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CONVEX SUBSETS OF PARTIAL MONOUNARY ALGEBRAS

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Partial monounary algebras were investigated in several papers, e.g., [1], [2], [7]–[12].

To each partial monounary algebra $\mathscr{A} = (A, f)$ there corresponds a directed graph $G(\mathscr{A}) = (A, E)$ without loops and multiple edges which is defined as follows: an ordered pair (a, b) of distinct elements of A belongs to E iff $f(a) = b$.

A subset B of A will be called *convex* (in $\mathscr{A}$) if, whenever a, b_1, b_2 are distinct elements of A having the property that $b_1, b_2 \in B$ and there is a path (in $G(\mathscr{A})$) going from b_1 to b_2 and containing the element a, then a belongs to B as well.

We denote by $\mathrm{Co}(\mathscr{A})$ the system of all convex subsets of $\mathscr{A}$. The system $\mathrm{Co}(\mathscr{A})$ is partially ordered by inclusion, and it is a lattice.

The aim of the present paper is to investigate the question to what extent the partial operation f on A is determined by the system $\mathrm{Co}(\mathscr{A})$. We shall describe all partial operations g on A such that $\mathrm{Co}(A, f) = \mathrm{Co}(A, g)$. In particular, necessary and sufficient conditions for f will be found under which f is uniquely determined by the system $\mathrm{Co}(A, f)$.

Convex subsets of partially ordered sets were studied by Birkhoff and Bennett [6]; cf. also Bennett and Birkhoff [5] and Bennett [3], [4]. Some results of the article [6] are applied in the present paper.

1. BASIC NOTATION

By a (partial) monounary algebra we understand a pair (A, f), where A is a non-empty set, $f\colon A \to A$ is a (partial) mapping. Let $\mathscr{U}$ be the class of all partial monounary algebras, $\mathscr{P}$ the class of all partially ordered sets and $\mathscr{P}_1$ the class of all partially ordered sets with the greatest element.

Let $(A, f) \in \mathscr{U}$, $x \in A$, $n \in N$, where N is the set of all positive integers. Put $f^0(x) = x$. Assume that $f^{n-1}(x)$ exists and $f^{n-1}(x) = z$. If $f(z)$ exists in (A, f), then we set $f^n(x) = f(z)$. Further, $f^{-n}(x) = \{y \in A : f^n(y) = x\}$. If $x, y \in A$, $f^n(x) = f^m(y)$ for some $n, m \in N \cup \{0\}$, then we write $x \equiv_f y$. The relation $\equiv_f$ is an equivalence relation on A. A partial monounary algebra is said to be connected, if $A/\equiv_f$ is a one-element set. If $X \in A/\equiv_f$, then X is called a *connected component* of (A, f).

If $x, y \in A$, $n \in N$, then $f^n(x) = y$ means that $f^n(x)$ exists and coincides with y; on the other hand, $f^n(x) \neq y$ means that if $f^n(x)$ exists, then it does not coincide with y.

1.1. Notation. Let $(N, f_i) \in \mathscr{U}$ (for $i = 0, 1, 2, 3$) be such that $f_0(n) = n + 1$, $f_1(n + 1) = f_2(n + 1) = f_3(n + 1) = n$ for each $n \in N$, $f_1(1) = 1$, $f_3(1) = 2$, and $f_2(1)$ does not exist. The class of all (A, f) which are isomorphic to a subalgebra of (N, f_i) for some $i \in \{0, \ldots, 3\}$ will be denoted by $\mathscr{U}_0$.

1.2. Notation. Let $(A, f) \in \mathscr{U} - \mathscr{U}_0$ be connected and such that one of the following conditions is valid:

(a) there is $c \in A$ with $f(c) = c$;

(b) there is $c \in A$ such that $f(c)$ does not exist;

(c) there is $c \in A$ with $f^{-1}(c) = \{f(c)\} \neq \{c\}$.

The class of all such (A, f) is denoted by the symbol $\mathscr{U}_1$.

1.3. Notation. Let $\mathscr{U}_2$ be the class of all connected monounary algebras which possess a two-element cycle and do not belong to $\mathscr{U}_0 \cup \mathscr{U}_1$. Further, let $\mathscr{U}_3$ be the class of all connected monounary algebras which contain a cycle with more than 2 elements, and $\mathscr{U}_4$ the class of all connected monounary algebras which are not in $\mathscr{U}_0 \cup \mathscr{U}_1 \cup \mathscr{U}_2 \cup \mathscr{U}_3$.

1.4. Remark. Observe that $(A, f) \in \mathscr{U}_4$ if and only if $(A, f) \in \mathscr{U}$ is connected, complete, contains no cycle and either (a) there is $\{a_n : n \in Z\} \subseteq A$ such that $f(a_{n+1}) = a_n$ for each $n \in Z$, or (b) there are $a, b \in A$, $a \neq b$, such that $f(a) = f(b)$.

Remark. We shall use the symbols $\vee$, $\wedge$, $\leqq$ for the lattice $\mathrm{Co}(A, f)$. Further, instead of $\{a\} \in \mathrm{Co}(A, f)$ we shall write $a \in \mathrm{Co}(A, f)$ and instead of $\emptyset \in \mathrm{Co}(A, f)$ we shall use also the symbol 0.

1.5. Lemma. *If $(A, f) \in \mathscr{U}$, then $\mathrm{Co}(A, f)$ is a complete lattice.*

Proof. The assertion is obvious.

1.6. Lemma. *If $(A, f) \in \mathscr{U}$, then $\mathrm{Co}(A, f)$ is an atomic lattice.*

Proof. Every nonempty convex subset of A is the join (in fact the union) of its (convex) one-element subsets $a \in A$. Since for each $a \in A$, a is an atom in $\mathrm{Co}(A, f)$, we obtain that $\mathrm{Co}(A, f)$ is atomic.

2. PARTIAL MONOUNARY ALGEBRAS AND POSETS

In this section we shall investigate whether $\mathrm{Co}(A, f) \cong \mathrm{Co}(P)$ for some $P \in \mathscr{P}$, provided $(A, f) \in \mathscr{U}_0 \cup \ldots \cup \mathscr{U}_4$ is given.

2.1. Lemma. *If $(A, f) \in \mathscr{U}_0 \cup \mathscr{U}_1$ and there is $c \in A$ such that either $f(c) = c$ or $f(c)$ does not exist, then there is $(A, \lesssim) \in \mathscr{P}_1$ such that $\mathrm{Co}(A, f) = \mathrm{Co}(A, \lesssim)$.*

Proof. Let $a, b \in A$. Put $a \lesssim b$ if and only if there is $n \in N \cup \{0\}$ with $f^n(a) = b$.

Then c is the greatest element of $(A, \leqq)$, $(A, \leqq) \in \mathscr{P}_1$. Further, it is obvious that $\mathrm{Co}(A, f) = \mathrm{Co}(A, \leqq)$.

2.2. Lemma. *If $(A, f) \in \mathscr{U}_0 \cup \mathscr{U}_1$ and there is $c \in A$ with $f^{-1}(c) = \{f(c)\} \neq c$, then there is $(A, \leqq) \in \mathscr{P}_1$ such that $\mathrm{Co}(A, f) = \mathrm{Co}(A, \leqq)$.*

Proof. For $a, b \in A$, $b \neq c$ put $a \leqq b$ if and only if there is $n \in N \cup \{0\}$ with $f^n(a) = b$. Further, let $a \leqq c$ for each $a \in A$. Then $\leqq$ is a partial order on A and c is the greatest element of $(A, \leqq)$. Obviously, $\mathrm{Con}(A, f) = \mathrm{Con}(A, \leqq)$.

2.3. Lemma. *Let $(A, f) \in \mathscr{U}_0 \cup \mathscr{U}_4$ and let neither the assumption of* 2.1, *nor the assumption of* 2.2 *be valid. Then there is $(A, \leqq) \in \mathscr{P} - \mathscr{P}_1$ such that $\mathrm{Co}(A, f) = \mathrm{Co}(A, \leqq)$.*

Proof. It follows from the assumption and from 1.4 that (A, f) is connected and possesses no cycle. For $a, b \in A$ put $a \leqq b$ if there is $n \in N \cup \{0\}$ such that $f^n(a) = b$. Then $(A, \leqq) \in \mathscr{P} - \mathscr{P}_1$ and $\mathrm{Co}(A, f) = \mathrm{Co}(A, \leqq)$.

