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Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 4, 730–744

Persistent URL: http://dml.cz/dmlcz/102269

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

CONTINUOUS AND COMPACT IMBEDDINGS OF WEIGHTED SOBOLEV SPACES I

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(Received October 1, 1986)

1. INTRODUCTION

In our paper we will establish some conditions on p, q and the *weight functions* v_0, v_1, w under which the *continuous imbedding*

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)\,, \tag{1.1}$$

or the *compact imbedding*

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w) \tag{1.2}$$

takes place.

Similar problems have been studied by various authors. As to the problem of compact imbeddings we refer the reader to our paper [7]. The problem of continuous imbedding is studied for $p = q$ by A. Kufner in the book [2]. The case $1 < p \leqq q < \infty$ is investigated by P. I. Lizorkin and M. Otelbaev in [5], by W. Zajaczkowski for power-type weights in [8] and in our paper [7].

Throughout the paper we will suppose that Ω is a bounded domain in $\mathbb{R}^N$ with the boundary $\partial\Omega$, $1 \leqq p \leqq q < \infty$, $1/N \geqq 1/p - 1/q$. If $x \in \Omega$ then we set $\mathrm{d}(x) = \mathrm{dist}\,(x, \partial\Omega)$. By $\mathscr{W}(\Omega)$ we denote the set of *weight functions* on Ω, i.e. the set of all measurable, a.e. in Ω positive and finite functions.

For $w \in \mathscr{W}(\Omega)$, $1 \leqq r < \infty$ the *weighted Lebesgue space* $L^r(\Omega; w)$ is the set of all measurable functions u defined on Ω with a finite norm

$$\|u\|_{r,\Omega,w} = \left(\int_\Omega |u(x)|^r\, w(x)\, \mathrm{d}x\right)^{1/r}. \tag{1.3}$$

Throughout the paper we assume that

$$v_0, v_1 \in \mathscr{W}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)\,, \quad v_0^{-1/p}, v_1^{-1/p} \in L^{p'}_{\mathrm{loc}}(\Omega) \left(p' = \frac{p}{p-1}\right). \tag{1.4}$$

We define the *weighted Sobolev space* $W^{1,p}(\Omega; v_0, v_1)$ as the set of all functions $u \in L^p(\Omega; v_0)$ which have on Ω distributional derivatives $\partial u/\partial x_i \in L^p(\Omega; v_1)$, $i =$

$= 1, 2, \ldots, N$. We can easily verify that the space $W^{1,p}(\Omega; v_0, v_1)$ with the norm

$$(1.5) \qquad \|u\|_{1,p,\Omega,v_0,v_1} = \left(\|u\|^p_{p,\Omega,v_0} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^p_{p,\Omega,v_1} \right)^{1/p}$$

is a Banach space. Further we define

$$W_0^{1,p}(\Omega; v_0, v_1) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,p,\Omega,v_0,v_1}}$$

(for details see [4]).

Given $x \in \mathbb{R}^N$ and $R > 0$, we put

$$B(x, R) = \{y \in \mathbb{R}^N;\ |x - y| < R\}\,.$$

If h is a positive number we use the notation

$$h\, B(x, R) = B(x, hR)\,.$$

By $\mathscr{C}^{0,1}$ we denote the class of all bounded domains in $\mathbb{R}^N$ with a *Lipschitz boundary* (in the sense of [3], Definition 5.5.6).

If $Q \subset \mathbb{R}^N$ then $|Q|$ is the *Lebesgue measure* of the set Q.

2. MAIN THEOREMS

In this section we present the main results of our paper. We will consider the case when the weight functions may have singularities or degenerations only on the boundary $\partial\Omega$ of the bounded domain Ω. (That is, on any bounded domain G, $\bar{G} \subset \Omega$, the weight functions are bounded from above and from below by positive constants, and thus we can use the fact that the classical Sobolev imbedding theorems take place on G.)

Proofs of the theorems presented in this section will be given in Section 3.

Throughout this paper we will suppose

2.1. Assumptions. (i) *Let* $\{\Omega_n\}_{n=1}^{\infty}$ *be a sequence of domains such that*

$$\Omega_n \in \mathscr{C}^{0,1}\,, \quad \{x \in \Omega;\ n^{-1} < d(x)\} \subset \Omega_n \subset \{x \in \Omega;\ (n+1)^{-1} < d(x)\} \text{ *)}$$

Put $\Omega^n = \operatorname{int}(\Omega \setminus \Omega_n)$.

(ii) *There exist* $n_0 \in \mathbb{N}$, $n_0 \geqq 3$, *a positive measurable function* r *defined on* Ω^{n_0} *and a constant* $c_r \geqq 1$ *such that*

$$r(x) \leqq d(x)/3\,, \quad x \in \Omega^{n_0}\,,$$

$$(2.1) \qquad x \in \Omega^{n_0}\,, \quad y \in B(x, r(x)) \Rightarrow c_r^{-1} \leqq \frac{r(y)}{r(x)} \leqq c_r\,. \text{ **)}$$

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*) Evidently $\Omega_n \subset \Omega_{n+1} \subsetneqq \Omega$, $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$.

**) This condition will appear to be very useful, but it is rather restrictive. How to get results without condition (2.1) will be shown in Section 6.

