Bohdan Zelinka

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PARTITIONABILITY OF TREES

Bohdan Zelinka, Liberec

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At the Third Czechoslovak Symposium on Graph Theory [1] in Prague in 1982, C. St. J. A. Nash-Williams proposed a problem to characterize k-partitionable finite graphs. Here we transfer the problem concerning graphs in general to a problem concerning trees only.

Let k be a positive integer. A finite graph G is called *k-partitionable*, if G has connected subgraphs $G_1, \ldots, G_r$ such that $|V(G_1)| = \ldots = |V(G_r)| = k$, and each vertex of G belongs to exactly one G_i. The set $\{G_1, \ldots, G_r\}$ will be called a *k-partition of G*.

In his comments to the problem the author noted that for $k = 2$ a graph has this property if and only if it has a linear factor. Obviously a necessary (not sufficient) condition for the k-partitionability of a graph G is that k divides the number n of vertices of G; then $r = n/k$.

Theorem 1. *Let G be a finite connected graph, let k be a positive integer. Then the graph G is k-partitionable if and only if G contains a k-partitionable spanning tree T.*

Proof. Let G be k-partitionable, let $\{G_1, \ldots, G_r\}$ be its k-partition. Each graph G_i for $i = 1, \ldots, r$ is connected and therefore we may choose its spanning tree T_i. Let G_0 be the graph whose vertex set is $\{G_1, \ldots, G_r\}$ and in which two vertices G_i, G_j for $i \neq j$ are adjacent if and only if in G there exists an edge joining a vertex of G_i with a vertex of G_j. As G is connected, so is G_0. Hence we may choose a spanning tree T_0 of G_0. If two vertices G_i, G_j are adjacent in T_0, we choose one edge of G joining a vertex of G_i with a vertex of G_j; we do this for all such pairs i, j and denote the set of edges chosen in this way by E_0. Now let T be the graph on the vertex set $V(G)$ of G whose edge set is the union of E_0 and of the edge sets of all trees T_i for $i = 1, \ldots, r$. Evidently T is a spanning tree of G and is k-partitionable; the k-partition is $\{T_1, \ldots, T_r\}$.

Now suppose that G contains a k-partitionable spanning tree T. Then there exist subtrees $T_1, \ldots, T_r$ of T such that $|V(T_i)| = k$ for $i = 1, \ldots, r$ and each vertex of T (i.e. of G) belongs to exactly one of them. For $i = 1, \ldots, r$ let G_i be the subgraph of G induced by the vertex set $V(T_i)$ of T_i. As G_i contains a spanning tree T_i, it is connected. Therefore G is k-partitionable. □

Now we will consider trees. We shall use the concept of median of a tree, as it was introduced in [2] and studied in [3].

Let T be a tree. For any two vertices x, y of T let $d(x, y)$ denote he distance between x and y in T, i.e. the length of the path connecting x and y in T. For each vertex x of T let $a(x) = (1/n) \sum_{y \in v(T)} d(x, y)$, where n is the number of vertices of T. A vertex at which the functional $a(x)$ attains its minimum is called a median of T. In [3] it was proved that every finite tree contains either exactly one median, or exactly two medians; if there are two medians, then they are adjacent.

Theorem 2. *Let T be a k-partitionable finite tree with n vertices. Then there exists a unique k-partition $\{T_1, \ldots, T_r\}$ of T.*

Remark. As above, the symbol r denotes n/k.

Proof. We shall proceed by induction according to r. The case $r = 1$ (i.e. $k = n$) is trivial. Now let $r_0 \geqq 2$ and suppose that the assertion is true for all r such that $1 \leqq r < r_0$. Let T be an (n/r_0)-partitionable tree and suppose that there exist two different k-partitions $\{T_1, \ldots, T_{r_0}\}$, $\{T'_1, \ldots, T'_{r_0}\}$ of T. Analogously as we have assigned the graph G_0 to the k-partition $\{G_1, \ldots, G_r\}$ in the proof of Theorem 1, now we assign the graph T_0 to the k-partition $\{T_1, \ldots, T_{r_0}\}$. Evidently T_0 is a tree and therefore it contains terminal vertices. Let T_i be a terminal vertex of T_0. Suppose that $T_i \notin \{T'_1, \ldots, T'_{r_0}\}$. Then there exist integers j, k from $\{1, \ldots, r_0\}$ such that $V(T_i) \cap V(T'_j) \neq \emptyset$, $V(T_i) \cap V(T'_k) \neq \emptyset$. There exists exactly one edge joining a vertex of T_i with a vertex of $V(T) - V(T_i)$ and this edge can be contained in at most one of the trees T'_j, T'_k; without loss of generality suppose that it is not contained in T'_j. Then T'_j must be a subtree of T_i; as $T_i \neq T'_j$, it is a proper subtree. But then $|V(T'_j)| < |V(T_i)| = k$, which is a contradiction. Hence T_i is one of the trees $T'_1, \ldots, T'_{r_0}$. Let T^* be the subtree of T induced by $V(T) - V(T_i)$. The tree T^* is k-partitionable, because the set of all T_j for $1 \leqq j \leqq r_0$, $j \neq i$ is a k-partition of T^*. The number of vertices of T^* is $n - k$, the number of trees in a k-partition of T^* is $r_0 - 1$, therefore by the induction hypothesis the set $\{T_1, \ldots, T_{r_0}\} - \{T_i\}$ is the unique k-partition of T^*. This implies that $\{T_1, \ldots, T_{r_0}\}$ is the unique k-partition of T. □