2.3.1. Remark. The partially ordered set $(A, \leqq)$ defined in the proofs of 2.1–2.3 will be called a *poset corresponding to a given $(A, f) \in \mathscr{U}_0 \cup \mathscr{U}_1 \cup \mathscr{U}_4$.*

Observe that $(A, \leqq)$ corresponding to (A, f) is uniquely determined except the case when $(A, f) \in \mathscr{U}_0$ is a two-element cycle; in the following proofs when (A, f) is a two-element cycle, take arbitrary $(A, \leqq)$ corresponding to (A, f).

2.4. Lemma. *If $(A, f) \in \mathscr{U}_3$, $P \in \mathscr{P}$, then $\mathrm{Co}(A, f)$ and $\mathrm{Co}(P)$ are not isomorphic.*

Proof. If $P \in \mathscr{P}$, then $\mathrm{Co}(P)$ is join-semidistributive (i.e. $a, b, c \in \mathrm{Co}(P)$, $a \vee b = a \vee c$ imply $a \vee b = (a \vee (b \wedge c)))$ according to [6]. Let C be a cycle of (A, f). Then card $C > 2$, thus there are distinct elements $a, b, c \in C$ and we have $a \vee c = C = a \vee b$, $C \neq a = a \vee 0 = a \vee (b \wedge c)$ (in $\mathrm{Co}(A, f)$).

2.5. Lemma. *If $(A, f) \in \mathscr{U}_2$, $P \in \mathscr{P}$, then $\mathrm{Co}(A, f)$ and $\mathrm{Co}(P)$ are not isomorphic.*

Proof. Since $(A, f) \in \mathscr{U}_2$, there are distinct elements $a, b, c, d \in A$ such that $f(a) = f(c) = b$, $f(b) = f(d) = c$. Then

(1) $b \leqq a \vee c$,

(2) $c \leqq b \vee d$.

Let $\mathrm{Co}(A, f) \cong \mathrm{Co}(P, \leqq)$ for some $(P, \leqq) \in \mathscr{P}$. Since A is the set of atoms of $\mathrm{Co}(A, f)$ and P is the set of atoms of $\mathrm{Co}(P)$, without loss of generality we may assume that $A = P$. Then (1) and (2) yield

(1′) either $a \leqq b \leqq c$ or $c \leqq b \leqq a$,

(2′) either $b \leqq c \leqq d$ or $d \leqq c \leqq b$.

Thus either

(3) $a \leqq b \leqq c \leqq d$,

or

(4) $d \leqq c \leqq b \leqq a$.

If (3) is valid, then $b \leqq a \vee d$ in $\mathrm{Co}(P)$, and $b \nleqq a \vee d$ in $\mathrm{Co}(A, f)$, a contradiction; if (4) holds, then $c \leqq a \vee d$ in $\mathrm{Co}(P)$, $c \nleqq a \vee d$ in $\mathrm{Co}(A, f)$, which is a contradiction.

3. COHERENCY OF PARTIAL MONOUNARY ALGEBRAS

In this section we shall prove some auxiliary results concerning coherency of partial monounary algebras.

3.1. Notation. If $(A,f) \in \mathscr{U}$, we put

$S_1(A,f) = \{x \in A : f^{-1}(x) = \emptyset, f(x) \neq x$ and either $f^2(x)$ does not exist or $f^2(x) = f(x)\} \cup \{x \in A$: the connected component containing x is a two-element cycle$\}$,
$S(A,f) = S_1(A,f) \cup \{f(x) : f^{-1}(f(x)) - f(x) \subseteq S_1(A,f)\} \cup \{x \in A$: the connected component containing x is a one-element set$\}$.

Example. If (A,f) is a partial monounary algebra from Fig. 1 or Fig. 2, then $S_1(A,f) = \{a, b\}$, $S(A,f) = \{a, b, c\}$
or $S_1(A,f) = \{a, b_1, b_2\} = S(A,f)$, respectively.

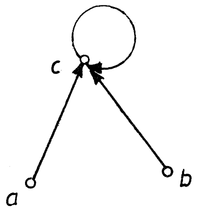

Fig. 1

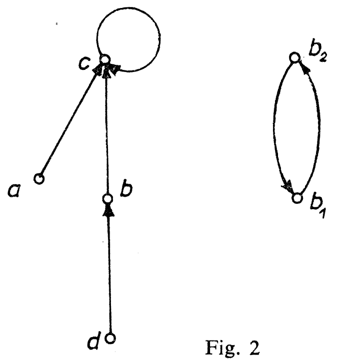

Fig. 2

3.1.1. Remark. If $(A,f) \in \mathscr{U}$, $x \in S(A,f)$, then x belongs to a connected component A' of (A,f) such that $(A',f) \in \mathscr{U}_0 \cup \mathscr{U}_1$ and (A',f) satisfies either the assumption of 2.1 or the assumption of 2.2.

3.1.2. Remark. If $(A,f) \in \mathscr{U}$, then $A - S(A,f)$ is closed under f.

3.2. Lemma. *Let $(A,f) \in \mathscr{U}$, $x \in A$. Then $x \in S(A,f)$ if and only if*
(i) *$x \vee y$ covers x and y (in $\mathrm{Co}(A,f)$) for each $y \in A - \{x\}$, and*
(ii) *if z_1, z_2 are distinct elements of $A - \{x\}$, then $x \not\leqq z_1 \vee z_2$.*

Proof. Let $x \in S(A,f)$. It follows from the definition that if $y \in A - \{x\}$, then $x \vee y = \{x, y\}$, which covers x and y. Now 3.1.1, 2.1 and 2.2 imply that there is $(A', \lesssim)$ corresponding to (A',f), where A' is the connected component of (A,f) containing x. Let $z_1, z_2 \in A - \{x\}$, $x \leqq z_1 \vee z_2$. Then $z_1, z_2 \in A'$ and either

(1) $z_1 \lesssim x \lesssim z_2$,

or

(2) $z_2 \lesssim x \lesssim z_1$.

We can assume that (1) is valid. This yields that $f(z_1) = x$ or $z_1 = x$, hence $f(z_1) = x$, and analogously, $f(x) = z_2$. Thus $x \notin S_1(A,f)$, since $f^{-1}(x) \neq \emptyset$ and x does not belong to a two-element cycle. We have $x \in S(A,f)$, therefore there is $y \in S_1(A,f)$ such that $x = f(y)$, $f^{-1}(f(y)) - f(y) \subseteq S_1(A,f)$. Since the connected component containing y is the connected component containing x and $x \notin S_1(A,f)$, we obtain that it is not a two-element cycle. Further,

$$z_2 = f(x) = f^2(y)\,,$$

thus from the definition of $S_1(A,f)$ we have that $f^2(y) = f(y) = x$, i.e. $z_2 = x$, which is a contradiction.

Suppose that (i) and (ii) hold. First assume that $f(x)$ does not exist. If $f^{-2}(x) \neq \emptyset$, there are distinct elements $x_1, x_2 \in A$ with $f(x_1) = x_2$, $f(x_2) = x$, and then $x_1 < x_1 \vee x_2 = \{x_1, x_2\} < \{x_1, x_2, x\} = x_1 \vee x$, a contradiction to (i). Hence $f^{-2}(x) = \emptyset$ and for $y \in f^{-1}(x)$ we obtain that $y \in S_1(A,f)$, thus $x \in S(A,f)$. If $f(x) = x$, the proof is analogous. Suppose that $f(x) \neq x$. If $f^2(x)$ exists and $f^2(x) \notin \{x, f(x)\}$, we get a contradiction, since

$$x < x \vee f(x) < x \vee f^2(x)\,.$$

Therefore

(3) either $f^2(x)$ does not exist or $f^2(x) \in \{x, f(x)\}$.

Let there be $z \in f^{-1}(x) - \{f(x)\}$. Then $x \leqq z \vee f(x)$, z and $f(x)$ are distinct, $z \neq x \neq f(x)$, a contradiction to (ii). This, according to (3), implies that $x \in S_1(A,f)$, thus $x \in S(A,f)$.

3.3. Definition. $(A,f) \in \mathcal{U}$ is said to be *coherent*, if it is connected and either $\operatorname{card} A = 1$ or $S(A,f) = \emptyset$.

A poset P is called *coherent*, if it is connected and no maximal element of P covers any minimal element of P (cf. [6]).

3.4. Lemma. (i) *If $(A,f) \in \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4$, then (A,f) is coherent.*

(ii) *If $(A,f) \in \mathcal{U}_0$, then (A,f) is coherent if and only if* card $A \neq 2$.