2.2. Theorem (sufficient conditions for the continuous imbedding). *Let the following conditions be fulfilled*:

C 1 $$W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Omega_n; w), \quad n \geqq n_0 .$$

C 2 *There exist positive measurable functions a_0, a_1 defined on Ω^{n_0} such that for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(x))$*

$$w(y) \leqq a_0(x), \quad a_1(x) \leqq v_1(y). \tag{2.2}$$

C 3 *There exist a constant $K_0 > 0$ such that*

$$v_1(x)\, r^{-p}(x) \leqq K_0\, v_0(x) \quad \text{for a.e.} \quad x \in \Omega^{n_0}. \tag{2.3}$$

C 4 $\lim_{n\to\infty} \mathscr{A}_n = \mathscr{A} < \infty$, *where*

$$\mathscr{A}_n = \sup_{x\in\Omega^n} \frac{a_0^{1/q}(x)}{a_1^{1/p}(x)}\, r^{(N/q)-(N/p)+1}(x). \; {}^{*)} \tag{2.4}$$

Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w). \tag{2.5}$$

2.3. Theorem (sufficient conditions for the compact imbedding). *Let us suppose $1/N > 1/p - 1/q$ and conditions* **C 2**, **C 3**. *Moreover, let the following conditions be fulfilled*:

C 1* $$W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega_n; w), \quad n \geqq n_0 .$$

C 4* $\lim_{n\to\infty} \mathscr{A}_n = 0$, *where $\mathscr{A}_n$ is defined by* (2.4). *Then*

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w). \tag{2.6}$$

Sufficient conditions for non-imbeddings are given by the two following theorems.

2.4. Theorem. *Let the following conditions be fulfilled*:

^C 2 *There exist positive measurable functions $\hat{a}_0, \hat{a}_1$ defined on Ω^{n_0} such that for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(x))$*

$$w(y) \geqq \hat{a}_0(x), \quad \hat{a}_1(x) \geqq v_1(y). \tag{2.7}$$

^C 3 *There exists a constant $k_0 > 0$ such that*

$$k_0\, v_0(x) \leqq v_1(x)\, r^{-p}(x) \quad \text{for a.e.} \quad x \in \Omega^{n_0}. \tag{2.8}$$

^C 4 $\lim_{n\to\infty} \hat{\mathscr{A}}_n = \infty$, *where*

$$\hat{\mathscr{A}}_n = \sup_{x\in\Omega^n} \frac{\hat{a}_0(x)^{1/q}}{\hat{a}_1(x)^{1/p}}\, r^{(N/q)-(N/p)+1}(x). \tag{2.9}$$

Then $W^{1,p}(\Omega; v_0, v_1)$ is not imbedded in $L^q(\Omega; w)$.

2.5. Theorem. *Suppose conditions* **^C 2** *and* **^C 3**. *Moreover, let the following conditions hold*:

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*) The sequence $\{\mathscr{A}_n\}_{n=1}^{\infty}$ is nonincreasing and so the limit exists.

$^\wedge$**C 4*** $\lim_{n\to\infty} \hat{\mathscr{A}}_n > 0$, *where* $\hat{\mathscr{A}}_n$ *is defined by* (2.9).

Then $W^{1,p}(\Omega; v_0, v_1)$ *is not compactly imbedded in* $L^q(\Omega; w)$.

Combining Theorem 2.2 and 2.4 (or 2.3 and 2.5) we easily get the following two theorems that give us necessary and sufficient conditions for the imbedding (2.5) (or (2.6), respectively) to take place.

2.6. Theorem (the continuous imbedding). *Let condition* **C 1** *and the following two conditions be fulfilled:*

$^\sim$**C 2** *There exist positive constants* $c_0 \leqq C_0$, $c_1 \leqq C_1$ *and positive measurable functions* a_0, a_1 *defined on* Ω^{n_0} *such that for every* $x \in \Omega^{n_0}$ *and for a.e.* $y \in B(x, r(x))$

$$c_0\, a_0(x) \leqq w(y) \leqq C_0\, a_0(x)\,, \quad c_1\, a_1(x) \leqq v_1(y) \leqq C_1\, a_1(x)\,. \tag{2.10}$$

$^\sim$**C 3** *There exist positive constants* $k_0 \leqq K_0$ *such that*

$$k_0\, v_0(x) \leqq v_1(x)\, r^{-p}(x) \leqq K_0\, v_0(x) \quad \text{for a.e.} \quad x \in \Omega^{n_0}\,. \tag{2.11}$$

Then the imbedding (2.5) *holds if and only if condition* **C 4** *is satisfied.*

2.7. Theorem (the compact imbedding). *Let* $1/N > 1/p - 1/q$ *and let conditions* **C 1***, $^\sim$**C 2**, $^\sim$**C 3** *be fulfilled. Then condition* **C 4*** *is necessary and sufficient for the imbedding* (2.6) *to take place.*

2.8. Remarks. Let us discuss some special cases of conditions **C 2**, $^\wedge$**C 2**, $^\sim$**C 2**, **C 4** and **C 4***.

(i) It is easy to see that in conditions **C 2**, $^\wedge$**C 2** we can take, for $x \in \Omega^{n_0}$:

$$a_0(x) = \operatorname*{ess\,sup}_{y\in B(x,r(x))} w(y)\,, \quad \hat{a}_0(x) = \operatorname*{ess\,inf}_{y\in B(x,r(x))} w(y)\,,$$
$$a_1(x) = \operatorname*{ess\,inf}_{y\in B(x,r(x))} v_1(y)\,, \quad \hat{a}_1(x) = \operatorname*{ess\,sup}_{y\in B(x,r(x))} v_1(y)\,.$$

(ii) Suppose that the functions $w(x)$, $v_1(x)$ are defined for all $x \in \Omega^{n_0}$. Moreover, let

$$c\, w(x) \leqq \operatorname*{ess\,inf}_{y\in B(x,r(x))} w(y) \leqq \operatorname*{ess\,sup}_{y\in B(x,r(x))} w(y) \leqq C\, w(x)\,,$$
$$c\, v_1(x) \leqq \operatorname*{ess\,inf}_{y\in B(x,r(x))} v_1(y) \leqq \operatorname*{ess\,sup}_{y\in B(x,r(x))} v_1(y) \leqq C\, v_1(x)$$

