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DIRECT DECOMPOSABILITY OF TOLERANCES AND CONGRUENCES ON SEMIGROUPS

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Let S be a semigroup. By a tolerance T on S we mean a reflexive and symmetric binary relation on S with the Substitution Property with respect to the multiplication in S, i.e., T is a subsemigroup of the direct product $S \times S$. The set of all tolerances on S constitutes an algebraic lattice $\operatorname{Tol}(S)$ with respect to the set inclusion [1]. The lattice $\operatorname{Con}(S)$ of all congruences on S is not its sublattice in general.

Let A and B be two semigroups, $A \times B$ their direct product and $T \in \operatorname{Tol}(A \times B)$. T is said to be *directly tolerance decomposable* if there exist $T_1 \in \operatorname{Tol}(A)$ and $T_2 \in \operatorname{Tol}(B)$ such that $T = T_1 \times T_2$. (We have $(x, y) \in T_1 \times T_2$ if and only if $(\mathrm{pr}_1 x, \mathrm{pr}_1 y) \in T_1$ and $(\mathrm{pr}_2 x, \mathrm{pr}_2 y) \in T_2$.) Analogously $\Theta \in \operatorname{Con}(A \times B)$ is called *directly congruence decomposable* if there exist $\Theta_1 \in \operatorname{Con}(A)$ and $\Theta_2 \in \operatorname{Con}(B)$ such that $\Theta = \Theta_1 \times \Theta_2$. If every tolerance (congruence) on $A \times B$ is directly tolerance (congruence) decomposable, we say that $A \times B$ has *directly decomposable tolerances (congruences)*. Abbreviated: $A \times B$ has DDT (DDC, respectively).

The aim of this paper is to describe all direct products of commutative semigroups which have DDT or DDC. Congruences and tolerances on direct products were studied in several papers ([2], [3], and [4]). Terminology and notation not defined here may be found in [5] and [6].

Let S be a semigroup. The notation S^1 stands for S if S has an identity, otherwise for S with an identity adjoined. For $a, b \in S$ we denote by $T_S(a, b)$ ($\Theta_S(a, b)$) the least tolerance (congruence) on S containing (a, b), i.e., $T_S(a, b)$ ($\Theta_S(a, b)$) is the principal tolerance (congruence) on S generated by (a, b).

It is very easy to verify the following.

Lemma 1. *Let S be a commutative group and let $a, b \in S$, $a \neq b$. Then for $x, y \in S$, $x \neq y$, we have $(x, y) \in T_S(a, b)$ if and only if there exist $z \in S$ and a positive integer m such that either $(x, y) = (a, b)^m (z, z)$ or $(x, y) = (b, a)^m (z, z)$.*

Theorem 1. *Let A, B be non-trivial semigroups. If $A \times B$ has DDT or DDC, then A and B are simple.*

Proof. We shall show that A is simple. Let $a_1, a_2 \in A$. Choose $b_1, b_2 \in B$ with $b_1 \neq b_2$. Put $R = (A^1 a_2 A^1 \times B) \times (A^1 a_2 A^1 \times B) \cup \mathrm{id}_{A \times B}$. It is easy to show that R is a congruence on $A \times B$ and $((a_2, b_1), (a_2, b_2)) \in R$.

Case 1. Suppose that $A \times B$ has DDT. Put $T = T_{A \times B}((a_2, b_1), (a_2, b_2))$. Then $T = T_1 \times T_2$, where $T_1 \in \mathrm{Tol}(A)$ and $T_2 \in \mathrm{Tol}(B)$. Hence we have $(b_1, b_2) \in T_2$ and so $((a_1, b_1), (a_1, b_2)) \in T \subseteq R$. Thus $a_1 \in A^1 a_2 A^1$, which means that A is simple.

Case 2. Assume that $A \times B$ has DDC. Using the same method of proof as in Case 1, we obtain that $((a_1, b_1), (a_1, b_2)) \in \Theta_{A \times B}((a_2, b_1), (a_2, b_2)) \subseteq R$ and so $a_1 \in A^1 a_2 A^1$. Therefore A is simple.

The following lemma shows that the simplicity of non-trivial semigroups A and B need not imply that $A \times B$ has DDT or DDC.

