to^{w - Lx^1} u(\cdot)$. This is possible by corollary II of theorem 3.3. Then we have that $x(\cdot, u_n) \to^{C_{X_w}} x(\cdot, u) \Rightarrow x(\tau, u_n) \to^w x(\tau, u)$ for all $\tau \in T$. But $x_n = x(t, u_n) \in R^e(t)$. So the first conclusion of the theorem follows. Since $R(t) \in P_{wk}(X)$ the Eberlein-Smulian theorem tells us that $R(t) = \text{cl}_w R^e(t)$.

Q.E.D.

Remark. Conditions under which hypothesis (A_4) is satisfied can be found in Cramer-Lakshmikantham-Mitchell [11] (theorem 3.2).

Assume that the same set of hypotheses is in effect. Let

$$(**_e) \qquad \dot{x}(t) = f(t, x(t)) + g(t, x(t))\, u(t)\,, \quad u(\cdot) \in S^1_{\text{ext}\, U}\,, \quad x(0) = x_0\,.$$

Call the solution set of $(**_e)$, $S^e(x_0)$. Then from the proof of theorem 4.3 we get that:

Theorem 4.4. *$S^e(x_0)$ is dense in $S(x_0)$ for the $C_{X_w}(T)$*-topology.

We continue our investigation on the properties of the control system $(**)$. Assume $X = \mathbb{R}^n$, $f: T \times \mathbb{R}^n \to \mathbb{R}^n$ is jointly continuous, locally Lipschitz in x and integrably bounded, $g(\cdot, \cdot) \in C_+(T \times \mathbb{R}^n)$, is locally Lipschitz in x and integrably bounded $U: T \to P_k(\mathbb{R}^n)$ has strictly convex values with nonempty interior and is Hausdorff continuous and integrably bounded and finally $0 \in f(t, x) + g(t, x)\, \text{int}\, U(t)$ for all $(t, x) \in T \times X$.

Theorem 4.5. *If the above hypotheses hold and $x(\cdot, u)$ is an extremal solution of $(**)$*

then $x(\cdot, u)$ solves $(**_e)$ i.e. $u(\cdot)$ is a bang-bang control.

Proof. Consider the following multivalued Cauchy problem

$$(***) \qquad \dot{x}(t) \in F(t, x(t))\,, \quad x(0) = x_0\,,$$

where $F(t, x) = f(t, x) + g(t, x)\, U(t)$. Clearly $x(\cdot, u)$ solves $(***)$. Furthermore from Stefani-Zecca [33] we know that the attainable sets for $(**)$ and $(***)$ are the same. Thus $x(\cdot, u)$ is an extremal solution for $(***)$ too. Then let $(t_n, x_n) \to (t, x)$ in $T \times X$. Then we have:

$$\begin{aligned} h(F(t_n, x_n), F(t, x)) &= h(f(t_n, x_n) + g(t_n, x_n)\, U(t_n), f(t, x) + g(t, x)\, U(t)) \leqq \\ &\leqq \|f(t_n, x_n) - f(t, x)\| + h(g(t_n, x_n)\, U(t_n), g(t, x)\, U(t)) \leqq \\ &\leqq \|f(t_n, x_n) - f(t, x)\| + h(g(t_n, x_n)\, U(t_n), g(t_n, x_n)\, U(t)) + \\ &+ h(g(t_n, x_n)\, U(t), g(t, x)\, U(t)) = \|f(t_n, x_n) - f(t, x)\| + \\ &+ g(t_n, x_n)\, h(U(t_n), U(t)) + |g(t_n, x_n) - g(t, x)|\, |U(t)| \end{aligned}$$

and this tends to 0 as $n \to \infty$. So $F(\cdot, \cdot)$ is Hausdorff continuous. Thus we can apply theorem 3.4 of Grasse [22] and get that

$$\dot{x}(t) \in \partial F(t, x(t)) \quad \text{a.e.}$$

But by hypothesis $F(\cdot, \cdot)$ has strictly convex values and so all boundary points are extreme points. Hence we can write that

$$\begin{aligned} &\dot{x}(t) \in \operatorname{ext} F(t, x(t)) \quad \text{a.e.} \\ &\Rightarrow \dot{x}(t) \in \operatorname{ext} [f(t, x(t)) + g(t, x(t))\, U(t)] \quad \text{a.e.} \\ &\Rightarrow \dot{x}(t) \in f(t, x(t)) + g(t, x(t)) \operatorname{ext} U(t) \quad \text{a.e.} \\ &\Rightarrow x(t) \in x_0 + \textstyle\int_0^t f(s, x(s))\, ds + \int_0^t g(s, x(s)) \operatorname{ext} U(s)\, ds \Rightarrow x(\cdot) = x(\cdot, u) \end{aligned}$$

with $u(\cdot)$ being a bang-bang control.