$(0 < c \leqq 1 \leqq C < \infty,\ x \in \Omega^{n_0})$. Then conditions **C 2** and $^\wedge$**C 2** are fulfilled with

$$a_0(x) = C\, w(x)\,, \quad a_1(x) = c\, v_1(x)\,,$$
$$\hat{a}_0(x) = c\, w(x)\,, \quad \hat{a}_1(x) = C\, v_1(x)\,, \quad x \in \Omega^{n_0}\,.$$

Condition $^\sim$**C 2** is fulfilled as well (with the constants $c_0 = c/C$, $C_0 = 1$, $c_1 = 1$, $C_1 = C/c$). So we can conclude that Theorem 2.6 (or Theorem 2.7) remains valid if condition $^\sim$**C 2** is replaced by the following one:

There exists positive constants c, C, $c \leqq C$, *such that for every* $x \in \Omega^{n_0}$ *and for* a.e. $y \in B(x, r(x))$

$$c\, w(x) \leqq w(y) \leqq C\, w(x)\,, \quad c\, v_1(x) \leqq v_1(y) \leqq C\, v_1(x)\,. \tag{2.12}$$

(iii) Now, let us consider the case when the functions w, v_1 have the special form

$$w(x) = \bar{w}(d(x))\,, \quad v_1(x) = \bar{v}_1(d(x)) \tag{2.13}$$

for all $x \in \Omega^{n_0}$, where $\bar{w}$ and $\bar{v}_1$ are non-negative measurable functions defined on the interval $(0, n_0^{-1})$. Then inequalities (2.12) have the form

$$c\, \bar{w}(d(x)) \leqq \bar{w}(d(y)) \leqq C\, \bar{w}(d(x))\,, \quad c\, \bar{v}_1(d(x)) \leqq \bar{v}_1(d(y)) \leqq C\, \bar{v}_1(d(x)) \tag{2.14}$$

for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(x))$.

Moreover, if

$$r(x) = \bar{r}(d(x))\,, \quad x \in \Omega^{n_0}\,, \tag{2.15}$$

where $\bar{r}$ is a positive measurable function defined on $(0, n_0^{-1})$, then it is possible to show that condition $\tilde{\mathbf{C}}\,\mathbf{2}$ can be replaced by the condition (cf. (2.14))

$\bar{\mathbf{C}}\,\mathbf{2}$ *There exist positive constants* c, C, $c \leqq C$, *such that for all* $t \in (0, n_0^{-1})$ *and for a.e.* $\tau \in (t - \bar{r}(t), t + \bar{r}(t))$

$$c\, \bar{w}(t) \leqq \bar{w}(\tau) \leqq C\, \bar{w}(t)\,, \quad c\, \bar{v}_1(t) \leqq \bar{v}_1(\tau) \leqq C\, \bar{v}_1(t)\,. \tag{2.16}$$

In this case we can replace condition **C 4** by the condition

$\bar{\mathbf{C}}\,\mathbf{4}$
$$\limsup_{t \to 0+} \frac{\bar{w}^{1/q}(t)}{\bar{v}_1^{1/p}(t)}\, \bar{r}^{(N/q)-(N/p)+1}(t) < \infty$$

and condition **C 4*** by the condition

$\bar{\mathbf{C}}\,\mathbf{4^*}$
$$\limsup_{t \to 0+} \frac{\bar{w}^{1/q}(t)}{\bar{v}_1^{1/p}(t)}\, \bar{r}^{(N/q)-(N/p)+1}(t) = 0\,.$$

In the rest of this section we will suppose that the functions $w(x)$, $v_1(x)$ satisfy (2.13) and $r(x)$ satisfies (2.15).

2.9. Corollary (the continuous imbedding). *Suppose conditions* **C 1**, $\bar{\mathbf{C}}\,\mathbf{2}$, $\tilde{\mathbf{C}}\,\mathbf{3}$ *are satisfied. Then the imbedding* (2.5) *holds if and only if condition* $\bar{\mathbf{C}}\,\mathbf{4}$ *is fulfilled.*

2.10. Corollary (the compact imbedding). *Let the inequality* $1/N > 1/p - 1/q$ *and conditions* **C 1***, $\bar{\mathbf{C}}\,\mathbf{2}$, $\tilde{\mathbf{C}}\,\mathbf{3}$ *be fulfilled. Then condition* $\bar{\mathbf{C}}\,\mathbf{4^*}$ *is necessary and sufficient for the imbedding* (2.6) *to take place.*

2.11. Remark. If the function $r(x)$ satisfies (2.15) then Assumption 2.1. (ii) can be replaced by the following one:

$$\bar{r}(t) \leqq t/3 \quad \textit{for} \quad t \in (0, n_0^{-1})$$

and

there exists a constant $c_{\bar{r}} \geqq 1$ *such that*

$$t \in (0, n_0^{-1}), \quad |t - \tau| < \bar{r}(t) \Rightarrow c_{\bar{r}}^{-1} \leqq \frac{\bar{r}(\tau)}{\bar{r}(t)} \leqq c_{\bar{r}}. \tag{2.17}$$

With regard to the inequality $\bar{r}(t) \leqq t/3$ for $t \in (0, n_0^{-1})$ we can see that condition (2.17) is satisfied if there exists a constant $c_{\bar{r}} \geqq 1$ such that

$$t \in (0, n_0^{-1}), \quad \frac{2}{3} < \frac{\tau}{t} < \frac{4}{3} \Rightarrow c_{\bar{r}}^{-1} \leqq \frac{\bar{r}(\tau)}{\bar{r}(t)} \leqq c_{\bar{r}}. \tag{2.18}$$

(Cf. with Property (H) from [2], Section 11.)

3. AUXILIARY ASSERTIONS

First, let us present some results by the second author [6] that we will use in the sequel.