(iii) *If $(A,f) \in \mathcal{U}_0 \cup \mathcal{U}_1$, then (A,f) is coherent if and only if the poset $(A, \leqq)$ corresponding to (A,f) is coherent.*

Proof. Let (A,f) be a connected partial monounary algebra. It follows from 3.1.1 that if $S(A,f) \neq \emptyset$, then $(A,f) \in \mathcal{U}_0 \cup \mathcal{U}_1$. Thus we obtain that (i) is valid. The assertion (ii) is obvious.

Let $(A,f) \in \mathcal{U}_0 \cup \mathcal{U}_1$.

If card $A = 1$, then (iii) holds. Let card $A > 1$. Suppose that (A,f) is not coherent, i.e. $S(A,f) \neq \emptyset$. Thus 3.1.1 implies that one of the following conditions is satisfied:

(a) there are $x, c \in A$ such that $f(c) = c$, $x \in f^{-1}(c)$, $f^{-1}(x) = \emptyset$;

(b) there are $x, c \in A$ such that $f(c)$ does not exist, $x \in f^{-1}(c)$, $f^{-1}(x) = \emptyset$;

(c) A is a two-element cycle.

In each of the cases (a)–(c) a minimal element of $(A, \leqq)$ is covered by a maximal element, therefore $(A, \leqq)$ is not coherent.

Let $(A, \leqq)$ be not coherent. Let $c, x \in A$ be such that c is a maximal element of

$(A, \preceq)$, x is a minimal element of $(A, \preceq)$ and x is covered by c. Then some of the following conditions is satisfied:

(a1) $f(c) = c$, $x \in f^{-1}(c)$, $f^{-1}(x) = \emptyset$;

(b1) $f(c)$ does not exist, $x \in f^{-1}(c), f^{-1}(x) = \emptyset$;

(c1) $A = \{x, c\}$, $f(x) = c$, $f(c) = x$.

Therefore $x \in S(A,f)$, and (A,f) is not coherent.

4. THE CLASSES $\mathscr{U}_i$ $(i = 0, 1, \ldots, 4)$

In this section we thoroughly investigate the classes $\mathscr{U}_i$ $(i = 0, 1, \ldots, 4)$ in order to obtain the following result. Let $i \in \{0, 1, \ldots, 4\}$, let (A,f) and (A, g) be partial monounary algebras with coherent connected components such that $\mathrm{Co}(A,f) = \mathrm{Co}(A, g)$. If A' is a connected component of (A,f) and $(A',f) \in \mathscr{U}_i$, then A' is a connected component of (A, g) and $(A', g) \in \mathscr{U}_i$.

4.1.1. Lemma. *Let the following conditions be satisfied:*

(i) (A,f) *is a partial monounary algebra with coherent connected components,* $n > 2$, $N_1 = \{1, 2, \ldots, n\}$ or $N_1 = N$, $A' = \{a_k \colon k \in N_1\} \subseteq A$;

(ii) (a) $a_1 \vee a_3 \neq a_2 \vee a_3$;

(b) $a_1 \vee a_k$ *covers* $a_1 \vee a_{k-1}$ *(in* $\mathrm{Co}(A,f)$*) for each* $k \in N_1 - \{1\}$;

(c) *if* $x \in A - A'$, $a \in A'$, *then* $x \vee a$ *covers* x *and* a (in $\mathrm{Co}(A,f)$).

Then A' is a connected component of (A,f).

Proof. Let the assumption hold. By A_1 we denote the connected component of (A,f) containing a_1. Assume that there is $x \in A_1 - A'$. We shall consider the following two cases:

(1) x belongs to some cycle C or $f(x)$ does not exist;

(2) (1) does not hold.

Let (1) be valid. If $a \in A'$, then (ii) (c) implies that some of the following conditions is satisfied:

(3.1) $\{a, x\} \subseteq C$;

(3.2) $f(a) = x$ and if x belongs to a cycle C, then $a \notin C$;

(3.3) $a \notin A_1$.

Thus for $a_1 \in A'$ (and analogously for a_2, a_3) either (3.1) or (3.2) or (3.3) holds. If $f(x)$ does not exist, then (3) (i.e. (3.1)–(3.3)) implies that $f(a_1) = f(a_2) = f(a_3) = x$, thus $a_1 \vee a_3$ does not cover $a_1 \vee a_2$, a contradiction to (ii) (b). Therefore x belongs to a cycle C and the only possibilities (in view of (3) and (ii) (b)) are:

(4.1) $a_1 \notin C$, $f(a_1) = x$, $\{a_2, a_3\} \subseteq C$;

(4.2) $\{a_1, a_2\} \subseteq C$, $a_3 \notin C$, $f(a_3) = x$.

If (4.1) holds, then $a_1 \vee a_2 = a_1 \vee a_3$, thus (ii) (b) is not valid. If (4.2) holds, then $a_1 \vee a_3 = a_2 \vee a_3$, a contradiction to (ii) (a). Hence we obtain that (1) is not valid, i.e.,

(*) $x \in A_1 - A'$ implies that $f(x)$ exists and x does not belong to a cycle.

For $a \in A'$, (ii) (c) yields that some of the following conditions is satisfied:

(5.1) $f(x) = a$,

(5.2) $f(a) = x$,

(5.3) $a \notin \{f^i(x): i \in N\}$, $x \notin \{f^i(a): i \in N\}$.

This is valid for $a_1, a_2, a_3 \in A'$ and since $a_2 < a_1 \vee a_3$ (in view of (ii) (b)), we obtain either

(6.1) $a_1 \notin \{f^i(x): i \in N\}$, $x \notin \{f^i(a_1): i \in N\}$

or

(6.2) $f(x) = a_1$.

Therefore

(6.3) $a_1 \notin \{f^i(x): i \in N,\ i > 1\}$, $x \notin \{f^i(a_1): i \in N\}$.

There are $j \in N$, $l \in N \cup \{0\}$ with

$$f^j(x) = f^l(a_1)\,, \quad f^{j_1}(x) \neq f^{l_1}(a_1) \quad \text{for} \quad (j_1, l_1) \neq (j, l)\,,$$
$$j_1 \leqq j\,, \quad l_1 \leqq l\,.$$

Pur $f^j(x) = b$. First, let $j > 1$. Then $l \geqq 1$, $b \notin A'$ (in view of the conditions (5)) and (ii) (c) implies that $l = 1$. From (*) for the element $b \in A_1 - A'$ we have that $f(b)$ exists and b does not belong to a cycle. Put $b' = f(b)$. From (5) we obtain that $b' \in A'$ (if we consider the elements b', a_1), $b' = a_i$ for some $i \in N - \{1\}$. Then

$$a_2 \leqq a_1 \vee a_3 \leqq \ldots \leqq a_1 \vee a_i = \{a_1, b, b'\}\,,$$

$b \notin A'$, hence $i = 2$. But

$$a_1 < b \vee a_1 < b' \vee a_1 = a_2 \vee a_1\,,$$

which is a contradiction to (ii) (b). Now let $j = 1$, $l > 1$. Then (ii) (c) (for b and a_1) yields that there is $i \in N$, $i \neq 1$ such that $b = a_i$. Since $a_1 \vee a_i$ does not cover a_1, we obtain that $i \neq 2$. From the assumption that each connected component of (A, f) is coherent we conclude that either

(7.1) there is $y \in A$ with $f(y) = x$, or

(7.2) $f(b) = b' \neq b$.