Q.E.D.

Remark. In general the attainable sets of $(**)$ and $(***)$ are not the same as the following simple counterexample illustrates:

$$\dot{x}(t) = x(t)\, u(t) \quad x \in \mathbb{R}, \quad u \in \mathbb{R},$$
$$x(0) = 0.$$

Then for $t > 0$ the attainable set for $(**)$ is $R_1(t) = \{0\}$, while the attainable set for $(***)$ is $R_2(t) = \mathbb{R}$.

We will conclude this paper with a bang-bang result concerning infinite dimensional linear control systems. We will show that all extremal attainable states can be reached using bang-bang controls.

Let X be a separable Banach space. By $\mathscr{L}(X)$ we will denote the space of continuous linear operators from X into itself. Also let $A(t)$ be a time dependent, linear not necessarily bounded operator on X which generates an evolution operator $\Phi: \Delta = \{(t, s) \in T \times T: 0 \leqq s \leqq t \leqq b\} \to \mathscr{L}(X)$ (see Tanabe [34]) s.t. $\|\Phi(t, s)\| \leqq M$, $B: T \to \mathscr{L}(X)$ is continuous for the strong operator topology and $U: T \to P_{wkc}(X)$ is an integrably bounded multifunction. Consider the linear control system governed by the following evoution equation:

$$(****) \qquad \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)\,,$$
$$u(\cdot) \in S^1_U\,, \quad x(0) = x_0\,.$$

We will assume that for each $u(\cdot) \in S^1_U$, $(****)$ admits a mild solution. Finally let $R(\cdot)$ denote the attainability multifunction of $(****)$.

Theorem 4.6. *If* $z \in \operatorname{ext} R(t)$
then $z = x(t, u)$ *where* $x(\cdot, u)$ *solves* $(****)$ *and* $u(\cdot)$ *is a bang-bang control.*

Proof. By hypothesis we have:

$$x(t, u) = \Phi(t, 0)\,x_0 + \int_0^t \Phi(t, s)\,B(s)\,u(s)\,ds$$

and note that

$$R(t) = \Phi(t, 0)\,x_0 + \int_0^t \Phi(t, s)\,B(s)\,U(s)\,ds \Rightarrow$$
$$\Rightarrow \operatorname{ext} R(t) = \Phi(t, 0)\,x_0 + \operatorname{ext} \int_0^t \Phi(t, s)\,B(s)\,U(s)\,ds\,.$$

Note that $\Phi(t, s)\,B(s)\,U(s) \in P_{wk}(X)$ and if $\{u_n(\cdot)\}_{n \geqq 1}$ is a Castaing representation for $U(\cdot)$, for all $y \in X$ we have:

$$d_{\Phi(t,s)B(s)U(s)}(y) = \inf_{n \geqq 1} \|y - \Phi(t, s)\,B(s)\,u_n(s)\| \Rightarrow s \to d_{\Phi(t,s)B(s)U(s)}(y) \text{ is measurable}$$
$$\Rightarrow s \to \Phi(t, s)\,B(s)\,U(s) \text{ is measurable and clearly integrably bounded}\,.$$

Then we can apply theorem 3.4 and get that

$$\operatorname{ext} R(t) \subseteq \Phi(t, 0)\,x_0 + \int_0^t \operatorname{ext} \Phi(t, s)\,B(s)\,U(s)\,ds \subseteq$$
$$\subseteq \Phi(t, 0)\,x_0 + \int_0^t \Phi(t, s)\,B(s) \operatorname{ext} U(s)\,ds\,.$$

Let $u(\cdot) \in S^1_{\operatorname{ext} U}$ s.t. $x(t) = \Phi(t, 0)\,x_0 + \int_0^t \Phi(t, s)\,B(s)\,u(s)\,ds \Rightarrow x(\cdot, u)$ solves $(****)$ with $u(\cdot)$ being a bang-bang control.

Q.E.D.

References

[1] *Z. Artstein:* Discrete and continuous bang-bang and facial spaces. Or: look for extreme points S.I.A.M. Rev. *22* (1980) pp. 172—185.

[2] *Z. Artstein - J. Burns:* Integration of compact set valued functions. Pacific J. Math. *58* (1975) pp. 297—307.

[3] *R. Aumann:* Integrals of set valued function. J. Math. Anal. Appl. 12 (1965) pp. 1—12.

[4] *M. Benamara:* Sections measurables extrémales d'une multiapplication. C.R. Acad. Sc. Paris t. *278* (1974) pp. 1249—1252.

[5] *M. Benamara:* Points extremaux, multiapplications et fonctionelles integrales. Thése de 3[eme] cycle, Université de Grenoble (1975).

[6] *V. Blagodatskikh:* Convexity of attainability spheres. Diff. Eq. *8* (1972) pp. 1661—1665.