3.1. Lemma. *Let* $p, q \in \langle 1, \infty)$, *let* $Q \subset \mathbb{R}^N$ *be an open set. Suppose that*

$$W^{1,p}(G; v_0, v_1) \hookrightarrow L^q(G; w) \tag{3.1}$$

for each domain G from a countable system of domains $\{G_n\}_{n=1}^{\infty}$ *such that* $G_n \subset G_{n+1} \subsetneqq Q$, $Q = \bigcup_{n=1}^{\infty} G_n$;

$$\lim_{n \to \infty} \sup_{\|u\|_{1,p,Q,v_0,v_1} \leqq 1} \|u\|_{q,Q \setminus G_n, w} < \infty. \tag{3.2}$$

Then

$$W^{1,p}(Q; v_0, v_1) \hookrightarrow L^q(Q; w). \tag{3.3}$$

Conversely, if the imbedding (3.3) *holds then condition* (3.2) *is satisfied.*

3.2. Remark. Lemma 3.1 *remains valid if conditions* (3.1)–(3.3) *are replaced by*

$$W^{1,p}(G; v_0, v_1) \hookrightarrow\hookrightarrow L^q(G; w), \tag{3.4}$$

$$\lim_{n \to \infty} \sup_{\|u\|_{1,p,Q,v_0,v_1} \leqq 1} \|u\|_{q,Q \setminus G_n, w} = 0, \tag{3.5}$$

$$W^{1,p}(Q; v_0, v_1) \hookrightarrow\hookrightarrow L^q(Q; w), \tag{3.6}$$

respectively.

3.3. Lemma (Besicovitch covering lemma). *Let A be a bounded set in* $\mathbb{R}^N$ *and* ϱ *a positive function defined on A. Then there exists a sequence* $\{x_k\}_{k=1}^{\infty} \subset A$ *such that the sequence of balls* $\{B_k\}_{k=1}^{\infty}$, $B_k = B_k(x_k, \varrho(x_k))$ *satisfies*

(i) $A \subset \bigcup_{k=1}^{\infty} B_k$.

(ii) *There exists a number* Θ *depending only on the dimension N such that for all* $z \in \mathbb{R}^N$ *we have* $\sum_{k=1}^{\infty} \chi_{B_k}(z) \leqq \Theta$ (*where* χ_{B_k} *is the characteristic function of* B_k).

(iii) *The sequence* $\{B_k\}$ *can be divided into* ξ *families of disjoint balls (the number* ξ *depends only on* N*).*

3.4. Remarks. Lemma 3.3 is valid also for *A an unbounded set* if we suppose that the function ϱ *is bounded on A.*

For the proof of Lemma 3.3 and Remark 3.4 see [1].

The next lemma shows how the imbedding constant in the Sobolev imbedding theorem $W^{1,p}(B) \hookrightarrow L^q(B)$ depends on the radius of the ball B.

3.5. Lemma. *Let* $1 \leqq p,\ q < \infty,\ 1/N \geqq 1/p - 1/q,\ R > 0,\ x \in \mathbb{R}^N$. *Then for* $u \in W^{1,p}(B(x, R))$,

$$(3.7) \qquad \left(\int_{B(x,R)} |u(y)|^q \,\mathrm{d}y\right)^{1/q} \leqq$$
$$\leqq K R^{N/q - N/p + 1} \left(R^{-p} \int_{B(x,R)} |u(y)|^p \,\mathrm{d}y + \int_{B(x,R)} |\nabla u(y)|^p \,\mathrm{d}y\right)^{1/p}$$

(*with* $K > 0$ *independent of* x, R *and* u).*)

Proof. The lemma is an easy consequence of the Sobolev imbedding theorem for $B(0, 1)$ and of the substitution theorem.

In the proofs of theorems from Section 2 we use the following

3.6. Lemma. *Suppose* $n \geqq 3m,\ m \in \mathbb{N},\ m \geqq 3$, *and* $B(x, r(x)) \cap \Omega^n \neq \emptyset$. *Then* $B(x, r(x)) \subset \Omega^m$.

Proof. Let $z \in B(x, r(x)),\ y \in B(x, r(x)) \cap \Omega^n$. Then we have

$$(3.8) \qquad d(z) \leqq d(y) + |y - z| < \frac{1}{n} + 2r(x) \leqq \frac{1}{n} + 2\,\frac{d(x)}{3}\,.$$

Further, we have

$$d(x) \leqq d(y) + |y - z| < \frac{1}{n} + r(x) \leqq \frac{1}{n} + \frac{d(x)}{3}\,,$$

therefore

$$(3.9) \qquad d(x) < \frac{3}{2n}\,.$$

Inequalities (3.8) and (3.9) imply

$$d(z) < \frac{2}{n} \leqq \frac{2}{3m} < \frac{1}{m + 1}\,.$$

This yields $z \in \Omega^m$ which completes the proof of Lemma 3.6.

4. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 2.2. By Lemma 3.1 it is sufficient to verify condition (3.2), where we set $Q = \Omega$ and $G_n = \Omega_{3n},\ n \in \mathbb{N}$.

*) We set $|\nabla u(x)|^p = \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}(x)\right|^p$.

