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LARGE NORM SOLUTIONS TO NONLINEAR ELLIPTIC PROBLEMS

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A number of problems with various physical background can be attacked from the standpoint of the calculus of variations. In his paper [5], M. Ôtani investigated nonzero solutions to the problem

$$\operatorname{div}\left(|\nabla u|_n^{p-2}\nabla u\right) + |u|^{q-2}u = 0 \quad \text{on} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \partial\Omega .$$

Taking advantage of the special form of the nonlinearities appearing in the equation (homogeneity in $\nabla u, u$), he succeeded in proving the existence of at least one nontrivial solution via the nonlinear eigenvalue theory.

In this paper we show how some simple ideas of nonlinear analysis combined with the monotone operator theory can be used to obtain somewhat better results for a wider class of elliptic equations. More precisely, we are going to establish the existence of an unbounded sequence of solutions to the problem

$$\text{(E)} \qquad \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x, \nabla u) + g(x, u) = 0\,, \quad x \in \Omega$$

$$\text{(B)} \qquad u(x) = 0\,, \quad x \in \partial\Omega$$

where $\Omega \subset R^n$ is a bounded domain with a smooth boundary.

Our aim is to formulate the problem in terms of the approximate variational method of Rayleigh-Ritz. To this end, we have to require $f_i = \partial F/\partial y_i$ for a certain potential F. In order to obtain infinitely many solutions, the nonlinearities we deal with are supposed to be odd in ∇u and u.

Note that there exists a considerable number of papers related to the semilinear case, i.e. to the case when f_i are linear in ∇u. Relevant references are e.g. [1], [6]. If the corresponding eigenvalue problems are involved, we refer to [3] or [1].

To agree upon notation, let us denote by R^n the usual Euclidean space with the norm $|\ |_n$. In what follows, the symbols c_i stand for positive numbers supposed to be constant in the given context.

1. MAIN RESULTS

Concerning the functions f_i, $i = 1, \ldots, n$, we assume

(F1) $$f_i(x, y) = \frac{\partial F(x, y)}{\partial y_i} \quad \text{for} \quad i = 1, \ldots, n$$

where $F \in C^1(\bar{\Omega} \times R^n)$, $F(x, 0) = 0$, $F(x, \cdot)$ is convex for each x, $F(x, -y) = F(x, y)$ for all x, y;

(F2) there is a fixed number $p > 1$ such that

$$|f_i(x, y)| \leqq c_1 |y|_n^{p-1} + c_2 ,$$

(F3) $$\sum_{i=1}^{n} f_i(x, y)\, y_i \geqq c_3 |y|_n^p - c_4 \quad \text{for all} \quad x, y .$$

As to the function g, we require

(G1) $$g \in C(\bar{\Omega} \times R^1) ,$$

$$g(x, -y) = -g(x, y) \quad \text{for all} \quad x, y ;$$

(G2) there is a fixed number $q > 1$ such that

$$|g(x, y)| \leqq c_5 |y|^{q-1} + c_6 \quad \text{for all} \quad x, y .$$

In view of (F2), (F3) the most convenient space in which the solution of (E), (B) is to be looked for is the Sobolev space $X = W_0^{1,p}(\Omega)$ defined as the completion of all smooth functions satisfying (B) with respect to the norm

$$\|v\|_X = \left(\int_\Omega |\nabla v|_n^p \,\mathrm{d}x\right)^{1/p} .$$

It is known that X is a separable, reflexive Banach space with a Schauder basis (see [3]).

Another space related to the function g is the Lebesgue space $L_q(\Omega)$ determined in the standard way. In the sequel, we will assume

(C1) $$q < \bar{q} \quad \text{where} \quad \bar{q} = \infty \quad \text{for} \quad p \geqq n , \qquad \bar{q} = np/(n - p) \quad \text{for} \quad p < n .$$

In other words, the embedding $X \hookrightarrow L_q(\Omega)$ is compact ([4]).

Finally, the missing a priori estimates are replaced by the following growth conditions:

(C2) there are constants ν, μ, $0 < \mu < \nu$ such that

$$F(x, y) \geqq \nu \sum_{i=1}^{n} f_i(x, y)\, y_i > 0 ,$$

$$\mu g(x, y)\, y \geqq G(x, y) =^{\text{def}} \int_0^y g(x, s)\, \mathrm{d}s > 0$$

for all x, y, $|y|_n \geqq c_7$.