[7] *C. Castaing - M. Valadier:* Convex Analysis and Measurable Multifunctions. Lecture Notes in Math. Vol. 580, Springer, Berlin (1977).

[8] *A. Cellina:* On the differential inclusion $\dot{x} \in [-1, 1]$. Rend. Acad. Naz. Lincei *69* (1980) pp. 1—6.

[9] *L. Cesari:* Convexity of the range of certain integrals. S.I.A.M. J. Contr. Optim. *13* (1975) pp. 666—676.

[10] *F. Clarke:* Optimization and Nonsmooth Analysis. Wiley, New York (1983).

[11] *E. Cramer - V. Laksmikantham - A. Mitchell:* On the existence of weak solutions of differential equations in nonreflexive Banach spaces. Nonl. Anal. T.M.A. *2* (1978) pp. 169—177.

[12] *R. Datko:* A general bang-bang principle and bang-bang approximations. J. Math. Anal. Appl. *10* (1965) pp. 284—194.

[13] *R. Datko:* On the integration of set valued mappings in Banach spaces. Fund. Math. *78* (1973) pp. 205—208.

[14] *J. L. Davy:* Properties of the solution set of a generalized differential equation. Bull. Austr. Math. Soc. *6* (1972) pp. 379—298.

[15] *F. DeBlasi - G. Pianigiani:* A Baire category approach to the existence of solutions of multivalued differential equations in Banach spaces. Funkc. Ekvac. *25* (1985) pp. 153—162.

[16] *F. DeBlasi - G. Pianigiani:* Remark on Hausdorff continuous multifunctions and selections Comm. Math. Univ. Carolinae *24* (1983) pp. 553—562.

[17] *G. Debreu:* Integration of correspondences Proc. of 5th Berkeley Symp. Math. Stat. and Probability I, Univ. of Calif. Press, Berkeley (1967) pp. 351—372.

[18] *J. Diestel - J. Uhl:* Vector Measures Math. Surveys, Vol. 15, A.M.S., Providence (1977).

[19] *J. Dugundji:* Topology Allyn and Bacon Inc., Boston (1966).

[20] *F. Hiai - H. Umegaki:* Integrals, conditional expectations and martingales of multivalued functions J. Multiv. Anal. *7* (1977) pp. 149—182.

[21] *C. Himmelberg:* Measurable relations Fund. Math. *87* (1975) pp. 53—72.

[22] *K. Grasse:* Some remarks on extremal solutions of multivalued differential equations J. Optim. The. Appl. *40* (1983) pp. 221—35.

[23] *J. Jarník - J. Kurzweil:* Integral of multivalued mappings and its connection with differential relations Časopis pro pěstování matematiky *108* (1983) pp. 8—28.

[24] *N. Kikuchi:* On some fundamental theorems of contigent equations in connection with control problems, Publ. R.I.M.S. Kyoto Univ *3* (1966), pp. 177—201.

[25] *I. Kluvánek - G. Knowles:* Vector Measures and Control Systems, Math. Studies, Vol. 20, North Holland, Amsterdam (1975).

[26] *J. Kurzweil:* Generalized ordinary differential equations and continuous dependence on a parameter, Czechoslovak Math. J. *7* (*82*) (1957), pp. 418—449.

[27] *J. Kurzweil:* Generalized ordinary differential equations, Czechoslovak Math. J. *8* (*83*) (1958), pp. 360—388.

[28] *N. S. Papageorgiou:* On the theory of Banach space valued multifunctions. Part 1: Integration and conditional expectation, J. Multiv. Anal. *17* (1985), pp. 185—207.

[29] *N. S. Papageorgiou:* Representation of set valued operators, Trans. Amer. Math. Soc. *292* (1985), pp. 557—572.

[30] *N. S. Papageorgiou:* Integral functionals on Souslin locally convex spaces, J. Math. Anal. Appl. *113* (1986), pp. 148—163.

[31] *N. S. Papageorgiou:* Random differential inclusions in Banach spaces, J. Diff. Eq. *65* (1986), pp. 287—303.

[32] *R. T. Rockafellar:* Integral functionals, normal integrands and measurable selections, Lecture Notes in Math. Vol. 543, Springer, Berlin (1976), pp. 157—207.

[33] *G. Stefani - P. Zecca:* Multivalued differential equations on manifolds with applications to control theory, Illinois J. Math. *24* (1980), pp. 560—575.

[34] *H. Tanabe:* Equations of Evolution, Pitman, London (1979).

[35] *D. Wagner:* Survey of measurable selection theorems, S.I.A.M. J. Contr. Optim. *15* (1977), pp. 859—903.

Authors' addresses: University of California, Department of Mathematics, Davis, California 95616, U.S.A.