Let us put $A = \Omega^{n_0}$ and $\varrho = r$ in Lemma 3.3. Then there exists a sequence $\{x_k\}_{k=1}^{\infty} \subset \Omega^{n_0}$ such that

$$\Omega^{n_0} \subset \bigcup_{k=1}^{\infty} B_k\,, \quad B_k = B(x_k, r(x_k))\,, \tag{4.1}$$

$$\sum_{k=1}^{\infty} \chi_{B_k}(z) \leqq \Theta\,, \quad z \in \mathbb{R}^N\,. \tag{4.2}$$

Fix $n \geqq n_0$ and denote

$$\mathscr{K}_n = \{k \in \mathbb{N};\ B_k \cap \Omega^{3n} \neq \emptyset\}\,. \ ^{*)}$$

By virtue of (4.1) we have

$$\|u\|^q_{q,\Omega\setminus G_n,w} = \|u\|^q_{q,\Omega\setminus\Omega_{3n},w} = \int_{\Omega^{3n}} |u(y)|^q\, w(y)\, \mathrm{d}y \leqq \sum_{k\in\mathscr{K}_n} \int_{B_k} |u(y)|^q\, w(y)\, \mathrm{d}y\,. \tag{4.3}$$

Using the first inequality in (2.2) and Lemma 3.5 we obtain

$$\begin{aligned} \int_{B_k} |u(y)|^q\, w(y)\, \mathrm{d}y &\leqq a_0(x_k) \int_{B_k} |u(y)|^q\, \mathrm{d}y \leqq \\ &\leqq [K a_0^{1/q}(x_k)\, r^{N/q - N/p + 1}(x_k)]^q\, [r^{-p}(x_k) \int_{B_k} |u(y)|^p\, \mathrm{d}y + \int_{B_k} |\nabla u(y)|^p\, \mathrm{d}y]^{q/p}\,, \end{aligned} \tag{4.4}$$

where $k \in \mathscr{K}_n$.

The second inequality in (2.2), (4.4), Assumption 2.1. (ii), (2.3) and (2.4) yield

$$\begin{aligned} &\int_{B_k} |u(y)|^q\, w(y)\, \mathrm{d}y \leqq \\ &\leqq \left[K \frac{a_0^{1/q}(x_k)}{a_1^{1/p}(x_k)}\, r^{N/q-N/p+1}(x_k)\right]^q \left[c_r^p \int_{B_k} |u(y)|^p \frac{v_1(y)}{r^p(y)}\, \mathrm{d}y + \int_{B_k} |\nabla u(y)|^p\, v_1(y)\, \mathrm{d}y\right]^{q/p} \leqq \\ &\leqq K^q \mathscr{A}_n^q [c_r^p K_0 \int_{B_k} |u(y)|^p\, v_0(y)\, \mathrm{d}y + \int_{B_k} |\nabla u(y)|^p\, v_1(y)\, \mathrm{d}y]^{q/p} \leqq \\ &\leqq K_1 \mathscr{A}_n^q [\int_{B_k} |u(y)|^p\, v_0(y)\, \mathrm{d}y + \int_{B_k} |\nabla u(y)|^p\, v_1(y)\, \mathrm{d}y]^{q/p}\,, \end{aligned} \tag{4.5}$$

where $K_1 = K^q(\max(c_r^p K_0, 1))^{q/p}$, $k \in \mathscr{K}_n$.

Inequalities (4.3), (4.5) and $q/p \geqq 1$, relations (4.2) and $\bigcup_{k=1}^{\infty} B_k \subset \Omega$ imply

$$\begin{aligned} &\|u\|^q_{q,\Omega\setminus G_n,w} \leqq \\ &\leqq K_1 \mathscr{A}_n^q \{\sum_{k\in\mathscr{K}_n} \int_{B_k} [|u(y)|^p\, v_0(y) + |\nabla u(y)|^p\, v_1(y)]\, \mathrm{d}y\}^{q/p} \leqq \\ &\leqq \Theta^{q/p} K_1 \mathscr{A}_n^q \|u\|^q_{1,p,\Omega,v_0,v_1}\,. \end{aligned} \tag{4.6}$$

Using (4.6) and condition **C 4** we get (3.2) and the theorem is proved.

Proof of Theorem 2.3. Inequality (4.6) and condition **C 4*** imply (3.5) and the proof is completed by Remark 3.2.

Proof of Theorem 2.4. By $^\wedge$**C 4** there exists an increasing sequence of natural

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*) By Lemma 3.6 we have $\bigcup_{k\in\mathscr{K}_n} B_k \subset \Omega^n \subset \Omega^{n_0}$ and this fact enables us to apply conditions **C 2** and **C 3** for points $y \in B_k$.

numbers $\{n_k\}_{k=1}^{\infty}$ and a sequence $\{x_k\}_{k=1}^{\infty}$, $x_k \in \Omega^{n_k}$,*) such that

$$\frac{\hat{a}_0(x_k)^{1/q}}{\hat{a}_1(x_k)^{1/p}} r^{N/q-N/p+1}(x_k) > k , \quad k \in \mathbb{N} . \tag{4.7}$$

Now let us put

$$u_k = \mathsf{R}_{r(x_k)/8}\, \chi_{3/4B_k} , \quad k = 1, 2, \ldots \tag{4.8}$$

(R_ε is a mollifier with radius ε, i.e. $(\mathsf{R}_\varepsilon f)(x) = \varepsilon^{-N} \int_{R^N} \Psi((x-y)/\varepsilon) f(y)\,dy$, where Ψ is a function such that $\Psi \in C_0^\infty(B(0,1))$, $\Psi \geqq 0$, $\int_{R^N} \Psi(x)\,dx = 1$). Functions u_k, $k = 1, 2, \ldots$, posses the following properties:

$$u_k \in C_0^\infty(B_k) , \quad 0 \leqq u_k \leqq 1 , \quad k = 1, 2, \ldots ; \tag{4.9}$$

$$u_k \equiv 1 \text{ on } \tfrac{1}{2}B_k , \quad k = 1, 2, \ldots ; \tag{4.10}$$

$$\exists c > 0 \Rightarrow \left|\frac{\partial u_k}{\partial x_i}(x)\right| \leqq \frac{c}{r(x_k)} , \quad x \in \Omega , \quad i = 1, 2, \ldots, N , \quad k = 1, 2, \ldots ; \tag{4.11}$$

$$u_k \in W^{1,p}(\Omega; v_0, v_1) , \quad k = 1, 2, \ldots . \tag{4.12}$$

Let $k \in \mathbb{N}$. By (4.10) and (2.7) we obtain

$$\begin{aligned}(\textstyle\int_\Omega |u_k(y)|^q\, w(y)\,dy)^{1/q} &\geqq (\textstyle\int_{(1/2)B_k} w(y)\,dy)^{1/q} \geqq \\ &\geqq 2^{-N/q} |B(0,1)|^{1/q}\, \hat{a}_0(x_k)^{1/q}\, r^{N/q}(x_k) .\end{aligned} \tag{4.13}$$