This also implies that a necessary condition for the k-partitionability of a tree T with n vertices is that there exist two vertex-disjoint subtrees of T, one having n/k and the other $n - n/k$ vertices. We shall prove a theorem concerning the existence of such trees.

Theorem 3. *Let T be a finite tree with n vertices, let q be a positive integer, $q < n$. Then the following two assertions are equivalent:*

(i) *There exist vertex-disjoint subtrees T_1, T_2 of T such that T_1 has q vertices and T_2 has $n - q$ vertices.*

(ii) *There exist adjacent vertices* v_1, v_2 *of* T *such that* $a(v_1) - a(v_2) = 1 - 2q/n$.

Proof. (i) $\Rightarrow$ (ii). Let (i) hold. Let v_1 (or v_2) be the vertex of T_1 (or of T_2) adjacent to a vertex of T_2 (or of T_1, respectively). Consider the functional $a(x)$. For any $x \in V(T_1)$ we have $d(v_1, x) = d(v_2, x) - 1$; for any $x \in V(T_2)$ we have $d(v_1, x) = d(v_2, x) + 1$. Hence $a(v_1) = a(v_2) + (1/n)(|V(T_2)| - |V(T_1)|) = (1/n)(n - 2q) = 1 - 2q/n$.

(ii) $\Rightarrow$ (i). Let (ii) hold. By deleting the edge v_1v_2 from T we obtain a graph with two connected components which are both trees. Let T_1 (or T_2) be the one of them which contains v_1 (or v_2, respectively). Again for any $x \in V(T_1)$ we have $d(v_1, x) = d(v_2, x) - 1$ and for any $x \in V(T_2)$ we have $d(v_1, x) = d(v_2, x) + 1$. This implies $a(v_1) = a(v_2) + (1/n)(|V(T_2)| - |V(T_1)|)$, therefore $a(v_1) - a(v_2) = (1/n)(|V(T_2)| - |V(T_1)|)$. As $a(v_1) - a(v_2) = 1 - 2q/n$, we have

$$|V(T_2)| - |V(T_1)| = n - 2q\,.$$

on the other hand,

$$|V(T_1)| + |V(T_2)| = n$$

and this yields $|V(T_1)| = q$, $|V(T_2)| = n - q$.

This enables us to recognize whether a given tree T is k-partitionable. We determine $a(x)$ for all $x \in V(T)$ and all differences $a(x) - a(y)$ for adjacent vertices x, y. If some of them equals $1 - 2k$, then there exists a subtree T' of T having $r = n/k$ vertices and such that the subgraph T_1 of T induced by $V(T) - V(T')$ is a tree. If such a tree exists, we continue doing the same with the tree T_1 as before with T. If it does not exist, we are sure that T is not k-partitionable. Thus we transfer the problem of k-partitionability of T to the problem of k-partitionability of a proper subtree of T. Continuing this process, after a finite number of steps we either find out that T is not k-partitionable, or arrive at a subtree of T having k vertices and thus trivially k-partitionable. In the second case T is k-partitionable.

From Theorem 3 an assertion on medians follows.

Theorem 4. *A finite tree* T *with* n *vertices is* $(n/2)$*-partitionable if and only if it has two medians.*

Proof. Let T be $(n/2)$-partitionable. Then (i) from Theorem 3 holds for $q = n/2$ and thus there exist adjacent vertices v_1, v_2 of T such that $a(v_1) = a(v_2)$. Now let w be a vertex of T_1 adjacent to v_1. The vertex v_1 is adjacent to both v_2 and w; thus according to [3], $a(v_1) < \max(a(v_2), a(w))$ and this implies $a(w) > a(v_1)$. If w' is a vertex of T_1 adjacent to w and distinct from v_1, then again $a(w) < \max(a(v_1), a(w'))$, which implies $a(w') > a(w)$. By induction we may prove that $a(x) > a(v_1)$ for all $x \in V(T_1) - \{v_1\}$ and analogously also $a(x) > a(v_2)$ for all $x \in V(T_2) - \{v_2\}$. This means that $a(x) > a(v_1) = a(v_2)$ for all $x \in V(T) - \{v_1, v_2\}$ and v_1, v_2 are medians of T.