If (7.1) holds, then (5) (for y, a_i) implies that $y \in A'$, $y = a_j$ for some $j \in N$. We get

$$a_2 \leqq a_1 \vee a_j = \{a_1, a_j\}\,,$$

thus $j = 2$ and

$$a_1 \vee a_2 = \{a_1, a_2\} \nleqq \{a_1, f(a_1), \ldots, f^l(a_1) = a_i\} = a_1 \vee a_i\,,$$

a contradiction. If (7.2) holds, then (5) (for b', a_1) yields that $b' \in A'$ and (5) (for x, b') implies that $b' \notin A'$, which is a contradiction. Now suppose that $j = 1$, $l = 1$. As above, (7.1) or (7.2) is valid. If $b \in A'$, $b = a_i$ for some $i \in N$, $i > 1$, then

$$a_2 \leqq a_1 \vee a_3 \leqq \ldots \leqq a_1 \vee a_i = \{a_1, a_i\}$$

and $i = 2$. If (7.1) holds, then $y \vee a_2 = \{y, x, a_2\}$ does not cover a_2, and (ii) (c) yields that $y \in A'$. Thus (ii) (b) implies $a_2 \leqq a_1 \vee y = \{a_1, y\}$, which is a contradiction. If (7.2) is valid, then $b' \vee a_1$ does not cover a_1, thus $b' \in A'$ according to (ii) (c). Then $x \vee b'$ does not cover x, which contradicts (ii) (c). Therefore $b \notin A'$. According to (*) for the element b we get that $f(b)$ exists and b does not belong to a cycle. Put $b' = f(b)$. Then (5) (for b' and a_1) implies that $b' \in A'$, but (5) (for x and b') yields that then $b' \notin A'$, a contradiction. Finally, assume that $j = 1, l = 0$, i.e. $b = a_1$. We have either (7.1) or (7.2); first let (7.1) be valid. According to (ii) (c) we get $y \in A'$, i.e., there is $j \in N, j > 1$ with $y = a_j$. Since $a_2 \leqq a_1 \vee a_j = \{a_1, x, a_j\}$, the relation $j = 2$ holds. In view of (5) (for x and a_3) we obtain that either $f(a_3) = x$ or $a_3 \notin \{f^i(x): i \in N\}$, $x \notin \{f^i(a_3): i \in N\}$. The both cases contradict (ii) (b) ($a_1 \vee a_3$ does not cover $a_1 \vee a_2$). Now let (7.2) hold. From (ii) (c) (for x and b') we get that $b' \notin A'$, and then (5) (for b' and a_3, for x and a_3) implies

(8) $a_3 \notin \{f^i(b'): i \in N\}$, $b' \notin \{f^i(a_3): i \in N\}$.

Therefore $a_1 \vee a_3 = \{a_1, a_3\}$, a contradiction to (ii) (b).

In each case we have got a contradiction, thus

(9) $A_1 - A' = \emptyset$, i.e., $A_1 \subseteq A'$.

Since $a_2 \leqq a_1 \vee a_i$ for each $i > 1$, we obtain that a_1, a_2, a_i belong to the same connected component (for $i \in N - \{1, 2\}$), hence

(10) $A' \subseteq A_1$.

Therefore (9) and (10) imply that $A_1 = A'$.

4.1.2. Lemma. *Let* (i) *and* (ii) *from* 4.1.1 *hold. Then* $(A', f) \in \mathcal{U}_0$ *and either* $f(a_k) = a_{k-1}$ *for each* $k \in N_1 - \{1\}$ *or* $N_1 = N$ *and* $f(a_k) = a_{k+1}$ *for each* $k \in N_1$.

Proof. 4.1.1 implies that A' is a connected component of (A, f). Assume that $(A', f) \notin \mathcal{U}_0$. Then either

(1) there are distinct elements $x, y \in A' - \{a_1\}$ such that $f(x) = a_1$, $f(a_1) = y$, or

(2) the condition (1) is not satisfied and there are $x, y \in A'$, $x \neq y$ such that $f(y) \neq x \neq f(x) = f(y) \neq y \neq f(x)$. If (1) holds, then $a_1 \vee x \nleqq a_1 \vee y \nleqq a_1 \vee x$, a contradiction to (ii). Thus either

(3) $f^{-1}(a_1) = \emptyset$.

or

(4) $f(a_1) = a_1$, or $f(a_1)$ does not exist, or $f^{-1}(a_1) = \{f(a_1)\} \neq \{a_1\}$.

According to (2) we may assume (without loss of generality) that $a_1 \vee y \leqq \leqq a_1 \vee x$, thus $y \leqq a_1 \vee x$. First let $y \neq a_1$. Then $x \neq a_1$ and (in view of (2)) there is a cycle C of (A', f) such that

(5) $\{y, a_1\} \subseteq C$, $a_1 \neq f(x) \in C$.

Then card $C \geqq 2, f^{-1}(a_1) \neq \emptyset$, thus (4) is valid and then card $C = 2$, $C = \{y, a_1\}$. Thus $f(y) = a_1 \neq f(x)$, a contradiction to (2). Now let $y = a_1$. Then (3) holds and there is $j \in N_1, j > 1$ with $f(a_1) = f(x) = a_j \in A'$. We obtain

$$a_1 \vee x \nleqq a_1 \vee a_j \nleqq a_1 \vee x ,$$

a contradiction to (ii) (b). Hence we have proved that $(A', j) \in \mathcal{U}_0$. Then it is obvious that (ii) (b) implies that either $f(a_k) = a_{k-1}$ for $k \in N_1 - \{1\}$ or $f(a_k) = a_{k+1}$ for each $k \in N$.

4.1.3. Lemma. *Let* (i) *from* 4.1.1 *hold. The following conditions are equivalent:*

(i) *A' is a connected component of (A, f), $(A', f) \in \mathscr{U}_0$ and either $f(a_k) = a_{k-1}$ for each $k \in N_1 - \{1\}$ or $N_1 = N$, $f(a_k) = a_{k+1}$ for each $k \in N_1$;*

(ii) (ii) from 4.1.1 is valid.

Proof. It is obvious that (i) yields (ii). The converse implication follows from 4.1.2.

4.1.4. Corollary. (a) *If $(A, f) \in \mathscr{U}$,* card $A = 1$, *then (A, f) is coherent and belongs to $\mathscr{U}_0$.*

(b) *Let $(A, f) \in \mathscr{U}$,* card $A > 1$. *Then (A, f) is coherent and belongs to $\mathscr{U}_0$ if and only if* card $A > 2$ *and*

(i) $S(A, f) = \emptyset$,

(ii) *there is $N_1 = \{1, 2, \ldots, n\}$, $n > 2$ or $N_1 = N$ such that $A = A' = \{a_k : k \in N_1\}$ fulfils* (ii) *from* 4.1.1.

4.2.1. Lemma. $(A, f) \in \mathscr{U}$, $C = \{c_1, c_2\} \subseteq A' \subseteq A$, $c_1 \neq c_2$. *The following conditions are equivalent:*

(i) *C is a cycle of (A', f), where A' is a connected component of (A, f), $(A', f) \in \mathscr{U}_2$;*

(ii) (a) $c_1 \vee c_2 = C$;

(b) *if $x \in A'$, then there is a unique $c = c(x) \in C$ with $x \vee c \ngeq C$;*

(c) $\{x \in A' : c(x) = c_1\} - \{c_1\} \neq \emptyset \neq \{x \in A' : c(x) = c_2\} - \{c_2\}$;

(d) *if $x \in A' - \{c_1\}$, $y \in A' - \{c_2\}$ and $c(x) = c_1$, $c(y) = c_2$, then $x \vee y \ngeq C$;*

(e) *$z \in A - A'$ implies that $z \vee c_1 \ngeq C$, $z \vee c_2 \ngeq C$.*

Proof. Let (i) hold. If $x \in A'$, then there exists the smallest $n \in N \cup \{0\}$ such that $f^n(x) \in C = \{c_1, c_2\}$; denote $c(x) = f^n(x)$. From this and from the definition of $\mathscr{U}_2$ we obtain that (ii) is valid.

Assume that (ii) is satisfied. Since the assumptions concerning c_1 and c_2 are symmetric, in view of (a) we can suppose that some of the following cases occurs: (1.1) $f^n(c_1) \neq c_2$, $f^n(c_2) \neq c_1$ for each $n \in N$; (1.2) $f(c_1) = c_2$, $f(c_2) = u \notin \{c_1, c_2\}$; (1.3) $f(c_1) = c_2 = f(c_2)$; (1.4) $f(c_1) = c_2$ and $f(c_2)$ does not exist; (1.5) $f(c_1) = c_2$, $f(c_2) = c_1$. Since $c(c_2) = c_2$, according to (c) we obtain that $A' - \{c_1, c_2\} \neq \emptyset$. Further, (A', f) is connected, because if not, then for $x \in A'$ which do not belong to the same component as e.g. c_2 we obtain

$$(2) \quad \begin{cases} x \vee c_2 = \{x, c_2\}, & \text{hence} \quad x \vee c_2 \ngeq C, \\ c_2 \notin x \vee c_1, & \text{hence} \quad x \vee c_1 \ngeq C, \end{cases}$$

which contradicts (b). If (1.1) holds, then we get a contradiction analogously by taking $x = f(c_1)$. Let (1.3) or (1.4) be valid. We either have

(3) there is $x \in A' - \{c_1, c_2\}$ with $f(x) = c_2$,

or (3) fails to hold. Then (3) implies $x \vee c_1 \ngeq C$, $x \vee c_2 \ngeq C$, which contradicts

(ii) (b). If (3) does not hold, then for each $x \in A' - \{c_1, c_2\}$ we have

$$x \vee c_1 \ngeqq C\,, \quad x \vee c_2 \geqq C\,, \quad \text{i.e.} \quad \{x \in A' \colon c(x) = c_2\} - \{c_2\} = \emptyset\,,$$

which contradicts (ii) (c). Assume that (1.2) holds. Take arbitrary elements $x, y \in A'$ with $c(x) = c_1 \neq x$, $c(y) = c_2 \neq y$ (they exist according to (ii) (c)). Then (ii) (d) yields

(4) $x \vee c_1 \ngeqq C$, $x \vee c_2 \geqq C$,

(5) $y \vee c_1 \geqq C$, $y \vee c_2 \ngeqq C$.