Now using (4.9), (4.11), (2.8), Assumption 2.1 (ii) and (2.7) we get

$$\begin{aligned}&(\textstyle\int_\Omega |u_k(y)|^p\, v_0(y)\,dy + \int_\Omega |\nabla u_k(y)|^p\, v_1(y)\,dy)^{1/p} \leqq \\ &\leqq (\textstyle\int_{B_k} v_0(y)\,dy + Nc^p \int_{B_k} r^{-p}(x_k)\, v_1(y)\,dy)^{1/p} \leqq \\ &\leqq (k_0^{-1} \textstyle\int_{B_k} v_1(y)\, r^{-p}(y)\,dy + Nc^p r^{-p}(x_k) \int_{B_k} v_1(y)\,dy)^{1/p} \leqq \\ &\leqq [(k_0^{-1} c_r^p + Nc^p)\, r^{-p}(x_k) \textstyle\int_{B_k} \hat{a}_1(x_k)\,dy]^{1/p} = \\ &= L r^{N/p-1}(x_k)\, \hat{a}_1(x_k)^{1/p} ,\end{aligned} \tag{4.14}$$

with $L = [(k_0^{-1} c_r^p + Nc^p)\, |B(0,1)|]^{1/p}$.

Now, suppose that

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w) . \tag{4.15}$$

Then (4.13) and (4.14) yield

$$2^{-N/q} |B(0,1)|^{1/q}\, \hat{a}_0(x_k)^{1/q}\, r^{N/q}(x_k) \leqq \tilde{K} L r^{N/p-1}(x_k)\, \hat{a}_1(x_k)^{1/p}$$

for $k \in \mathbb{N}$ ($\tilde{K}$ is the norm of the imbedding operator from (4.15)), which contradicts (4.7) and the theorem is prooved.

---

*) We can suppose that $n_k \geqq 3n_0$, $k = 1, 2, \ldots$, where n_0 is a number from Assumption 2.1. (ii). Then by Lemma 3.6 we have $B_k = B(x_k, r(x_k)) \subset \Omega^{n_0}$, $k = 1, 2, \ldots$, and we can use inequality (2.8) for $y \in B_k$.

Proof of Theorem 2.5. To prove the theorem it is sufficient to verify that condition (3.5) from Remark 3.2 is violated (provided we put $Q = \Omega$, $G_n = \Omega_n$).

By $^\wedge$**C 4*** there exist $\varepsilon > 0$, an increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$, $n_1 \geqq n_0$, and a sequence $\{x_k\}_{k=1}^{\infty}$, $x_k \in \Omega^{3n_k}$ such that

$$(4.16) \qquad \frac{\hat{a}_0(x_k)^{1/q}}{\hat{a}_1(x_k)^{1/p}}\, r^{N/q - N/p + 1}(x_k) \geqq \varepsilon\,, \quad k \in \mathbb{N}\,.$$

Further, by Lemma 3.6 we have

$$(4.17) \qquad B_k = B(x_k, r(x_k)) \subset \Omega^{n_k}\,, \quad k \in \mathbb{N}\,.$$

Set

$$(4.18) \qquad \tilde{u}_k = u_k / \|u_k\|_{1,p,\Omega,v_0,v_1}\,, \quad k \in \mathbb{N}\,,$$

where u_k are the functions from (4.8). By virtue of (4.9), (4.13) and (4.14) we have

$$\|\tilde{u}_k\|_{q,\Omega^{n_k},w} = \|\tilde{u}_k\|_{q,\Omega,w} \geqq L_1 \frac{\hat{a}_0(x_k)^{1/q}}{\hat{a}_1(x_k)^{1/q}}\, r^{N/q - N/p + 1}(x_k)\,, \quad k \in \mathbb{N}\,,$$

with $L_1 = 2^{-N/q} L^{-1} |B(0, 1)|^{1/q}$, and (4.16) yields

$$\sup_{\|u\|_{1,p,\Omega,v_0,v_1} \leqq 1} \|u\|_{q,\Omega^{n_k},w} \geqq \|\tilde{u}_k\|_{q,\Omega^{n_k},w} \geqq L_1 \varepsilon\,, \quad k \in \mathbb{N}\,,$$

which contradicts (3.5). Theorem 2.5 is proved.

5. EXAMPLES

In the following examples we will give some applications of Corollaries 2.9 and 2.10.

5.1. Example. Let α, β be real numbers. For $x \in \Omega$ we put

$$(5.1) \qquad w(x) = d^{\alpha}(x)\,, \quad v_0(x) = d^{\beta - p}(x)\,, \quad v_1(x) = d^{\beta}(x)\,.$$

Further, let us take $n_0 = 3$ (n_0 from Assumption 2.1.(ii)) and

$$(5.2) \qquad r(x) = d(x)/3\,, \quad x \in \Omega^{n_0}\,.$$

The function r has the form (2.15) with $\bar{r}(t) = t/3$. By Remark 2.11, Assumption 2.1.(ii) is fulfilled if the function $\bar{r}$ satisfies inequality (2.18). It is easy to see that this inequality is true with the constant $c_{\bar{r}} = 3/2$. Condition $^\sim$**C 3** is obviously fulfilled. Functions w, v_1 have the form (2.13), where

$$\bar{w}(t) = t^{\alpha}\,, \quad \bar{v}_1(t) = t^{\beta}\,.$$

Putting

$$c = \min\,\{(2/3)^{\alpha}, (4/3)^{\alpha}, (2/3)^{\beta}, (4/3)^{\beta}\}\,,$$
$$C = \max\,\{(2/3)^{\alpha}, (4/3)^{\alpha}, (2/3)^{\beta}, (4/3)^{\beta}\}$$

we can see that condition $^-$**C 2** is fulfilled as well.