Following Rabinowitz [6], we can deduce from (C2)

(1.1) $$G(x, y) \geqq c_8 |y|^{1/\mu} - c_9$$

and

(1.2) $$F(x, y) \leqq c_{10}|y|_n^{1/v} + c_{11}$$

for all x, y. As a consequence, we obtain

(1.3) $$1 < p \leqq 1/v < 1/\mu \leqq q < \bar{q}\,.$$

To relate the critical point theory to our problem, we introduce the energy functional I,

$$I(v) = -\int_\Omega F(\cdot, \nabla v)\,dx + \int_\Omega G(\cdot, v)\,dx = -\mathscr{F}(v) + \mathscr{G}(v)\,.$$

Now (E) is an Euler-Lagrange equation obtained by setting an arbitrary variation of I on X equal to zero, i.e.

(1.4) $$\langle I'(v), w\rangle = -\int_\Omega \sum_{i=1}^{n} f_i(\cdot, \nabla v)\frac{\partial w}{\partial x_i} + g(\cdot, v)\,w\,dx = 0 \quad \text{for all} \quad w \in X\,.$$

We are able to formulate our main result.

Theorem 1. *Let Ω be a bounded domain with a smooth boundary. Let the conditions* (F1)–(F3), (G1), (G2), (C1), (C2) *be satisfied.*

Then for an arbitrary number d there exists a weak solution $u \in X$ of (E), (B) *(i.e., u satisfies* (1.4)) *such that*

(1.5) $$\|u\|_{L_q(\Omega)} \geqq d\,.$$

The remaining part of the paper is devoted to the proof of Theorem 1. In Section 2, the Rayleigh-Ritz method is used to get the sequence of approximate solutions. The limit process represents a standard application of the monotone operator theory and is carried out in Section 3.

2. THE RAYLEIGH-RITZ METHOD

To begin with, consider the Schauder basis $\{e_i\}_{i=1}^\infty$ of X. Every function $v \in X$ has a unique representation of the form

$$v = \sum_{i=1}^{\infty} a_i(v)\,e_i\,, \quad a_i \in X^*\,.$$

Now the sequence of subspaces $X_n \subset X$

$$X_n =^{\text{def}} \operatorname{span}\{e_i \mid i \leqq n\}$$

represents a suitable platform for using an approximate variational method.

We intend to find *approximate solutions* of the problem (E), (B) as critical points of the energy functional I restricted to X_n. For this purpose, the following result is needed.

Lemma 1. *Let H be a Banach space,* $\dim(H) < \infty$. *Suppose there are subspaces* H_1, H_2,

$$H = H_1 + H_2\,, \quad \{0\} \neq H_1 \cap H_2 \neq H_2\,.$$

Denote $S(r) = \{v \mid \|v\|_H = r\}$. *Let* J *be a functional on* H, $J \in C^1(H)$, J *being even* $(J(-v) = J(v))$ *and satisfying*

(2.1) $$\lim_{\|v\|_H \to \infty} J(v) = +\infty .$$

Suppose we have

(2.2) $$J \leqq b \quad on \quad S(r) \cap H_2 ,$$

(2.3) $$J > a \quad on \quad H_1 .$$

Then there exists at least one critical point u *of* J *such that*

(2.4) $$J'(u) = 0 ,$$

(2.5) $$J(u) \in [a, b] .$$

We postpone the proof to Section 4.

To apply this assertion to our situation, some auxiliary results are of interest.

Lemma 2. *Denote* $R_n v = \sum_{i=n}^{\infty} a_i(v)\, e_i$.

Then we have the estimate

(2.6) $$\|R_n v\|_{L_q(\Omega)} \leqq \varepsilon(n)\, \|R_n v\|_X \quad for\ all \quad v \in X \quad where \quad \lim_{n \to \infty} \varepsilon(n) = 0 .$$

Proof. Assume the contrary. Then there is a sequence

(1) $$w_{n_k} = \frac{R_{n_k} v_{n_k}}{\|R_{n_k} v_{n_k}\|_X} \quad \text{satisfying}$$

(2) $$\|w_{n_k}\|_{L_q(\Omega)} > \delta > 0 , \quad \|w_{n_k}\|_X = 1 .$$

By virtue of reflexivity, we can suppose

$$w_{n_k} \rightharpoonup w \quad \text{weakly in } X .$$

According to (1), we get

$$a_i(w_{n_k}) = 0 \quad \text{whenever} \quad n_k > i .$$

Consequently, $a_i(w) = 0$ for all i, thus $w = 0$. On the other hand, $w_{n_k} \to w$ strongly in $L_g(\Omega)$ due to (C1), which contradicts (2). Q.E.D.