This implies that $x \in f^{-n}(c_1)$ for some $n \in N \cup \{0\}$, $y = f^m(c_2)$ for some $m \in N \cup \{0\}$, but then $x \vee y \geqq C$, a contradiction to (d). Therefore (1.5) is valid. Since we have shown that (A', f) is connected and since (e) holds, we get that A' is a connected component of (A, f) and (i) is valid.

4.2.2. Corollary. *Let $(A, f) \in \mathscr{U}$. Then $(A, f) \in \mathscr{U}_2$ if and only if there is $C = \{c_1, c_2\} \subseteq A$, $c_1 \neq c_2$, such that* 4.2.1 (ii) *is valid, where $A' = A$.*

4.3.1. Lemma. *Let $(A, f) \in \mathscr{U}$, $C \subseteq A' \subseteq A$, $2 <$ card $C < \aleph_0$. The following conditions are equivalent:*

(i) *C is a cycle of (A, f), A' is a connected component of (A, f), $(A', f) \in \mathscr{U}_3$;*

(ii) (a) *$c_1 \vee c_2 = C$ for each $c_1, c_2 \in C$, $c_1 \neq c_2$;*

(b) *if $x \in A'$, then there is a unique $c = c(x) \in C$ with $x \vee c \ngeqq C$;*

(c) *if $z \in A - A'$, $c \in C$, then $x \vee c \ngeqq C$.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. Let (ii) hold. If for distinct elements $c_1, c_2, c_3 \in A$ the relation $c_1 \vee c_2 = c_1 \vee c_3$ is valid, then c_2, c_3 belong to some cycle C_1. Therefore (ii) (a) implies that $C \subseteq C_1$, where C_1 is a cycle. Then

$$c_1 \vee c_2 = C_1 \quad \text{for each} \quad c_1, c_2 \in C\,, \quad c_1 \neq c_2\,,$$

and hence $C_1 = C$. Let A_1 be the connected component containing C, $x \in A_1$. There exists the least nonnegative integer n such that $f^n(x) \in C$; put $c = f^{n+1}(x)$. Then

$$x \vee c \geqq \{x\} \cup C \geqq C$$

and (ii) (c) yields that $x \in A'$. Hence

(1) $A_1 \subseteq A'$.

Suppose that there is $y \in A' - A_1$. Then $y \vee c \ngeqq C$ for each $c \in C$, a contradiction to (ii) (b). According to (1) we obtain

(2) $A_1 = A'$,

thus (i) is valid.

4.3.2. Corollary. *Let $(A, f) \in \mathscr{U}$. Then $(A, f) \in \mathscr{U}_3$ if and only if there is $C \subseteq A$ with $2 <$ card $C < \aleph_0$ such that* 4.3.1 (ii) *is valid, where $A' = A$.*

4.5.1. Notation. If $(A, f) \in \mathscr{U}$, $x, y, z \in A$, then by the symbol $\langle x, y, z\rangle$ we denote the sublattice of $\mathrm{Co}(A, f)$ which is generated by x, y and z (considered as atoms in $\mathrm{Co}(A, f)$). Further, $\mathbf{3} \in \mathscr{P}_1$ denotes a 3-element chain.

4.5.2. Lemma. (a) (cf. [6], p. 228) *Let $P \in \mathscr{P}$, $a, b, c \in P$. Then* $\langle a, b, c\rangle \cong \mathrm{Co}(\mathbf{3})$ *if and only if a, b, c form a 3-element chain in P.*

(b) (cf. [6], Thm. 10) *Let $P \in \mathscr{P}$, $c \in P$. Then c is maximal or minimal in P if and only if*

$$(X \vee Y) \wedge c = (X \wedge c) \vee (Y \wedge c) \quad \textit{for each} \quad X, Y \in \mathrm{Co}(P)\,.$$

4.6.1. Lemma. *Let (A,f) be a partial monounary algebra with coherent connected components, $A' \subseteq A$. The following conditions are equivalent:*

(i) *A' is a connected component of (A,f), $(A',f) \in \mathscr{U}_i$;*

(ii) (a) *A' does not contain any connected component B' of (A,f) with the property $(B',f) \in \mathscr{U}_0 \cup \mathscr{U}_2 \cup \mathscr{U}_3$;*

(b) *there is $c \in A'$ such that*

(b1) *$(X \vee Y) \wedge c = (X \wedge c) \vee (Y \wedge c)$ for each $X, Y \in \mathrm{Co}(A,f)$;*

(b2) *if $x, y, z \in A'$, $x \neq c \neq z$, $\langle x, y, z\rangle \cong \mathrm{Co}(\mathbf{3})$, then $\langle c, x, z\rangle \cong \mathrm{Co}(\mathbf{3})$;*

(c) *if $z \in A - A'$, $x \in A'$, then $x \vee z$ covers x and z.*

Proof. Let (i) hold. According to 2.1 and 2.2 there exists $(A', \leqq) \in \mathscr{P}_1$ corresponding to (A',f). Let c be the greatest element of $(A', \leqq)$. Then 4.5.2 (b) implies that (b1) is valid. Assume that $x, y, z \in A'$, $x \neq c \neq z$, $\langle x, y, z\rangle \cong \mathrm{Co}(\mathbf{3})$. Without loss of generality we may suppose that $x \leqq z$. Then $x \leqq z \leqq c$, hence 4.5.2 (a) yields that $\langle c, x, z\rangle \cong \mathrm{Co}(\mathbf{3})$. The assertions (ii) (a) and (ii) (c) are obvious.

Conversely, let (ii) hold. It follows from (a) and 2.1 – 2.3 that there exists $(A', \leqq) \in \mathscr{P}$ corresponding to (A',f) ($(A', \leqq)$ need not be connected). Since (b1) is valid, according to 4.5.2 (b) we obtain that c is minimal or maximal in $(A', \leqq)$. Let A_1 be the connected component containing c. Then (a) yields that $(A_1,f) \in \mathscr{U}_1 \cup \mathscr{U}_4$. Assume that $A' - A_1 \neq \emptyset$ and let $A_2 \subseteq A' - A_1$ be a connected component of (A,f). Then $(A_2,f) \in \mathscr{U}_1 \cup \mathscr{U}_4$. Since each connected component of (A,f) is coherent, there exist distinct elements $x, y, z \in A_2$ with $x \leqq y \leqq z$ and thus (in view of 4.5.2 (a)) $\langle x, y, z\rangle \cong \mathrm{Co}(\mathbf{3})$. Then (b2) yields $\langle c, x, z\rangle \cong \mathrm{Co}(\mathbf{3})$, a contradiction to 4.5.2 (a), because $x, z \notin A_1$, $c \in A_1$. Therefore $A' \subseteq A_1$. Since A_1 is a coherent connected component of (A,f), $(A_1,f) \in \mathscr{U}_1 \cup \mathscr{U}_4$, then there is $z \in A_1 - A'$ and $x \in A'$ such that $x \vee z$ does not cover x, which contradicts (c). Hence $A' = A_1$, A' is a connected component of (A,f). Suppose that $(A',f) \in \mathscr{U}_4$. Then c must be minimal. Since $(A',f) \notin \mathscr{U}_0$, there is $x_1 \in A'$ such that $c \leqq x_1$ does not hold. Denote $y = f(x_1)$, $z = f(y)$. We obtain (in view of 4.5.2 (a))

$$\langle x, y, z\rangle \cong \mathrm{Co}(\mathbf{3})\,, \quad \langle c, x, z\rangle \ncong \mathrm{Co}(\mathbf{3})\,,$$

a contradiction to (b2). Therefore $(A',f) \in \mathscr{U}_1$.