A. Continuous imbedding

Condition **C 1** is an immediate consequence of the *classical Sobolev imbedding theorem*. By Corollary 2.9 the imbedding

$$W^{1,p}(\Omega; d^{\beta-p}, d^{\beta}) \hookrightarrow L^q(\Omega; d^{\alpha}) \tag{5.3}$$

takes place if and only if condition $^{-}$**C 4** holds or equivalently, if

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 \geqq 0\,. \tag{5.4}$$

B. Compact imbedding

Let $1/N > 1/p - 1/q$. Condition **C 1*** is an immediate consequence of the *Rellich-Kondrashov theorem*. Now, Corollary 2.10 implies that the imbedding

$$W^{1,p}(\Omega; d^{\beta-p}, d^{\beta}) \hookrightarrow\hookrightarrow L^q(\Omega; d^{\alpha}) \tag{5.5}$$

holds if and only if condition $^{-}$**C 4*** holds or equivalently, if

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0\,. \tag{5.6}$$

5.2. Example. Let $\alpha, \beta, \gamma, \delta$ be real numbers. For $x \in \Omega$ we put

$$\begin{aligned} w(x) &= \begin{cases} d^{\alpha}(x)\,|\log d(x)|^{\gamma} & \text{if } d(x) < 1/3 \\ 3^{-\alpha}\log^{\gamma} 3 & \text{if } d(x) \geqq 1/3\,, \end{cases} \\ v_0(x) &= \begin{cases} d^{\beta-p}(x)\,|\log d(x)|^{\delta} & \text{if } d(x) < 1/3 \\ 3^{p-\beta}\log^{\delta} 3 & \text{if } d(x) \geqq 1/3\,, \end{cases} \\ v_1(x) &= \begin{cases} d^{\beta}(x)\,|\log d(x)|^{\delta} & \text{if } d(x) < 1/3 \\ 3^{-\beta}\log^{\delta} 2 & \text{if } d(x) \geqq 1/3\,. \end{cases} \end{aligned} \tag{5.7}$$

Further, let us take $n_0 = 3$,

$$r(x) = d(x)/3\,, \quad x \in \Omega^{n_0}\,. \tag{5.8}$$

A. Continuous imbedding

By Corollary 2.9 the imbedding

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w) \tag{5.9}$$

takes place if and only if

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0 \quad \text{or} \tag{5.10}$$

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 = 0 \quad \text{and} \quad \frac{\gamma}{q} - \frac{\delta}{p} \leqq 0\,.$$

B. Compact imbedding

Let $1/N > 1/p - 1/q$. Now Corollary 2.10 implies that the imbedding

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w) \tag{5.11}$$

holds if and only if

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0 \quad \text{or} \tag{5.12}$$

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 = 0 \quad \text{and} \quad \frac{\gamma}{q} - \frac{\delta}{p} < 0\,.$$

5.3. Example. Let α, β be real numbers. For $x \in \Omega$ we put

$$w(x) = \exp\left(\alpha/d(x)\right), \quad v_0(x) = d^{-2p}(x) \exp\left(\beta/d(x)\right), \tag{5.13}$$
$$v_1(x) = \exp\left(\beta/d(x)\right).$$

Further, let us take $n_0 = 3$,

$$r(x) = d^2(x)\,, \quad x \in \Omega^{n_0}\,. \text{ *)} \tag{5.14}$$

A. Continuous imbedding

By Corollary 2.9 the imbedding

$$W^{1,p}(\Omega; d^{-2p} \exp(\beta/d), \exp(\beta/d)) \hookrightarrow L^q(\Omega; \exp(\alpha/d)) \tag{5.15}$$

takes place if and only if

$$\alpha/q - \beta/p \leqq 0\,. \tag{5.16}$$

B. Compact imbedding

Let $1/N > 1/p - 1/q$. Now Corollary 2.10 implies that the imbedding

$$W^{1,p}(\Omega; d^{-2p} \exp(\beta/d), \exp(\beta/d)) \hookrightarrow\hookrightarrow L^q(\Omega; \exp(\alpha/d)) \tag{5.17}$$

holds if and only if condition (5.16) is fulfilled.

6. IMBEDDINGS UNDER WEAKENED ASSUMPTIONS

In the proof of Theorems 2.2 and 2.3 we have used "the right hand side of inequality" (2.1).**) If condition (2.1) is not satisfied, Theorem 2.2 can be reformulated in the following way.

6.1. Theorem (sufficient conditions for the continuous imbedding). *Let the fol-*

*) If we put $r(x) = d(x)/M$ with $M \geqq 3$, then condition **C 2** would not be fulfilled.