As an easy consequence of the above result, we obtain

Lemma 3. *Let* $b \in R^1$ *be a fixed number. Then there exist* $r(b) > 0$ *and* $m(b)$ *such that*

(2.7) $$I(v) \leqq b \quad whenever \quad v \in S(r) \cap \operatorname{span}\{e_i \mid i \geqq m\} .$$

Proof. Choose $v \in S(r) \cap \operatorname{span}\{e_i \mid i \geqq m\}$. Then (F3), (G2), and (C2) yield

$$I(v) \leqq -c_{12}\|v\|_X^p + c_{13}\|v\|_{L_q(\Omega)}^q + c_{14} .$$

By virtue of (2.6), we obtain

$$I(v) \leqq -c_{12}\|v\|_X^p + c_{13}\, \varepsilon(m)\, \|v\|_X^q + c_{14} = -c_{12} r^p + c_{13}\, \varepsilon(m)\, r^q + c_{14} .$$

In view of Lemma 2, $r(b)$, $m(b)$ can be chosen in such a way that (2.7) holds for an arbitrary fixed number b. Q.E.D.

Lemma 4. *For a fixed number n, we have*

$$\lim_{\|v\|_{X_n}\to\infty} I(v) = +\infty \quad \textit{for} \quad v \in X_n . \tag{2.8}$$

Proof. By virtue of (F2), (1.1) we have

$$I(v) \geqq -c_{15}\|v\|_X^p + c_{16}\|v\|_{L_{1/\mu}(\Omega)}^{1/\mu} - c_{17} .$$

Observing that $1/\mu > p$ and that all norms are equivalent on X_n, we obtain (2.8). Q.E.D.

Now we are about to use Lemma 1. Set $H = X_n$, $J = I$. Let us choose $b < 0$ arbitrarily. In view of Lemma 3, we find $m(b)$, $r(b)$ such that (2.7) holds. We can set $H_2 = \operatorname{span}\{e_i \mid n \geqq i \geqq m\}$, $H_1 = X_m \subsetneqq X_n$ provided that n is sufficiently large. By virtue of Lemmas 3 and 4, the conditions (2.1), (2.2) are fulfilled. Moreover, it follows from (2.8) that $I \geqq a(m)$ on $H_1 = X_m$ for some $a \in R^1$ (in fact $a = a(b)$). Applying Lemma 1 for each $n \geqq m$, we get a sequence $\{u_n\}_{n\geqq m}$ of approximate solutions satisfying

$$-\int_\Omega \sum_{i=1}^n f_i(\cdot, \nabla u_n) \frac{\partial v}{\partial x_i} + g(\cdot, u_n)\, v \,\mathrm{d}x = 0 \quad \text{for all} \quad v \in X_n , \tag{2.9}$$

$$-\int_\Omega F(\cdot, \nabla u_n)\,\mathrm{d}x + \int_\Omega G(\cdot, u_n)\,\mathrm{d}x \in [a, b] . \tag{2.10}$$

3. PASSING TO THE LIMIT

In this section, our aim is to demonstrate that the sequence $\{u_n\}_{n\geqq m}$ constructed in Section 2 possesses a weak limit $u \in X$ – the solution of (E), (B). To this end choose ζ such that $\mu < \zeta < \nu$. Setting $v = u_n$ in (2.9), multiplying by $-\zeta$ and adding to (2.10), we get

$$\int_\Omega F(\cdot, \nabla u_n) - \zeta \sum_{i=1}^n f_i(\cdot, \nabla u_n) \frac{\partial u_n}{\partial x_i} \,\mathrm{d}x +$$
$$+ \zeta \int_\Omega g(\cdot, u_n)\, u_n - G(\cdot, u_n)\,\mathrm{d}x \leqq -a , \quad (-a > 0) .$$

From (C2), (F3) we conclude that

$$\|u_n\|_X \leqq c_{18} \quad \text{for all} \quad n \geqq m . \tag{3.1}$$

Using (1.1), (2.10), we deduce the estimate

$$\int_\Omega \sum_{i=1}^n f_i(\cdot, \nabla u_n) \frac{\partial u_n}{\partial x_i} \,\mathrm{d}x \geqq -b - c_{19} . \tag{3.2}$$