4.6.2. Corollary. *Let $(A,f) \in \mathscr{U}$. Then (A,f) is coherent and belongs to $\mathscr{U}_1$ if and only if $S(A,f) = \emptyset$,* card $A > 1$ *and* 4.6.1 (ii) *holds, where $A' = A$.*

4.7.1. Lemma. *Let (A,f) be a partial monounary algebra with coherent connected components, $\emptyset \neq A' \subseteq A$. The following conditions are equivalent:*

(i) *A' is a connected component of (A,f), $(A',f) \in \mathscr{U}_4$;*

(ii) (a) *A' does not contain any connected component B' of (A,f) such that $(B',f) \in \mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$;*

(b) *if A' is a disjoint union of nonempty A_1 and A_2, then there are $a_1 \in A_1$, $a_2 \in A_2$ such that $a_1 \vee a_2$ does not cover a_1;*

(c) *if $z \in A - A'$, $x \in A'$, then $x \vee z$ covers x and z.*

Proof. Let (i) hold. It is obvious that (a) and (c) are valid. Suppose that the assumption of (b) is satisfied. The relation $(A',f) \in \mathcal{U}_4$ implies that there are $x, y, z \in$ $\in A'$ such that

(1) $f(x) = y$, $f(y) = z$,

and either

(2.1) $\{x, y\} \subseteq A_i$, $z \in A_j$,

or

(2.2) $x \in A_i$, $\{y, z\} \subseteq A_j$,

or

(2.3) $\{x, z\} \subseteq A_i$, $y \in A_j$,

where $\{i, j\} = \{1, 2\}$. If (2.1) or (2.2) holds, put $a_i = x$, $a_j = z$. Then $a_i \vee a_j =$ $= \{x, y, z\}$ and it covers neither a_i nor a_j. Let (2.3) hold. If $\{f^n(z): n \in N\} \subseteq A_i$, we can take $a_j = y$, $a_i = f(z)$; then

$$a_i \vee a_j = \{y, z, f(z)\} = \{a_j, a_i, f(z)\}$$

and it covers neither a_i nor a_j. If $v \in \{f^n(z): n \in N\} \cap A_j \neq \emptyset$, put $a_j = v$, $a_i = x$; then

$$a_i \vee a_j = \{x, f(x), \ldots, v\},$$

and it covers neither a_i nor a_j. Therefore (b) is valid.

Now assume that (ii) holds. Let $a \in A'$ and let B_1 be the connected component containing a, $A_1 = B_1 \cap A'$, $A_2 = A - A_1$. If $A_2 \neq \emptyset$, then $a_1 \vee a_2$ covers a_1 and a_2 for each $a_1 \in A_1$, $a_2 \in A_2$ (since a_2 does not belong to the same connected component as a_1), which is a contradiction to (b). Thus $A_2 = \emptyset$, i.e.,

(3) $A' \subseteq B_1$.

Since B_1 is a coherent connected component of (A,f), according to (3) we obtain that there are $z \in B_1 - A'$ and $x \in A'$ such that $x \vee z$ covers neither x nor z, which contradicts (c). Hence A' is a connected component of (A,f) and (a) yields that $(A',f) \in \mathcal{U}_4$.

4.7.2. Corollary. *Let $(A,f) \in \mathcal{U}$. Then $(A,f) \in \mathcal{U}_4$ if and only if $S(A,f) = \emptyset$ and* 4.7.1 (ii) *is valid, where $A' = A$.*

4.8. Corollary. *Let (A,f) and (A, g) be partial monounary algebras with coherent connected components,* $\operatorname{Co}(A,f) = \operatorname{Co}(A, g)$ *and $i \in \{0, 1, \ldots, 4\}$. Then $(A,f) \in \mathcal{U}_i$ if and only if $(A, g) \in \mathcal{U}_i$.*

Proof. If $i = 0$, then the assertion follows from 4.1.4 (the conditions considered are expressed merely by $\mathrm{Co}(A,f)$ in view of 3.2 and 4.1.1). For $i = 2$ and $i = 3$ the result is obtained in view of 4.2.2 and 4.3.2. Since the condition (ii) (a) of 4.6.1 can be expressed in terms of $\mathrm{Co}(A,f)$ (with respect to 4.1.3, 4.2.1 and 4.3.1), the assertion holds for $i = 1$ according to 4.6.2. Analogously, 4.7.2 implies that it is valid for $i = 4$.

4.9. Theorem. *Let (A,f) and (A, g) be partial monounary algebras with coherent connected components,* $\mathrm{Co}(A,f) = \mathrm{Co}(A, g)$, *$A' \subseteq A$. Then A' is a connected component of (A,f) if and only if A' is a connected component of (A, g). Moreover, if $i \in \{0, 1, \ldots, 4\}$, then $(A',f) \in \mathcal{U}_i$ if and only if $(A', g) \in \mathcal{U}_i$.*

Proof. The assertion follows from 4.1.3, 4.2.1, 4.3.1, 4.6.1 and 4.7.1 (because all conditions considered can be expressed merely in terms of the system $\mathrm{Co}(A,f)$).

5. COHERENT (A,f), (A,g) WITH $\mathrm{Co}(A,f) = \mathrm{Co}(A,g)$

In this part we investigate the relations between (A,f) and (A,g) under the assumption that $\mathrm{Co}(A,f) = \mathrm{Co}(A, g)$, where (A,f), (A, g) are coherent and (A,f), $(A, g) \in \mathcal{U}_i$ $(i = 0, 1, \ldots, 4)$.

Notation. Let $(A,f) \in \mathcal{U}_4$ and $f(a) = f(b)$ imply $a = b$ for each $a, b \in A$. Put $h(x) = y$ if and only if $f(y) = x$. Then (A, h) is a monounary algebra, and will be denoted by the symbol $(A,f)^-$.

5.1. Lemma. *Let $(A,f), (A, g) \in \mathcal{U}_4$ and* $\mathrm{Co}(A,f) = \mathrm{Co}(A, g)$.

(i) *If $f(a) = f(b)$ implies $a = b$ for each $a, b \in A$, then either $(A, g) = (A,f)$ or $(A, g) = (A,f)^-$.*

(ii) *If there are distinct elements $a, b \in A$ with $f(a) = f(b)$, then $(A, g) = (A,f)$.*

Proof. In [6], p. 231–232 it was proved:

(BB1) If $(P, \leqq_1)$ and $(P, \leqq_2)$ are coherent, $\mathrm{Co}(P, \leqq_1) = \mathrm{Co}(P, \leqq_2)$, then either $(P, \leqq_2) = (P, \leqq_1)$ or $(P, \leqq_2) = (P, \leqq_1)^-$, where $(P, \leqq_1)^-$ is the dual of $(P, \leqq_1)$.

According to 2.3 there is $(A, \leqq_1)$ corresponding to (A,f) and $(A, \leqq_2)$ corresponding to (A, g). Then $\mathrm{Co}(A, \leqq_1) = \mathrm{Co}(A, \leqq_2)$, hence either $(A, \leqq_2) = (A, \leqq_1)$ or $(A, \leqq_2) = (A, \leqq_1)^-$. Thus (i) holds. If the assumption of (ii) is valid, there is no $\mathcal{B} \in \mathcal{U}$ such that $(A, \leqq_1)^-$ corresponds to $\mathcal{B}$. Hence we get that $(A, \leqq_2) = (A, \leqq_1)$, i.e., $(A, g) = (A,f)$.

5.2. Theorem. *Let $(A,f), (A, g) \in \mathcal{U}_4$.*

(i) *If the assumption of* (ii) *from* 5.1 *is valid, then* $\mathrm{Co}(A,f) = \mathrm{Co}(A, g)$ *if and only if $(A,f) = (A, g)$.*

(ii) *If the assumption of* (i) *from* 5.1 *holds, then* $\mathrm{Co}(A,f) = \mathrm{Co}(A, g)$ *if and only if $(A, g) \in \{(A,f), (A,f)^-\}$.*

Proof. The assertion follows from 5.1 and from the fact that $\mathrm{Co}(A,f)^{-} = \mathrm{Co}(A,f)$.