**) "The left hand side of inequality" (2.1) was used in the proofs of Theorems 2.4 and 2.5.

lowing conditions be fulfilled:

C 1# $$W^{1,p}(\Omega_n; v, v) \hookrightarrow L^q(\Omega_n; w), \quad n \geqq n_0 . \;^{*})$$

C 2# *There exist positive measurable functions a_0, a_1 defined on Ω^{n_0} such that for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(x))$*

$$w(y) \leqq a_0(x), \quad a_1(x) \leqq v(y). \tag{6.1}$$

C 4# $$\lim_{n \to \infty} \mathscr{B}_n = \mathscr{B} < \infty, \quad \textit{where}$$

$$\mathscr{B}_n = \sup_{x \in \Omega^n} \frac{a_0^{1/q}(x)}{a_1^{1/p}(x)} r^{N/q - N/p}(x). \tag{6.2}$$

Then

$$W^{1,p}(\Omega; v, v) \hookrightarrow L^q(\Omega; w). \tag{6.3}$$

Proof is a modification of that of Theorem 2.2. Inequality (4.4) is now replaced by the estimate

$$\int_{B_k} |u(y)|^q \, w(y) \, dy \leqq \tag{6.4}$$
$$\leqq [K a_0^{1/q}(x_k) \, r^{N/q - N/p}(x_k)]^q \, [\int_{B_k} |u(y)|^p \, dy + r^p(x_k) \int_{B_k} |\nabla u(y)|^p \, dy]^{q/p},$$

where $k \in \mathscr{K}_n$.

From the assumptions that Ω is bounded and $r(x) \leqq d(x)/3$, $x \in \Omega^{n_0}$, we obtain

$$r^p(x_k) \leqq (\operatorname{diam} \Omega/6)^p, \quad k \in \mathscr{K}_n.$$

Analogously as in the proof of Theorem 2.2 we get

$$\|u\|^q_{q, \Omega \setminus G_n, w} \leqq \Theta^{q/p} K_1 \mathscr{B}_n^q \|u\|^q_{1,p,\Omega,v,v} \tag{6.5}$$

($K_1 = K^q[\max(1, (\operatorname{diam} \Omega/6)^p)]^{q/p}$), which together with **C 4#** completes our proof.

The next theorem is an analogue of Theorem 2.4.

6.2. Theorem. *Let the following conditions be fulfilled:*

^C 2# *There exist positive measurable functions $\hat{a}_0$, $\hat{a}_1$ defined on Ω^{n_0} such that for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(x))$*

$$w(y) \geqq \hat{a}_0(x), \quad \hat{a}_1(x) \geqq v(y). \tag{6.6}$$

^C 4 $$\lim_{n \to \infty} \hat{\mathscr{A}}_n = \infty, \quad \textit{where } \hat{\mathscr{A}}_n \textit{ is defined by } (2.9).$$

Then $W^{1,p}(\Omega; v, v)$ is not imbedded in $L^q(\Omega; w)$.

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*) Let us recall that we still suppose that the weight function v ($v = v_0 = v_1$) satisfies assumption (1.4).

Proof is analogous to that of Theorem 2.4. Using (4.9), (4.11) and (6.6) we get

$$(6.7)\qquad \left(\int_\Omega |u_k(y)|^p v(y)\,\mathrm{d}y + \int_\Omega |\nabla u_k(y)|^p v(y)\,\mathrm{d}y\right)^{1/p} \leqq$$
$$\leqq \left(\int_{B_k} v(y)\,\mathrm{d}y + Nc^p \int_{B_k} r^{-p}(x_k)\, v(y)\,\mathrm{d}y\right)^{1/p} \leqq$$
$$\leqq \left(\left(\int_{B_k} v(y)\,\mathrm{d}y\right)\left(1 + \frac{Nc^p}{r^p(x_k)}\right)\right)^{1/p} \leqq$$
$$\leqq \left(\int_{B_k} \hat{a}_1(x_k)\,\mathrm{d}y\right)^{1/p}\left(1 + \frac{Nc^p}{r^p(x_k)}\right)^{1/p} \leqq L r^{N/p-1}(x_k)\, \hat{a}_1(x_k)^{1/p}$$

(with $L = (2N|B(0, 1)|)^{1/p} c)$ for $k \in \mathbb{N}$ such that $n_k \geqq 1/(3cN^{1/2})$.

If we suppose that

$$(6.8)\qquad W^{1,p}(\Omega; v, v) \hookrightarrow L^q(\Omega; w)\,,$$

than (4.13) and (6.7) give

$$\frac{\hat{a}_0(x_k)^{1/q}}{\hat{a}_1(x_k)^{1/p}}\, r^{N/q-N/p+1}(x_k) \leqq \tilde{K} L |B(0, 1)|^{-1/q}$$

($\tilde{K}$ is the norm of the imbedding operator in (6.8)), which contradicts (4.7). The theorem is proved.

For completeness we shall formulate the corresponding theorems in the case of compact imbeddings. Their proofs are left to the reader.

6.3. Theorem (sufficient conditions for the compact imbedding). *Let us suppose $1/N > 1/p - 1/q$ and condition* **C 2**$^{\#}$. *Moreover, let the following conditions be fulfilled:*

C 1$^{\#*}$ $\qquad W^{1,p}(\Omega_n; v, v) \hookrightarrow\hookrightarrow L^q(\Omega_n; w)\,, \quad n \geqq n_0\,.$

C 4$^{\#*}$ $\qquad \lim_{n\to\infty} \mathscr{B}_n = 0\,,$ *where $\mathscr{B}_n$ is defined by* (6.2).

Then

$$(6.9)\qquad W^{1,p}(\Omega; v, v) \hookrightarrow\hookrightarrow L^q(\Omega; w)\,.$$

6.4. Theorem. *Suppose condition* $^{\wedge}$**C 2**$^{\#}$. *Further, let the following condition hold:*

$^{\wedge}$**C 4*** $\qquad \lim_{n\to\infty} \hat{\mathscr{A}}_n > 0\,,$ *where $\hat{\mathscr{A}}_n$ is defined by* (2.9).

Then $W^{1,p}(\Omega; v, v)$ is not completely imbedded in $L^q(\Omega; w)$.

It is the intention of the authors to develop the present topics in further papers, including imbedding theorems for special weight functions, for weight functions which have singularities or degenerations on some part of the boundary $\partial\Omega$ or inside Ω, and imbedding theorems on unbounded domains.

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