It is well known, due to convexity of F, that $\mathscr{F}'$ is a monotone, demicontinuous operator mapping bounded sets of X into bounded sets in X^* (see [4]). Keeping (3.1) in mind, we can pass to the subsequence (denoted $\{u_n\}_{n=1}^\infty$ for simplicity) such that

$$(3.3) \qquad u_n \rightharpoonup u \quad \text{weakly in} \quad X\,,$$

$$(3.4) \qquad \mathcal{F}'(u_n) \rightharpoonup h \quad \text{weakly in} \quad X^*\,,$$

$$(3.5) \qquad u_n \to u \quad \text{strongly in} \quad L_q(\Omega)\,,$$

$$(3.6) \qquad g(\cdot, u_n) \to g(\cdot, u) \quad \text{strongly in} \quad L_{q'}(\Omega)\,, \quad 1/q + 1/q' = 1$$

where the last assertion is due to the Krasnoselskij theorem.

Keeping $v \in X_n$ fixed, we are able to pass to the limit in (2.9), that is

$$(3.7) \qquad -\langle h, v\rangle + \int_\Omega g(\cdot, u)\, v \,\mathrm{d}x = 0$$

for all $v \in \bigcup_{n=1}^{\infty} X_n$, and therefore for all $v \in X$.

We wish the function u to be a solution of (E), (B), i.e. we are to show that

$$\langle h, v\rangle = \int_\Omega \sum_{i=1}^{n} f_i(\cdot, \nabla u) \frac{\partial v}{\partial x_i} \,\mathrm{d}x \quad \text{for all} \quad v \in X\,.$$

By virtue of the well known results of the monotone operator theory (see [4]), it suffices to prove

$$(3.8) \qquad \lim_{n\to\infty} \int_\Omega \sum_{i=1}^{n} f_i(\cdot, \nabla u_n) \frac{\partial u_n}{\partial x_i} \,\mathrm{d}x = \langle h, u\rangle\,.$$

Substituting $v = u_n$ in (2.9), we obtain

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^{n} f_i(\cdot, \nabla u_n) \frac{\partial u_n}{\partial x_i} \,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega g(\cdot, u_n)\, u_n \,\mathrm{d}x\,.$$

But in view of (3.5), (3.6) the limit on the right-hand side equals $\int_\Omega g(\cdot, u)\, u \,\mathrm{d}x$. Combining this with (3.7), we obtain (3.8). Consequently, u is a solution of (E), (B) in the weak sense.

On the point of conclusion, we show (1.5). Passing to the limit in (3.2), we get

$$\int_\Omega g(\cdot, u)\, u \,\mathrm{d}x = \int_\Omega \sum_{i=1}^{n} f_i(\cdot, \nabla u) \frac{\partial u}{\partial x_i} \,\mathrm{d}x \geqq -b - c_{19}\,.$$

By (G2) we have

$$\|u\|^q_{L_q(\Omega)} \geqq -b - c_{19}\,.$$

Thus the choice of b sufficiently small ($b < 0$) leads to (1.5).

We have proved Theorem 1.

4. PROOF OF LEMMA 1

To begin with, we claim that Lemma 1 is a special case of Theorem (0.1) in [2]. Nevertheless, we are going to present a simple proof based on the well known Borsuk-Ulam theorem.

Assume there are no critical points of J with critical values in $[a, b]$. Since (2.1) holds and $\dim(H) < \infty$, it is a matter of routine to construct a homotopy h such that

(1) $$h \in C(H \times [0, 1], H), \quad h(v, 0) = v,$$

(2) $$h(-v, t) = -h(v, t),$$

(3) $$h(\{v \mid J(v) \leqq b\}, 1) \subset \{v \mid J(v) \leqq a\}$$

(see [2] for details).

The space H can be decomposed as a direct sum

$$H = \tilde{H}_1 \oplus H_1 \cap H_2 \oplus \tilde{H}_2 \quad \text{where} \quad \tilde{H}_1 \subset H_1, \quad \tilde{H}_2 \subset H_2.$$

Consider the corresponding projection P on the space $\tilde{H}_2$.

We define a mapping

$$s: S(r) \cap H_2 \to S(r) \cap \tilde{H}_2,$$

$$s(v) = \frac{Ph(v, 1)}{\|Ph(v, 1)\|_H} r.$$

We claim that s is well defined. Indeed, if $Ph(v, 1) = 0$ then we would have $h(v, 1) \in H_1$, which is impossible due to (2.2), (2.3), (3).

On the other hand, s is an odd mapping of a finite dimensional sphere into its proper subsphere. We have obtained a contradiction with the Borsuk-Ulam theorem [3].

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