Lemma 2. *Let L be a two-element left zero semigroup. Then $L \times L$ has neither DDT nor DDC.*

Proof. Let $L = \{e, f\}$. It is clear that $\mathrm{Tol}(L) = \mathrm{Con}(L) = \{\mathrm{id}_L, L \times L\}$. We have $\mathrm{card}\, \mathrm{id}_L = 2$ and $\mathrm{card}\, L \times L = 4$. Put $\Theta = \mathrm{id}_{L \times L} \cup \{((e, e), (f, f)), ((f, f), (e, e))\}$. Evidently $\Theta \in \mathrm{Con}\,(L \times L) \subseteq \mathrm{Tol}\,(L \times L)$ and $\mathrm{card}\, \Theta = 6$. If $\Theta = T_1 \times T_2$, where $T_1, T_2 \in \mathrm{Tol}\,(L)$, then $\mathrm{card}\, \Theta \in \{4, 8, 16\}$, which is a contradiction.

Lemma 3. *Let A, B be semigroup and let $a_1, a_2 \in A$, $b_1, b_2 \in B$. Suppose that $A \times B$ has DDT. If $(x, y) \in T_A(a_1, a_2)$ and $(u, v) \in T_B(b_1, b_2)$, then $((x, u), (y, v)) \in$ $\in T_{A \times B}((a_1, b_1), (a_2, b_2))$.*

Proof. Put $T = T_{A \times B}((a_1, b_1), (a_2, b_2))$. If $A \times B$ has DDT, then $T = T_1 \times T_2$, where $T_1 \in \mathrm{Tol}\,(A)$ and $T_2 \in \mathrm{Tol}\,(B)$. Hence we have $(a_1, a_2) \in T_1$, $(b_1, b_2) \in T_2$ and so $(x, y) \in T_1$, $(u, v) \in T_2$. Therefore $((x, u), (y, v)) \in T_1 \times T_2 = T$.

For any element x of a semigroup S we denote by $\langle x \rangle$ the subsemigroup of S generated by x. If S is a periodic group, then by the order of $x \in S$ we mean $\mathrm{ord}\, x =$ $= \mathrm{card}\, \langle x \rangle$.

Theorem 2. *Let A, B be non-trivial commutative semigroups. Then the following conditions are equivalent:*

(i) *$A \times B$ has DDT;*

(ii) *$A \times B$ has DDC;*

(iii) *A, B are periodic groups and* $\mathrm{ord}\, a$, $\mathrm{ord}\, b$ *are relatively prime whenever $a \in A$, $b \in B$.*

Proof. (i) or (ii) $\Rightarrow$ (iii). It is well known that every commutative simple semigroup S is a group and $\mathrm{Tol}\,(S) = \mathrm{Con}\,(S)$. According to Theorem 1, A and B are groups and $A \times B$ has DDT. Let $e = e^2 \in A$ and $f = f^2 \in B$.

Now we shall show that the groups A and B are periodic. By way of contradiction, we assume that a is an aperiodic element of A. Choose $b \in B$ with $b \neq f$. It follows from Lemma 3 that $((a^2, f), (a^4, b)) \in T_{A \times B}((a, f), (a^2, b))$. According to Lemma 1, we have the following possibilities:

Case 1. $(a^2, f) = (a, f)^m (x, y)$ and $(a^4, b) = (a^2, b)^m (x, y)$ for some positive integer m, $x \in A$ and $y \in B$. Then $a^2 = a^m x$, $a^4 = a^{2m} x$ and so $a^4 = a^{m+2}$. Hence we have $m = 2$ and so $f = fy = y$, $b = b^2 y = b^2$. Consequently $b = f$, which is a contradiction.

Case 2. $(a^2, f) = (a^2, b)^m (x, y)$ and $(a^4, b) = (a, f)^m (x, y)$ for some positive integer m, $x \in A$ and $y \in B$. Thus we have $a^2 = a^{2m} x$, $a^4 = a^m x$ and so $a^{m+4} = a^2$. Hence $a^{m+2} = e$, which is a contradiction.

Consequently, A is a periodic group. Analogously we can show that B is a periodic group.

Finally, we shall prove that ord a, ord b are relatively prime whenever $a \in A$, $b \in B$. We can suppose that $a \neq e$ and $b \neq f$. Let k be a positive integer such that k divides ord a and ord b. According to Lemma 3, we have $((a, f), (e, f)) \in T_{A \times B}((a, f), (e, b))$. Lemma 1 implies the following two possibilities:

Case 1. $(a, f) = (a, f)^m (x, y)$ and $(e, f) = (e, b)^m (x, y)$ for some positive integer m, $x \in A$ and $y \in B$. Then $x = e$, $y = f$ and so $a = a^m$, $f = b^m$. Hence k divides $m - 1$ and m. Consequently, k divides 1.

Case 2. $(a, f) = (e, b)^m (x, y)$ and $(e, f) = (a, f)^m (x, y)$ for some positive integer m, $x \in A$ and $y \in B$. Thus we have $x = a$, $y = f$ and so $e = a^{m+1}$, $f = b^m$. Hence k divides $m + 1$ and m. Consequently, k divides 1.