5.3.1. Notation. Let $(A,f) \in \mathscr{U}_0$ and (A,f) be coherent, i.e., there is $N_1 = \{1, \ldots, n\}$, $n > 2$ or $N_1 = N$ such that $A = \{a_k : k \in N_1\}$ and one of the following conditions is satisfied:

(a) $N_1 = N, f(a_k) = a_{k+1}$ for each $k \in N_1$,

(b) $f(a_k) = a_{k-1}$ for each $k \in N_1 - \{1\}, f(a_1) = a_1$,

(c) $f(a_k) = a_{k-1}$ for each $k \in N_1 - \{1\}, f(a_1)$ does not exist,

(d) $f(a_k) = a_{k-1}$ for each $k \in N_1 - \{1\}, f(a_1) = a_2$.

If $N_1 = N$, then the partial operation on A fulfilling the condition (a) ((b), (c), (d)) will be denoted by the symbol f_0 (f_1, f_2, f_3). Let $N_1 \neq N$. By the symbol $f_1(f_2, f_3)$ we denote the partial operation on A such that $f_1(f_2, f_3)$ fulfils the condition (b) ((c), (d)).

Further, put

$$f_4(a_k) = f_5(a_k) = f_6(a_k) = a_{k+1} \quad \text{for each} \quad k \in N_1 - \{n\}\,,$$

$$f_4(a_n) = a_n,\ f_5(a_n) \quad \text{does not exist}\,, \quad f_6(a_n) = a_{n-1}\,.$$

5.3.2. Theorem. *Let (A,f) and (A,g) be coherent, (A,f), $(A,g) \in \mathscr{U}_0$,* card $A = \aleph_0$. *Then* $\mathrm{Co}(A,f) = \mathrm{Co}(A,g)$ *if and only if* $(A,g) \in \{(A,f_i) : i \in \{(0, 1, 2, 3\}\}$.

Proof. It is obvious that if $(A,g) = (A,f_i)$ for some $i \in \{0, 1, 2, 3\}$, then $\mathrm{Co}(A,f) = \mathrm{Co}(A,g)$. Assume that $\mathrm{Co}(A,f) = \mathrm{Co}(A,g)$. Denote by $(A, \leqq_1)$ and $(A, \leqq_2)$ the posets corresponding to (A,f) and (A,g) (they are uniquely determined in view of 2.3.1). Then $\mathrm{Co}(A, \leqq_1) = \mathrm{Co}(A, \leqq_2)$ and (BB1) yields that either $(A, \leqq_2) = (A, \leqq_1)$ or $(A, \leqq_2) = (A, \leqq_1)^{-}$. Consider the first case. If (a) from 5.3.1 is valid, then $(A,g) = (A,f)$. If (b) or (c) or (d) holds, then $(A,g) \in \{(A,f_i) : i \in \{1, 2, 3\}\}$. In the latter case, (a) from 5.3.1 implies that $(A,g) \in \{(A,f_i) : i \in \{1, 2, 3\}\}$ and if (b) or (c) or (d) is valid, then $(A,g) = (A,f_0)$.

5.3.3. Theorem. *Let (A,f) and (A,g) be coherent, (A,f), $(A,g) \in \mathscr{U}_0$ and* card $A < \aleph_0$. *Then* $\mathrm{Co}(A,f) = \mathrm{Co}(A,g)$ *if and only if* $(A,g) \in \{(A,f_i) : i \in \{1, 2, \ldots, 6\}\}$.

Proof. It follows from 5.3.1 that (A,f) fulfils some of the conditions (b), (c) or (d). It is clear that if $(A,g) = (A,f_i)$ for some $i \in \{1, 2, \ldots, 6\}$, then $\mathrm{Co}(A,f) = \mathrm{Co}(A,g)$. Now let $\mathrm{Co}(A,f) = \mathrm{Co}(A,g)$. According to 2.3.1 there are posets $(A, \leqq_1)$ and $(A, \leqq_2)$ corresponding to (A,f) and (A,g). Hence $\mathrm{Co}(A, \leqq_1) = \mathrm{Co}(A, \leqq_2)$ and (BB1) imply that either $(A, \leqq_2) = (A, \leqq_1)$ or $(A, \leqq_2) = (A, \leqq_1)^{-}$. If $(A, \leqq_2) = (A, \leqq_1)$, then $(A,g) \in \{(A,f_i) : i \in \{1, 2, 3\}\}$. If $(A, \leqq_2) = (A, \leqq_1)^{-}$, then $(A,g) \in \{(A,f_i) : i \in \{4, 5, 6\}\}$.

5.4.1. Notation. Let $(A,f) \in \mathscr{U}_1$ be coherent. Assume that there is $c \in A$ such that one of the following conditions is satisfied:

(a) $f(c) = c$, card $f^{-1}(c) = 2$,

(b) $f(c)$ does not exist, $\operatorname{card} f^{-1}(c) = 1$,

(c) $f^{-1}(c) = \{f(c)\} \neq \{c\}$.

Denote $f_1(x) = f_2(x) = f_3(x) = f(x)$ for each $x \in A - \{c\}$, $f_1(c) = c$, $f_2(c)$ does not exist and $f_3(c) = a$, where $\{a\} = f^{-1}(c) - \{c\}$ (such a exists and is uniquely determined).

5.4.2. Theorem. *Let the assumption of* 5.4.1 *hold, let* $(A, g) \in \mathscr{U}_1$ *be coherent. Then* $\operatorname{Co}(A, f) = \operatorname{Co}(A, g)$ *if and only if* $(A, g) \in \{(A, f_i): i \in \{1, 2, 3\}\}$.

Proof. It is obvious that if $(A, g) \in \{(A, f_i): i \in \{1, 2, 3\}\}$, then $\operatorname{Co}(A, f) = \operatorname{Co}(A, g)$. Let $\operatorname{Co}(A, f) = \operatorname{Co}(A, g)$. Denote by $(A, \leqq_1)$ and $(A, \leqq_2)$ the posets corresponding to (A, f) and (A, g). According to (BB1) either $(A, \leqq_2) = (A, \leqq_1)$or $(A, \leqq_2) = (A, \leqq_1)^-$. Since $(A, f) \in \mathscr{U}_1$, there are distinct elements $a, b \in A$ with $f(a) = f(b)$. Then there is no $\mathscr{B} \in \mathscr{U}$ such that $(A, \leqq_1)^-$ corresponds to $\mathscr{B}$ and therefore $(A, \leqq_2) = (A, \leqq_1)$. This yields that $(A, g) \in \{(A, f_i): i \in \{1, 2, 3\}\}$.

5.5.1. Notation. Let $(A, f) \in \mathscr{U}_1$ be coherent and let the assumption of 5.4.1 fail to be fulfilled. It follows from the definition of $\mathscr{U}_1$ that there is $c \in A$ such that $\operatorname{card}(f^{-1}(c) - \{c\}) \geqq 2$ and either $f(c) = c$ or $f(c)$ does not exist. Put $f_1(x) = f_2(x) = f(x)$ for each $x \in A - \{c\}$, $f_1(c) = c$ and $f_2(c)$ does not exist.

5.5.2. Theorem. *Let the assumption of* 5.5.1 *hold, let* $(A, g) \in \mathscr{U}_1$ *be coherent. Then* $\operatorname{Co}(A, f) = \operatorname{Co}(A, g)$ *if and only if* $(A, g) \in \{(A, f_i): i \in \{1, 2\}\}$.

Proof. Analogously as 5.4.2.

5.6.1. Notation. Let $(A, f) \in \mathscr{U}_2 \cup \mathscr{U}_3$, let $C = \{c_1, c_2, \ldots, c_n\}$ be a cycle of (A, f). The set of all permutations of $1, 2, \ldots, n$ will be denoted by S_n. If $\beta \in S_n$, put $f_\beta(x) = f(x)$ for each $x \in A - C$, $f_\beta(c_{\beta(i+1)})$ for each $i \in \{1, 2, \ldots, n-1\}$, $f_\beta(c_{\beta(n)}) = c_{\beta(1)}$.

5.6.2. Theorem. *Let the assumption of* 5.6.1 *hold,* $(A, g) \in \mathscr{U}_2 \cup \mathscr{U}_3$. *Then* $\operatorname{Co}(A, f) = \operatorname{Co}(A, g)$ *if and only if* $(A, g) \in \{(A, f_\beta): \beta \in S_n\}$.