Now suppose that T has two medians v_1, v_2. Then $a(v_1) = a(v_2)$ and thus (ii) from Theorem 3 holds for $q = n/2$. This implies (i) from Theorem 3 for $q = n/2$ and T is $(n/2)$-partitionable. □

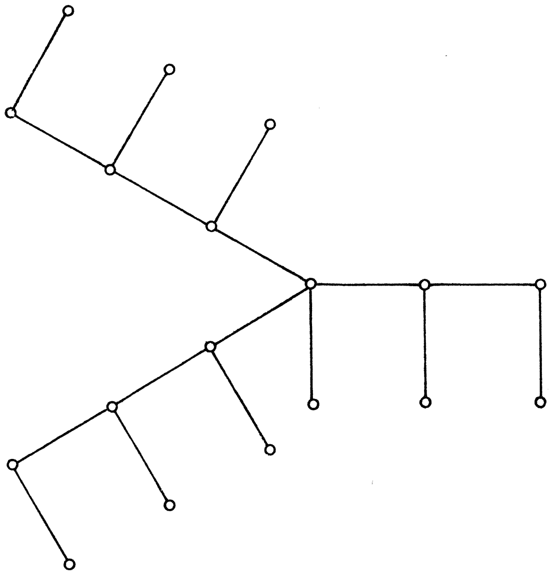
Let us again consider the tree whose vertex set is the k-partition of a tree T and in which two vertices are adjacent if and only if there exists an edge of T joining a vertex of one of them with a vertex of the other. Theorem 2 implies that this tree is uniquely determined by the tree T and the number k; thus it is natural to denote it by T/k.

Theorem 5. *Let T be a finite tree, let k, m be positive integers. Let T be k-partitionable. Then T is (km)-partitionable if and only if T/k is m-partitionable. In this case $T/(km) \cong (T/k)/m$.*

Proof. Suppose that T/k is m-partitionable and consider the graph $(T/k)/m$. Its vertices are the subtrees of T/k forming the m-partition of T/k. Any such tree consists of m vertices; each of these vertices is a subtree of T having k vertices. To any such subtree of T/k we assign a subgraph of T which is induced by the union of vertex sets of all subtrees of T which are vertices of the above mentioned subtree of T/k. This subgraph is evidently connected, i.e., it is a subtree of T, and has km vertices. Thus these subtrees form a (km)-partition of T and T is (km)-partitionable. This consideration also implies $T/(km) \cong (T/k)/m$.

Now suppose that T is (km)-partitionable and consider the graphs T/k and $T/(km)$. We shall proceed by induction on $r = n/(km)$. For $r = 1$ the assertion is trivial. Let $r_0 \geqq 2$ and suppose that the assertion is true for all r such that $1 \leqq r < r_0$. Suppose that for our numbers k, m the equality $r_0 = n/(km)$ holds. Let T' be the subtree of T which is a teminal vertex of $T/(km)$. Suppose that there exists a tree T'' from the k-partition of T such that $V(T'') \cap V(T') \neq \emptyset$, $V(T'') - V(T') \neq \emptyset$. Then the tree T'' contains the edge joining a vertex of T' with a vertex not in T'. This implies that there is only one tree with this property. Let $|V(T') \cap V(T'')| = p$. Then $1 \leqq p < k$. Any other tree from the k-partition of T which has a non-empty intersection with T' must be a subtree of T', because there is only one edge joining a vertex of T' with a vertex not in T'. Therefore there are trees having k vertices with the property that each vertex of $V(T') - V(T'')$ belongs to exactly one of them. But $|V(T') - V(T'')| = km - p$. As $1 \leqq p < k$, the number $km - p$ is not divisible by k and this is a contradiction. We have the result that any tree which is a terminal vertex of $T/(km)$ is k-partitionable and its k-partition is a subset of the k-partition of T. Now let T''' be the tree obtained from T by deleting all vertices of T'. This tree is (km)-partitionable with $n - km$ vertices, thus the value of r for it is $r_0 - 1$. According to the induction hypothesis T'''/k is m-partitionable. If we add a new tree which is T'/k to the m-partition of T'''/k, we obtain an m-partition of T/k. Therefore T/k is m-partitionable. □

Note that if T is k-partitionable and (km)-partitionable, it need not be m-partitionable. The tree in Fig. 1 is 2-partitionable and 6-partitionable, but not 3-partitionable.



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Author's address: 461 17 Liberec 1, Studentská 1292, Czechoslovakia (katedra tváření a plastů VŠST).