Therefore ord a and ord b are relatively prime.

(iii) $\Rightarrow$ (i) and (ii). Let T be a tolerance on $A \times B$, where A and B are periodic commutative groups. Put $(a, c) \in T_1$ if and only if there exist $u, v \in B$ such that $((a, u), (c, v)) \in T$. Analogously we put $(b, d) \in T_2$ if and only if $((x, b), (y, d)) \in T$ for some $x, y \in A$. It is easy to show that $T_1 \in \mathrm{Tol}(A)$, $T_2 \in \mathrm{Tol}(B)$ and $T \subseteq T_1 \times T_2$.

Now, we shall prove that $T_1 \times T_2 \subseteq T$. Let $(a, c) \in T_1$ and $(b, d) \in T_2$. Then $((a, u), (c, v)) \in T$ and $((x, b), (y, d)) \in T$ for some $x, y \in A$ and $u, v \in B$. Hence we have $((ac^{-1}, uv^{-1}), (e, f)) \in T$ and $((xy^{-1}, bd^{-1}), (e, f)) \in T$, where $e = e^2 \in A$ and $f = f^2 \in B$. By hypothesis ord ac^{-1} and ord uv^{-1} are relatively prime and so there exist integers i and j such that

$$(1) \qquad i \operatorname{ord} ac^{-1} + j \operatorname{ord} uv^{-1} = 1 .$$

Analogously we can get

$$(2) \qquad r \operatorname{ord} xy^{-1} + s \operatorname{ord} bd^{-1} = 1$$

for some integers r and s. Put $k = j \operatorname{ord} uv^{-1}$ and $m = r \operatorname{ord} xy^{-1}$. Then by (1) and (2) we have $((ac^{-1}, f), (e, f)) = ((ac^{-1}, uv^{-1})^k, (e, f)^k) \in T$ and $((e, bd^{-1}, (e, f)) = ((xy^{-1}, bd^{-1})^m, (e, f)^m) \in T$. Therefore $((ac^{-1}, bd^{-1}), (e, f)) \in T$ and so $((a, b), (c, d)) \in T$. This means that $T_1 \times T_2 \subseteq T$. Consequently $T = T_1 \times T_2$ and so $A \times B$ has DDT. Since A, B are groups, $A \times B$ has DDC.

Let A_i $(i = 1, 2, \ldots, n)$ be a semigroup, by induction we can define that their their direct product $\bigtimes_{i=1}^{n} A_i$ has directly decomposable tolerances (congruences). Abbreviated: $\bigtimes_{i=1}^{n} A_i$ has DDT (DDC, respectively).

Theorem 3. *Let A_i $(i = 1, 2, \ldots, n)$ be non-trivial commutative semigroups. Then the following conditions are equivalent*

(i) $\bigtimes_{i=1}^{n} A_i$ *has DDT;*

(ii) $\bigtimes_{i=1}^{n} A_i$ *has DDC;*

(iii) A_i $(i = 1, 2, \ldots, n)$ *is a periodic group and* $\operatorname{ord} a_j$, $\operatorname{ord} a_k$ *are relatively prime whenever* $a_j \in A_j$, $a_k \in A_k$ *and* $j, k \in \{1, 2, \ldots, n\}$, $j \neq k$.

The proof follows from Theorem 2 by induction.

If V is a class of semigroups such that for every pair $A, B \in V$, $A \times B$ has DDT (DDC), V is said to have *directly decomposable tolerances* (congruences). Abbreviated: V has DDT (DDC, respectively).

Theorem 4. *Every variety of semigroups having DDT or DDC is trivial.*

Proof. Let V be a variety of semigroups having DDT or DDC. By way of contradiction we suppose that there exists A of V such that $\operatorname{card} A \geqq 2$. Let $a \in A$. Then $\langle a \rangle \in V$ and so $\langle a \rangle \times \langle a \rangle$ has DDT or DDC. It follows from Theorem 2 that $\operatorname{card} \langle a \rangle = 1$ and so $a^2 = a$. Hence A is a band. It is well known that A is a semilattice S of rectangular bands. Thus we have $S \in V$ and so $S \times S$ has DDT or DDC. Theorem 2 implies $\operatorname{card} S = 1$ and so A is a rectangular band. Then there exists either a two-element left zero subsemigroup L of A or a two-element right zero subsemigroup R of A. Thus we have either $L \in V$ or $R \in V$, which is a contradiction, (see Lemma 2 and its dual).

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