Proof. If $(A, g) = (A, f_\beta)$ for some $\beta \in S_n$, then obviously $\operatorname{Co}(A, f) = \operatorname{Co}(A, g)$. Assume that $\operatorname{Co}(A, f) = \operatorname{Co}(A, g)$. It follows from 4.2.1 and 4.3.1 that C is a cycle of (A, g). Define an equivalence relation θ on A as follows:

$$(1) \quad x\theta = \begin{cases} \{x\}, & \text{if } x \in A - C, \\ C, & \text{if } x \in C. \end{cases}$$

(Instead of $x\theta = \{x\}$ we shall write also $x\theta = x$.) Then θ is a congruence of (A, f) and of (A, g), $(A, f)/\theta \in \mathscr{U}_0 \cup \mathscr{U}_1$, $(A, g)/\theta \in \mathscr{U}_0 \cup \mathscr{U}_1$. Denote $(A, f)/\theta = (A', f')$, $(A, g)/\theta = (A', g')$, where $A' = (A - C) \cup \{C\}$. Since C is a cycle of (A, f) and (A, g), we obtain that $f(c) \in C$ and $g(c) \in C$ for each $c \in C$ and hence

(2) $f'(C) = C$, $g'(C) = C$.

According to 2.1–2.3 there are $(A', \leqq_1)$ and $(A', \leqq_2)$ corresponding to (A', f') and (A', g'). Thus $\operatorname{Co}(A', \leqq_1) = \operatorname{Co}(A', \leqq_2)$ and (BB1) implies that either

$(A', \leqq_2) = (A', \leqq_1)$ or $(A', \leqq_2) = (A', \leqq_1)^-$. Now (2) implies that then $(A', \leqq_2) = = (A', \leqq_1)$, $g'(x) = f'(x)$ for each $x \in A' - \{C\}$. Therefore $g(x) = f(x)$ for each $x \in A - C$. Since C is a cycle of (A, g), this yields $(A, g) \in \{(A, f_\beta): \beta \in S_n\}$.

6. THE GENERAL CASE

In the present section we proceed as follows. Let (A, f) and (A, g) be partial monounary algebras. We have to decide whether

(1) $\mathrm{Co}(A, f) = \mathrm{Co}(A, g)$

is valid.

First we show that there exists a uniquely defined partial monounary algebra (A, f_1) such that

(i) each connected component of (A, f_1) is coherent,

(ii) $\mathrm{Co}(A, f) = \mathrm{Co}(A, f_1)$.

In the same way we can construct a partial monounary algebra (A, g_1).

For deciding whether (1) holds it suffices now to decide whether

(2) $\mathrm{Co}(A, f_1) = \mathrm{Co}(A, g_1)$

is valid. In view of (2) and 4.9, (A, f_1) and (A, g_1) have the same connected components $\{A_j\}_{j \in J}$, and $(A_j, f_1) \in \mathcal{U}_i$ if and only if $(A_j, g_1) \in \mathcal{U}_i$ $(i = 0, 1, \ldots, 4)$; for deciding whether

(3) $\mathrm{Co}(A_j, f_1) = \mathrm{Co}(A_j, g_1)$

Section 5 can be used. The results are summarized in Theorem 6.3.

The notion introduced in the next definition is analogous to that given in the paper by Birkhoff and Bennett [6].

6.1. Definition. Let $(A, f), (A, g) \in \mathcal{U}$. If $x \in A - (S(A, f) \cap S(A, g))$ implies $g(x) = f(x)$, then we say that (A, f) and (A, g) are *isomedic* and write $(A, f) \approx (A, g)$.

6.2.1. Notation. Let $(A, f) \in \mathcal{U}$. Put

$$(1) \quad f_1(x) = \begin{cases} f(x) & \text{if} \quad x \in A - S(A, f), \\ x & \text{if} \quad x \in S(A, f). \end{cases}$$

6.2.2. Lemma. *Let $(A, f) \in \mathcal{U}$. Then $(A, f_1) \in \mathcal{U}$ and*

(i) *each connected component of (A, f_1) is coherent;*

(ii) $(A, f) \approx (A, f_1)$;

(iii) $\mathrm{Co}(A, f) = \mathrm{Co}(A, f_1)$.

Proof. Definition 3.1 of $S(A, f)$ yields that $(A, f_1) \in \mathcal{U}$ and $S(A, f_1) = S(A, f)$. Further, each connected component of (A, f_1) is coherent. Then (1) yields that (ii) is valid. Let $U \in \mathrm{Co}(A, f)$, $a, b \in U$, $a \neq b$, $k \in N$ be such that

(2) $f^k(a) = b$, $f_1^{k_1}(a) \neq b$ for each $k_1 < k$.

Further, let $c = f_1^m(a)$, $0 < m < k$. Since $f_1(a) \neq a$, $f_1^2(a) \neq f_1(a), \ldots, f_1^k(a) \neq$ $\neq f_1^{k-1}(a)$, we get (according to (1))

(3) $f_1(a) = f(a)$, $f_1^2(a) = f^2(a), \ldots, f_1^m(a) = f^m(a), \ldots, f_1^k(a) = f^k(a) = b$.

Then (2) and (3) imply

(4) $f^k(a) = b$, $f^{k_1}(a) \neq b$ for each $k_1 < k$, $c = f^m(a)$,

and the relation $U \in \mathrm{Co}(A,f)$ implies that $c \in U$. Therefore

(5) $\mathrm{Co}(A,f) \subseteq \mathrm{Co}(A,f_1)$.

Now assume that $V \in \mathrm{Co}(A,f_1)$, $a, b \in V$, $a \neq b$, $k \in N$ and

(6) $f^k(a) = b$, $f^{k_1}(a) \neq b$ for each $k_1 < k$.

Let $c = f^m(a)$, $0 < m < k$. Then $k \geqq 2$ and then $a \notin S(A,f)$, $f(a) \notin S(A,f), \ldots$ $\ldots, f^k(a) \notin S(A,f)$, thus (1) yields

(7) $f_1(a) = f(a), \ldots, f_1^m(a) = f^m(a), \ldots, f_1^k(a) = f^k(a) = b$.

Since $V \in \mathrm{Co}(A,f_1)$, we obtain that $c \in V$ and this implies $V \in \mathrm{Co}(A,f)$, i.e.

(8) $\mathrm{Co}(A,f_1) \subseteq \mathrm{Co}(A,f)$.

Therefore (iii) is valid by virtue of (5) and (8).

6.3. Theorem. *Let $(A,f), (A,g) \in \mathscr{U}$. Then* $\mathrm{Co}(A,f) = \mathrm{Co}(A,g)$ *if and only if the following conditions are satisfied:*

(i) *(A,f_1) and (A,g_1) have the same connected components $\{A_j\}_{j \in J}$;*

(ii) $\mathrm{Co}(A_j,f_1) = \mathrm{Co}(A_j,g_1)$ *for each $j \in J$.*

Proof. Assume that $\mathrm{Co}(A,f) = \mathrm{Co}(A,g)$. According to 6.2.2, (A,f_1) and (A,g_1) have coherent connected components and we have

(1) $\mathrm{Co}(A,f_1) = \mathrm{Co}(A,f) = \mathrm{Co}(A,g) = \mathrm{Co}(A,g_1)$.

It follows from 4.9 that (i) is valid. Then (1) and (i) imply that (ii) holds.

Conversely, let the conditions (i) and (ii) be satisfied. Without loss of generality it suffices to show that $\mathrm{Co}(A,f) \subseteq \mathrm{Co}(A,g)$. In view of 6.2.2 (iii) we are to prove that $\mathrm{Co}(A,f_1) \subseteq \mathrm{Co}(A,g_1)$. Let $U \in \mathrm{Co}(A,f_1)$. Then U is the disjoint sum of $U_j =$ $= U \cap A_j$, $j \in J$ and we have

(2) $U_j \in \mathrm{Co}(A_j,f_1)$ for each $j \in J$.

Then (iii) implies

(3) $U_j \in \mathrm{Co}(A_j,g_1)$ for each $j \in J$

and therefore U (the disjoint sum of U_j, $j \in J$) belongs to $\mathrm{Co}(A,g_1)$. Hence $\mathrm{Co}(A,f_1) \subseteq \mathrm{Co}(A,g_1)$.

6.3.1. Remark. Let us recall that the validity of the condition (ii) from 6.3 can be decided by applying 4.8, 5.2, 5.3.2, 5.3.3, 5.4.2, 5.5.2 and 5.6.2.

6.4.1. Corollary. *A partial monounary algebra (A,f) is uniquely determined by* $\mathrm{Co}(A,f)$ *if and only if each connected component A' of (A,f) belongs to $\mathscr{U}_2 \cup \mathscr{U}_4$ and there are distinct elements $a, b \in A'$ with $f(a) = f(b)$.*

Proof. The assertion follows from 6.3 and from the theorems of Section 5.

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